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Stability Analysis for Parametric Vector Optimization Problems

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Preface

We study stability of minimal points and solutions to parametric (or perturbed) vector optimization problems in the framework of real topological vector spaces and, if necessary, normed spaces. Because of particular importance of finite-dimensional problems, called multicriteria optimization problems, which model various real-life phenomena, a special attention is paid to the finite-dimensional case. Since one can hardly expect the sets of minimal points and solutions to be singletons, set-valued mappings are natural tools for our studies.

Vector optimization problems can be stated as follows. Let X be a topological space and let Y be a topological vector space ordered by a closed convex pointed cone $\mathcal{K} \subset Y$. Vector optimization problem

$$\begin{aligned} &\mathcal{K} - \min \ f_0(x) \\ &\text{subject to } x \in A_0\,, \qquad (P_0) \end{aligned}$$

where $f : X \to Y$ is a mapping, and $A_0 \subset X$ is a subset of X, relies on finding the set $\mathrm{Min}(f_0, A_0, \mathcal{K}) = \{y \in f_0(A_0) \mid f_0(A_0) \cap (y - \mathcal{K}) = \{y\}\}$ called the **Pareto** or **minimal point** set of (P_0), and the **solution set** $S(f_0, A_0, \mathcal{K}) = \{x \in A_0 \,| f_0(x) \in \mathrm{Min}(f_0, A_0, \mathcal{K})\}$. We often refer to problem (P_0) as the **original problem** or **unperturbed one**. The space X is the **argument** space and Y is the **outcome** space.

Let U be a topological space. We embed the problem (P_0) into a family (P_u) of vector optimization problems parametrised by a parameter $u \in U$,

$$\begin{aligned} &\mathcal{K} - \min \ f(u, x) \\ &\text{subject to } x \in A(u)\,, \qquad (P_u) \end{aligned}$$

where $f : U \times X \to Y$ is the parametrised objective function and $A : U \rightrightarrows Y$, is the feasible set multifunction, (P_0) corresponds to a parameter value u_0. The performance multifunction $\mathcal{M} : U \rightrightarrows Y$, is defined as $\mathcal{M}(u) = \text{Min}(f(u,\cdot), A(u), \mathcal{K})$, and the solution multifunction $\mathcal{S} : U \rightrightarrows Y$, is given as $\mathcal{S}(u) = S(f(u,\cdot), A(u), \mathcal{K})$, and $f : U \times X \to Y$, $A(u) \subset X$.

Our aim is to study continuity properties of $\mathcal{M}$ and $\mathcal{S}$ as functions of the parameter u. Continuous behaviour of solutions as functions of parameters is of crucial importance in many aspects of the theory of vector optimization as well as in applications(correct formulation of the model and/or approximation) and numerical solution of the problem in question.
We investigate continuity in the sense of Hausdorff and Hőlder of the multivalued mappings of minimal points $\mathcal{M}(u)$ and solutions $\mathcal{S}(u)$ as functions of the parameter u under possibly weak assumptions. We attempt to avoid as much as possible compactness assumptions which are frequently over-used (see eg [83]).
It is a specific feature of vector optimization that the outcome space is equipped with a partial order generated by a cone the properties of which are important for stability analysis. In many spaces cones of nonnegative elements have empty interiors and because of this we derive stability results for cones with possibly empty interior. This kind of results are specific for vector optimization and do not have their counterpart in scalar optimization.
We introduce two new concepts: the notion of containment(with some variants for cones with empty interiors),[16], and the notion of strict minimality, [12].

The containment property (CP), defined in topological vector spaces, is introduced to study upper semicontinuities (in the sense of Hausdorff) of minimal points, [11, 16]. It is a variant of the domination property (DP), which appears frequently in the context of stability of solutions to parametric vector optimization problems. Although it is not a commonly adopted view point, the domination property may be accepted as a solution concept which generalizes the standard concept of a solution to scalar optimization problem. In consequence, the containment property (CP) may also be seen as a solution concept in vector optimization. To investigate more deeply this aspect we interpret the containment property as a generalization of the concept of the set of ϕ–local solutions appearing in the

context of Lipschitz continuity of solutions to scalar optimization problems. Under mild assumptions the containment property imply that the set weakly minimal points equals the set of minimal points. This equality, in turn, is a typical ingradient of standard finite-dimensional sufficient conditions for upper semicontinuity of minimal points.
To study Hőlder upper continuity of minimal points we define the rate of containment of a set with respect to a cone, which is a real-valued function of a scalar argument, see [14, 15]. The rate of growth of this function influence decisively the rate of Hőlder continuity of minimal points, [15].

Strictly minimal points are introduced to study lower semicontinuities (lower Hausdorff, lower Hőlder) of minimal points [20, 13]. The definition of a strictly minimal point is given in topological vector spaces and it is a generalization of the notion of a super efficient point in the sense of Borwein and Zhuang defined in normed spaces. We discuss strict minimality in vector optimization by proving that it is a vector counterpart of the concept of $\phi-$ local solution to scalar optimization problem.

Theory of vector optimization may be considered as an abstract study of optimization problems with mappings taking values in the outcome space equipped with a partial order structure. As such, it contains many concepts and results which generalize and/or have their counterparts in scalar optimization. The very definition of the set of minimal points of vector optimization problem in the outcome space may serve as an example here. This is a counterpart of the optimal value of scalar optimization problem. Another example is the concept of well-posed optimization problem. In subsequent developments we often compare our results and considerations with the corresponding approaches in scalar optimization. For instance, we define several classes of well-posed vector optimization problems by generalizing the concept of scalar minimizing sequence and in these classes we investigate continuity of solutions. For scalar optimization problems, the existing approaches and results on well-posedness are extensively discussed in the monograph by Dontchev and Zolezzi [33].

Convergence and rates of convergence of solutions to perturbed optimization problems is one of crucial topics of stability analysis in optimization both from theoretical and numerical points of view. For scalar optimization it was investigated by many authors see eg., [72], [32], [47], [78], [55], [81], [59], [60], [82], [2], and many others. An exhaustive survey of current state of research is given in the recent monograph by Bonnans and Shapiro [26]. In vector optimization the results on Lipschitz continuity of solutions are not so numerous, and concern some classes of problems, for linear case see eg.,[28], [29], [30], for convex case see eg., [25], [31].

The organization of the material is as follows. In Chapter 2 we investigate upper Hausdorff continuity of the multivalued mapping M, $M(u) = \mathrm{Min}(\Gamma(u)|\mathcal{K})$ assigning to a given parameter value u from a topological space U the set of minimal points of the set $\Gamma(u) \subset Y$ with respect to cone $\mathcal{K} \subset Y$, where for any subset A of a topological vector space Y the set of minimal points is defined as $\mathrm{Min}(A|\mathcal{K}) = \{y \in A \mid A \cap (y - \mathcal{K}) = \{y\}\}$, and $\Gamma : U \rightrightarrows Y$, is a given multivalued mapping. The main tool which allows us to obtain the general result is the containment property (CP). Some infinite-dimensional examples are discussed. A special attention is paid to the containment property (CP) in finite-dimensional case, when $Y = R^m$.

In Chapter 3 we discuss upper Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the rate of containment δ which is a one-variable nondecreasing function, defined for a given set A and the order generating cone $\mathcal{K}$. The assumption of sufficiently fast growth rate of this function appears to be the crucial assumption for all upper Hőlder stability results of Chapter 3.

In Chapter 4 we apply the results obtained in Chapters 2 and 3 to derive conditions for upper Hausdorff and upper Hőlder stability of minimal points to parametric vector optimization problems by taking $\Gamma(u) = f(u, A(u))$. Moreover, we introduce the concept of $\Phi-$ strong solutions to vector optimization problem (P_0), which is a generalization of the concept of a $\phi-$local minimizer to scalar optimization problem, the latter being introduced by Attouch and

Wets [6].

In Chapter 5 we investigate the lower continuity and lower Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the notion of strict minimality mentioned above and the rate of strict minimality. In Section 5.5 we apply the results obtained in Chapter 5 to parametric vector optimization problems and we derive sufficient conditions for lower and lower Hőlder continuity of Pareto point multivalued mapping $\mathcal{M}$. An important tool here is the notion of $\Phi-$ strict solution to vector optimization problem introduced in Section 6.1. This notion can be interpreted as another possible generalization of the concept of $\phi-$local minimizer.

In Chapter 6 we propose several definitions of a well-posed vector optimization problem. All these definitions are based on properties of $\varepsilon-$solutions to vector optimization problems. For well-posed vector optimization problems we prove upper Hausdorff continuity of solution multivalued mapping S, $S(u) = S(f(u,\cdot), A(u), \mathcal{K})$.

6

Well-posedness in vector optimization and continuity of solutions

We propose several definitions of well-posedness for vector optimization problems in topological spaces. These definitions are based on the properties of ε-solutions to vector optimization problems. In the resulting classes of well-posed problems stability of Pareto solutions is investigated.

The notion of well-posedness and its various generalizations appear to be very fruitfull in the scalar optimization especially in proving stability results. Well-posedness is also important in establishing convergence of algorithms solving scalar optimization problems.
In vector optimization there is no a commonly accepted definition of well-posed problem. Some attempts in this direction has been already done, see Lucchetti, Bednarczuk.

In this section we propose several approaches to well-posedness in vector optimization. The definitions we introduce are analysed mainly from the point of view of stability of well-posed problems under perturbations.

Let Y be a topological vector space ordered by a partial ordering relation $\preceq$ generated by a closed convex pointed cone $\mathcal{K}$, $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$ with nonempty interior, $int\mathcal{K} \neq \emptyset$, and $x \preceq y \overset{\text{def}}{\Longleftrightarrow} y - x \in \mathcal{K}$.

Let $f : X \to Y$ be a function defined on a topological space X, and let $A \subset X$ be a subset of X.

The minimization problem

$$(P) \qquad \text{minimize } f \text{ } on \text{ } A$$

consists in finding the set $S(f, A, \mathcal{K})$ of all **Pareto solutions**, ie. all $\underline{x} \in A$ such that there is no $x \in A$ satisfying $f(\underline{x}) - f(x) \in \mathcal{K} \setminus \{0\}$. The image of the set $S(f, A, \mathcal{K})$ under the mapping f is called the **Pareto set** of (P) and is denoted by $\mathcal{M} = Min(f, A, \mathcal{K})$. If, in the above definitions, instead of cone $\mathcal{K}$ we use cone $\mathcal{K}\prime = \{0\} \cup int\mathcal{K}$, then we obtain **weak Pareto solutions** $WS(f, A, \mathcal{K})$ and **weak Pareto points** $WMin(f, A, \mathcal{K})$. Whenever possible we shall use the simplified notations S, Min, WS, $WMin$.

In the sequel we shall often use **ε-Pareto-solutions** of (P), as defined eg. in [53] and [56]. We recall that a point $\underline{r} \in A$ is an **ε-Pareto solution** of (P) if there is no $r \in A$ such that $f(\underline{r}) - \varepsilon - f(r) \in \mathcal{K} \setminus \{0\}$. The set of all ε-Pareto solutions will be denoted by $S_\varepsilon(f, A)$ and the set of all the ε-Pareto points (ie. the image of the $S_\varepsilon(f, A)$ under f) will be denoted by $Min_\varepsilon(f, A)$, or shortly S_ε, Min_ε.

We adopt the standard definitions of lower ($l.c.$) and upper ($u.c.$) continuities as defined eg. by Kuratowski [52]. Following Nikodem [?] we use $\mathcal{K}$-semicontinuities. We say that a multifunction $F : X \rightrightarrows Y$ is **$\mathcal{K}$-upper Hausdorff continuous** ($\mathcal{K} - u.H.c.$) at x_0 if for every $0-$neighbourhood V in Y there exists a neighbourhood W of x_0 in X such that $F(x) \subset F(x_0) + V + \mathcal{K}$ for all $x \in W$. $\{0\}$-upper Hausdorff continuity is called **upper Hausdorff continuity**. In the context of vector optimization $\mathcal{K}$-semicontinuities has been investigated by Sterna-Karwat [?].

We also use the following variants of lower semicontinuity. A multifunction $F : X \rightrightarrows Y$ is **inf-lower continuous** ($i.l.c.$) at (x_0, y_0) if for each neighbourhood V of y_0 there exists a neighbourhood U of x_0 such that for each $x \in U$ one has $F(x) \cap (V + \mathcal{K}) \neq \emptyset$.

A multifunction F is **sup-lower continuous** ($s.l.c.$) at (x_0, y_0) if one has $F(x) \cap (V - \mathcal{K}) \neq \emptyset$ for all $x \in U$. We say that F is **sup- or inf- lower continuous at** x_0 if it sup- or inf- lower continuous at all (x_0, y), $y \in F(x_0)$. The above definitions were introduced by Penot and Sterna-Karwat [?], (see also [?]). Moreover, a multifunction $F : X \rightrightarrows Y$ is **uniformly sup-lower continuous** at x_0 if for each $0-$neighbourhood V in Y there exists a neighbourhood U of x_0 such that for each $y_0 \in F(x_0)$ we have $F(x) \cap (y_0 + V - \mathcal{K}) \neq \emptyset$. This property can also be considered on proper subsets of $F(x_0)$. In a similar way we can also define uniform inf-lower continuity.

A function $f : X \to Y$ is $\mathcal{K}$-**lower continuous** at x_0 if for each $0-$neighbourhood W in Y there exists a neighbourhood O of x_0 in X such that $f(x) \in f(x_0) + W + \mathcal{K}$ for all $x \in O$. Analogously, $f : X \to Y$ is $\mathcal{K}$-**upper continuous** at x_0 if for each $0-$neighbourhood W in Y there exists a neighbourhood O of x_0 in X such that $f(x) \in f(x_0) + W - \mathcal{K}$ for all $x \in O$ (see also [40], [58]).

Theorem 6.0.1 *Let X, U be any topological spaces and let Y be a topological vector space. Let $f : X \to Y$ be a $\mathcal{K}$-upper continuous (respectively, $\mathcal{K}$-lower continuous) function on X and let $\mathcal{A} : U \to X$ be a lower semicontinuous multifunction at $u_0 \in U$. Then the multifunction $\mathcal{FA} : U \rightrightarrows Y$ defined as $\mathcal{FA}(u) = f(\mathcal{A}(u))$ for $u \in U$, is sup-lower continuous (respectively, inf-lower continuous) at u_0.*

Proof. Let $y_0 \in \mathcal{FA}(u_0)$. Let us take any open $0-$neighbourhoood Q in Y. There exists an $x_0 \in \mathcal{A}(u_0)$ such that $f(x_0) = y_0$ and, by the upper continuity of f, (respectively, lower continuity of f) there exists an open neighbourhood W of x_0 such that $f(W) \subset y_0 + Q - \mathcal{K}$ (respectively, $f(W) \subset y_0 + Q + \mathcal{K}$). Since $\mathcal{A}$ is lower semicontinuous at u_0, there exists a neighbourhood U of u_0 such that

$$W \cap \mathcal{A}(u) \neq \emptyset \text{ for } u \in U.$$

Now, by taking any $x \in \mathcal{A}(u)$, $x \in W$, $u \in U$, we obtain that $f(x) \in \mathcal{FA}(u)$, $f(x) \in y_0 + Q - \mathcal{K}$, (respectively, $f(x) \in y_0 + Q + \mathcal{K}$ and hence $(y_0 + Q - \mathcal{K}) \cap \mathcal{FA}(u) \neq \emptyset$ (respectively, $(y_0 + Q + \mathcal{K}) \cap \mathcal{FA}(u) \neq \emptyset$) for $u \in U$.

□

For $X = R^n$, $Y = R^p$ an analogous result was proved by Tanino, Sawaragi and Nakayama [83].

Let U be a topological space.

We shall investigate the family of parametric problems of the form

$$P_u \qquad minimize \quad f \ on \ \mathcal{A}(u)$$

in a neighbourhood of a certain $u_0 \in U$. We have $Min(f, \mathcal{A}(u)|\mathcal{K}) = \mathcal{M}(u)$, $WMin(f, \mathcal{A}(u)|\mathcal{K}) = \mathcal{WM}(u)$, $S(f, \mathcal{A}(u)) = \mathcal{S}(u)$, $WS(f, \mathcal{A}(u)) = \mathcal{WS}(u)$, where $\mathcal{A} : U \rightrightarrows X$ is the **feasible set multifunction,** $\mathcal{M} : U \rightrightarrows Y$ is the **optimal value multifunction** , $\mathcal{S} : U \rightrightarrows X$ is the **solution multifunction**. Moreover, $\mathcal{A}(u_0) = A$, $\mathcal{M}(u_0) = Min(f, A, \mathcal{K})$, $\mathcal{S}(u_0) = S(f, A)$.

6.1 Well-behaved vector optimization problems.

6.2 Well-posed vector optimization problems.

Now we are going to introduce several concepts of well-posedness for vector optimization problems. These concepts are relatively close to each other and all are based on the properties of ε-solutions.

For $y \in \mathcal{K}$ and $\eta \in Min(f, A, \mathcal{K})$ we consider the multifunction $\Pi^\eta : \mathcal{K} \rightrightarrows X$ defined as

$$\Pi^\eta(y) = \{x \in A | \ \eta - f(x) + y \in \mathcal{K}\}.$$

For $y = 0$ we have $\bigcup_{\eta \in Min} \Pi^\eta(0) = S$, where $\Pi^\eta(0)$ is the subset of the solution containing the elements $x \in S(f, A)$ for which $f(x) = \eta$. The sets $\Pi^\eta(y)$ has been already used by [86] to investigate some stability properties of sequences of vector optimization problems.

For $y \in \mathcal{K}$ we define the multifunction $\Pi : \mathcal{K} \rightrightarrows X$ by the formula

$$\Pi(y) = \bigcup_{\eta \in Min} \Pi^\eta(y).$$

We call this multifunction the **ε-Pareto-solution multifunction**. Now the definition of well–posed vector optimization problems can be introduced in the following way.

Definition 6.2.1 *The problem* (P) *is* **η-well–posed** *if*
$(i) Min(f, A, \mathcal{K}) \neq \emptyset$,
(ii) *the multifunction* Π^η *is upper continuous at* $\varepsilon = 0$.

Definition 6.2.2 *The problem* (P) *is* **well–posed** *if*
(i) $Min(f, A, \mathcal{K}) \neq \emptyset$,
(ii) *the multifunction* Π *is upper continuous at* $\varepsilon = 0$

The set $\Pi(\varepsilon)$ contains all the ε-solutions, ie. $\Pi(\varepsilon) = S_\varepsilon$ and $\Pi(0) = S(f, A)$.

Definition 6.2.3 *Let (x_n) be a sequence of feasible elements ie. $x_n \in A$, for n=1,... . The sequence (x_n) is said to be a* **minimizing sequence** *of the problem (P) if for each n there exist $y_n \in \mathcal{K}$ and $\eta_n \in Min(f, A, \mathcal{K})$ such that $f(x_n) \leq \eta_n + y_n$, $\lim_n y_n = 0$*

Proposition 6.2.1 *Let X and Y be topological vector spaces with Y satisfying the first countability axiom. The following conditions are equivalent:*

(i) the problems (P) is well-posed in the sense of Definition 6.2.2,

(ii) $Min(f, A, \mathcal{K}) \neq \emptyset$, any minimizing sequence (x_n), $(x_n) \subset A \setminus S(f, A)$, contains a convergent subsequence with the limit point belonging to $S(f, A)$.

Proof.$(ii) \rightarrow (i)$. Suppose on the contrary that the problem (P) is not well-posed. This means that there exists an open set Q containing $\Pi(0)$, a sequence $(y_n) \subset \mathcal{K}$ tending to 0, and some elements $x_n \in \Pi(y_n)$ such that $x_n \notin Q$. Hence, there exists a sequence $(x_n) \in Min$ such that $\eta_n - f(x_n) + y_n \in \mathcal{K}$. But it must be also $cl((x_n)) \cap \Pi(0) = \emptyset$ since $x_n \notin Q$. This, however, contradicts (ii).

$(i) \rightarrow (ii)$. Follows directly from the definitions.

□

Definition 6.2.4 *The problem (P) is* **weakly well–posed** *if*

(i) $Min(f, A, \mathcal{K}) \neq \emptyset$,

(ii) the multifunction Π is u.H.c. at $y = 0$.

Analogously as above we can prove the following

Proposition 6.2.2 *The following conditions are equivalent:*

(i) the problem (P) is weakly well-posed

(ii) $Min(f, A) \neq \emptyset$, any minimizing sequence (x_n), $(x_n) \subset A \setminus S(f, A)$ has the property that for every neighbourhood W of zero

$$x_n \in S(f, A) + W$$

for all n sufficiently large.

6.2.1 Conditions for well-posedness in the objective space

Let A_0 be a subset in the objective space Y. The multifunction $\tilde{\Pi} : \mathcal{K} \rightrightarrows Y$,

$$\tilde{\Pi}(\varepsilon) = \bigcup_{\eta \in Min(A_0|\mathcal{K})} \{y \in A_0 \mid y \preceq \eta + \varepsilon\}$$

is called the ε–solution multifunction.

Obviously, $\tilde{\Pi}(0) = Min(A_0|\mathcal{K})$.

Proposition 6.2.3 *If* (CP) *holds for* A_0*, then* $\tilde{\Pi}$ *is u.H.c. at* $\varepsilon = 0$.

Proposition 6.2.4 *If* (DP) *holds for* A_0*, and* $Min(A_0|\mathcal{K}) = clW\, Min(A_0|\mathcal{K})$*, then* $\tilde{\Pi}$ *is u.H.c. at* $\varepsilon = 0$.

Proof Suppose that $\tilde{\Pi}$ is not u.H.c. at $\varepsilon = 0$. This means that there exists a 0–neighbourhood $\bar{W}$ such that in every 0–neighbourhood O there exist ε_o and $\eta_o \in Min(A_0|\mathcal{K})$ such that for some $y_o \in A_0$, $y_o \preceq \eta_o + \varepsilon_o$, we have

$$y_o \notin Min(A_0|\mathcal{K}) + \bar{W}.$$

This implies, in view of the assumption, that there exists $\bar{k}_o \in \mathcal{K}$, and 0–neighbourhoods $\bar{O}_o$ such that $\bar{k}_o + \bar{O}_o \subset \mathcal{K}$, and $y_o = \bar{\eta}_o + \bar{k}_o$, $\bar{k}_o \notin \bar{W}$.

Hence,

$$\bar{\eta}_o + \bar{k}_o \preceq \eta_o + \varepsilon_o,$$

which is a contradiction.

□

Proposition 6.2.5 *If* (DP) *holds for* A_0*, then* $\tilde{\Pi}$ *is* $\mathcal{K}$*–u.H.c. at* $\varepsilon = 0$.

6.3 Stability results

Lemma 6.1 *Let* X*,* Y *be Hausdorff topological vector spaces satisfying the first countability axiom, let* U *be a topological space, and* *let $\mathcal{K}$ be a convex closed pointed cone in Y with a nonempty interior. Let $f : X \to Y$, be a $\mathcal{K}$-lower continuous function, $u_0 \in U$. Suppose that*

(i) $\mathcal{FA}$ is sup-lower continuous at (x, u_0), for $x \in Min(f, A)$,

(ii) the set$f(A)$ has the domination property and $Min(f, A) = WMin(f, A)$.

If (z_m) is a sequence of Pareto solutions,

$$z_m \in \mathcal{S}(u_m), \quad m = 1, 2, ..., \ lim_m u_m = u_0,$$

and

$$lim_m z_m = z_0, \ z_0 \in A,$$

then $z_0 \in S(f, A)$, ie. the multifunction $\mathcal{S}$ is closed at u_0.

Proof. Suppose on the contrary that $z_0 \notin S$, ie. $y = f(z_0) \notin M(f, A)$. Hence, by (ii), there exists $\overline{y} \in M(f, A)$ such that $y \in \overline{y} + int\mathcal{K}$, ie. there exists a neighbourhood O of zero such that $y = \overline{y} + k, \ \ k + O \in \mathcal{K}$.

By $\mathcal{K}$-lower continuity of f at z_0 we have

$$f(z_m) \in y + 1/2O + \mathcal{K}$$

for all m sufficiently large. By sup-lower continuity of $\mathcal{FA}$ at u_0

$$(\overline{y} + 1/2O - \mathcal{K}) \cap \mathcal{FA}(u_m) \neq \emptyset$$

for all m sufficiently large. Hence, there exists a sequence (w_m) such that

$$f(w_m) \in \mathcal{FA}(u_m), \quad f(w_m) \in \overline{y} + 1/2O - \mathcal{K}$$

for all m sufficiently large. And consequently,

$$\begin{aligned} f(z_m) - f(w_m) &= \\ &[f(z_m) - y] + [y - \overline{y}] + [\overline{y} - f(w_m)] \\ &\in 1/2O + k + 1/2O + \mathcal{K} \subset \mathcal{K} + \mathcal{K} \end{aligned}$$

for all m sufficiently large. This, however, contradicts the assumption that $z_m \in \mathcal{S}(u_m)$.

□

The following result is a simple consequence of Lemma 6.1.

Proposition 6.3.1 *Let X, Y, and U, $\mathcal{K}$ be as in Lemma 6.1, $f : X \to Y$ be a $\mathcal{K}$-continuous function on X, $u_0 \in U$.*
Suppose that
(*i*) *$\mathcal{A}$ is closed at u_0, $\mathcal{FA}$ is sup-lower continuous at u_0,*
(*ii*) *$M(f, A) = WM(f, A)$.*
Then the multifunction $\mathcal{S}$ is closed at u_0.

Proof. Since $\mathcal{A}$ is closed $z_0 \in A$. Observe that in the proof of Proposition 6.3.1 we need the domination property to assure that $\overline{y} \in M(f, A)$ which , in turn allow us to make use of the sup-lower continuity assumption of $\mathcal{FA}$ at (x, u_0), where $x \in M(f, A)$. Since by (i) we have sup-lower continuity of $\mathcal{FA}$ at u_0 we do not need to have $\overline{y} \in M(f, A)$ but it is enough to take any $\overline{y} \in f(A)$ such that $y \in \overline{y} + int\mathcal{K}$. The remainder of the proof is the same .

□

This proposition can be regarded as a counterpart of Theorem 4.2.1 of Tanino, Nakayama, Sawaragi [83] in the case when there is no parametrization in the domination structure.

Lemma 6.2 *Let X, Y be Hausdorff topological vector spaces satifying the first countability axiom, U be a topological space satisfying the first countability axiom, and $\mathcal{K}$ be a convex closed pointed cone in Y with a nonempty interior.*
Let $f : X \to Y$, be a function, $u_0 \in U$,
Suppose that
(*i*) *$\mathcal{FA}$ is sup-lower continuous at (x, u_0), $x \in M(f, A)$ and uniformly on $M(f, A)$,*
(*ii*) *the set $f(A)$ has the domination property (DP), and the containment property (CP),*
(*iii*) *$WM(f, A) = M(f, A)$.*
If (x_n) is a sequence of elements of A such that

$$f(x_n) \in \mathcal{FA}(u_0) \cap \mathcal{FA}(u_n), \ n = 1, 2, ...$$

for a certain sequence (u_n), $lim_n u_n = u_0$
then
either (x_n) *contains a minimizing sequence*
or $(x_n) \notin S(u_n)$, $n = 1, 2, ...$

Proof. Suppose first that for each neighbourhood W of zero in Y there exists an index $\overline{N}$ such that

$$f(x_n) \in M(f, A) + W \quad for \quad n \geq \overline{N}.$$

Hence, there exists a sequence $(\eta_n) \subset M(f, A)$ such that $f(x_n) \in \eta_n + W$, $for\ n \geq \overline{N}$. We shall prove that (x_n) contains a minimizing sequence. Indeed, let us consider open neighbourhoods $W_l = \{z| - c_l \leq z \leq c_l\}$, where $(c_l) \subset int\mathcal{K}$, $lim_l c_l = 0$. Thus, there exists a subsequence $(x_{n_l}) \subset (x_n)$ such that

$$f(x_{n_l}) \leq \eta_{n_l} + c_l, \quad \text{for } l = 1, 2, ...$$

This means, however, that (x_{n_l}) is a minimizing sequence.

Another possibility is that there exists a neighbourhood $\overline{W}$ of zero such that for an infinite set of indices, say L, we have

$$f(x_n) \notin M(f, A) + \overline{W} \ for \ n \in L.$$

Since $f(A)$ has the domination property, for each $n \in L$ there exists $\eta_n \in M(f, A)$ such that $p_n = f(x_n) - \eta_n \in \mathcal{K}$, $\quad n \in L$.

Since $p_n \notin \overline{W}$, by the containment property (CP), there exists a neighbourhood $\overline{O}$ of zero in Y such that $p_n + \overline{O} \in \mathcal{K}$, for $n \in L$.

Now, by the uniform sup-lower continuity of $\mathcal{FA}$ at u_0, we have

$$(\eta_n + \overline{O} - \mathcal{K}) \cap \mathcal{FA}(u_n) \neq \emptyset, \ for \ n \in L.$$

ie, there exists a sequence (w_n) such that

$$f(w_n) \in \mathcal{FA}(u_n) \ and \ f(w_n) \in \eta_n + \overline{O} - \mathcal{K}$$

for $n \in L$. This gives

$$f(x_n) - f(w_n) = [f(x_n) - \eta_n] + [\eta_n - f(w_n)] \in p_n + \overline{O} + \mathcal{K} \in \mathcal{K} + \mathcal{K} \subset \mathcal{K}$$

for $n \in L$. This means, however, that $x_n \notin S(u_n)$, $\quad n \in L$.

□

6.4 Continuity of solutions to vector optimization problems

7 Stability results

Now we prove the following stability results.

Theorem 7.0.1 *Let X, Y, U , Y , and $\mathcal{K}$ be as in Lemma 6.1, . Assume that $u_0 \in U$ and*

(i) $\mathcal{A}$ is u.s.c. at u_0 and $\mathcal{F}\mathcal{A}$ is sup-lower continuous at (x, u_0) for $x \in M(f, A)$ uniformly on $M(f, A)$, f is $\mathcal{K}$-lower continuous on X,

(ii) (P) is well-posed in the sense of Definition 6.2.2,

(iii) the set $f(A)$ has the containment property, and the domination property,

(iv) $M(f, A) = WM(f, A)$.

Then $\mathcal{S}$ is u.s.c. at u_0.

Proof. Suppose that $\mathcal{S}$ is not u.s.c. at u_0. This means that there exist a closed subset $F \subset X$, $F \cap \mathcal{S}(u_0) = \emptyset$, and a sequence (u_k), $lim_k u_k = u_0$ such that $F \cap \mathcal{S}(u_k) \neq \emptyset$. Hence, there exists a sequence (x_k), $x_k \in F \cap \mathcal{S}(u_k)$, $k = 1, 2,$.

Two situations are possible now:

(1) there exists a subsequence (x_{k_m}) of the sequence (x_k) such that $x_{k_m} \in \mathcal{A}(u_{k_m}) \setminus \mathcal{A}(u_0)$, $m = 1, 2, ...$

(2) for all k sufficiently large $x_k \in \mathcal{A}(u_k) \cap \mathcal{R}(u_0)$.

Consider the case (1) and denote $z_m = x_{k_m}$, $m = 1, 2, ...$ If (z_m) does not contain any convergent subsequence or contains a subsequence with a limit point x_0 not belonging to $\mathcal{A}(u_0)$, then $\mathcal{A}$ is not u.s.c. at u_0 which contradicts (i). Hence, suppose that (z_m) contains a convergent subsequence with a limit point $z_0 \in \mathcal{A}(u_0)$. Without loss of generality we can assume that $lim_m z_m = z_0$. By Lemma 6.1, z_0 must be in $\mathcal{S}(u_0)$, which contradicts the assumption that $F \cap \mathcal{S}(u_0) = \emptyset$.

Consider now the case (2), ie. for all k sufficiently large $x_k \in \mathcal{A}(u_k) \cap \mathcal{A}(u_0)$. Hence, we have

$$f(x_k) \in \mathcal{F}\mathcal{A}(u_k) \cap \mathcal{F}\mathcal{A}(u_0).$$

References

[1] Alexiewicz A., Analiza Funkcjonalna, Monografie Matematyczne, PWN, Warszawa 1969

[2] Amahroq T., Thibault L., On proto-differentiability and strict proto-differentiability of multifunctions of feasible points in perturbed optimization problems, Numer. Functional Analysis and Optimization, 16(1995), 1293-1307

[3] Arrow K.J., Barankin E.W., Blackwell D., Admissiblem points of convex sets, Contribution to the Theory of Games, ed. by H.W.Kuhn, A.W. Tucker, Princeton University Press, Princeton, Newe Jersey, vol.2(1953) 87-91

[4] H.Attouch, H.Riahi, Stability results for Ekeland's ε-variational principle and cone extremal solutions, Mathematics of OR **18** (1993), 173-201

[5] Attouch H., Wets R., Quantitative Stability of Variational systems: I. The epigraphical distance, Transactions of the AMS, 328(1991), 695-729

[6] Attouch H., Wets R.,Quantitative Stability of Variational systems: II. A framework for nonlinear conditioning, IIASA Working paper 88-9, Laxemburg, Austria, February 1988

[7] Attouch H., Wets R.,Lipschitzian stability of the $\varepsilon-$approximate solutions in convex optimization, IIASA Working paper WP-87-25, Laxemburg, Austria, March 1987

[8] Aubin J-P, Applied Functional Analysis, Wiley Interscience, New York 1979

[9] Aubin J.-P., Frankowska H., Set-valued Analysis, Birkhauser, 1990

[10] Barbu V., Precupanu T., Convexity and Optimization in Banach spaces, Editura Academiei, Bucharest, Romania, 1986

[11] Bednarczuk E., Berge-type theorems for vector optimization problems, optimization **32**, (1995), 373-384

[12] Bednarczuk E., A note on lower semicontinuity of minimal points, to appear in Nonlinear Analysis and Applications

[13] Bednarczuk E., On lower Lipschitz continuity of minimal points Discussiones Mathematicae, Diffferential Inclusion, Control and Optimization, 20(2000), 245-255

[14] Bednarczuk E., Upper Hőlder continuity of minimal points, to appear in Journal on Convex Analysis

[15] Bednarczuk E., Hőlder-like behaviour of minimal points in vector optimization, submitted to Control and Cybernetics

[16] Bednarczuk E., Some stability results for vector optimization problems in partially ordered topological vector spaces, Proceedings of the First World Congress of Nonlinear Analysts, Tampa, Florida, August 19-26 1992, 2371-2382, ed V.Lakshimikantham, Walter de Gruyter, Berlin, New York 1996

[17] Bednarczuk E., An approach to well-posedness in vector optimization: consequences to stability, Control and Cybernetics 23(1994), 107-122

[18] Bednarczuk E., Well-posedness of vector optimization problems, in: Recent Advances and Historical Developments of Vector Optimization problems, Springer Verlag, Berlin, New York 1987

[19] Bednarczuk E.,Penot J.-P.(1989) - Metrically well-set optimization problems, accepted for publication in Applied Mathematics and Optimization

[20] Bednarczuk E.,Song W., PC points and their application to vector optimization, Pliska Stud.Math.Bulgar.12(1998), 1001-1010

[21] Bednarczuk E.M., Song W., Some more density results for proper efficiency, Journal of Mathematical Analysis and Applications, 231(1999) 345-354

[22] Bednarczuk E., Stability of minimal points for cones with possibly empty interiors, submitted

[23] Bednarczuk E., On variants of the domination property and their applications, submitted

[24] Berge C., Topological spaces, The Macmillam Company, New York 1963

[25] Bolintineanu N., El-Maghri A., On the sensitivity of efficient points, Revue Roumaine de Mathematiques Pures et Appliques, 42(1997), 375-382

[26] Bonnans J.F., Shapiro A., Perturbation Analysis of Optimization Problems, Springer Series in Operations Research, Springer, New York, Berlin, 2000

[27] J.M. Borwein, D.Zhuang, Super efficiency in vector optimization, Trans. of the AMS **338**, (1993), 105-122

[28] Davidson M.P., Lipschitz continuity of Pareto optimal extreme points, Vestnik Mosk. Univer. Ser.XV, Vychisl.Mat.Kiber. 63(1996),41-45

[29] Davidson M.P., Conditions for stability of a set of extreme points of a polyhedron and their applications, Ross. Akad.Nauk, Vychisl. Tsentr, Moscow 1996

[30] Davidson M.P., On the Lipschitz stability of weakly Slater systems of convex inequalities, Vestnik Mosk. Univ., Ser.XV, 1998, 24-28

[31] Deng-Sien, On approximate solutionss in convex vector optimization, SIAM Journal on Control and Optimization, 35(1997), 2128-2136

[32] Dontchev A., Rockafellar T., Characterization of Lipschitzian stability, pp.65-82, Mathematical Programming with Data Perturbations, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker 1998

[33] Dontchev A., Zolezzi T., Well-Posed Optimization Problems, Lecture Notes in Mathematics 1543, Springer, New York, Berlin,1993,

[34] Dauer J.P., Gallagher R.J., Positive proper efficiency and related cone results in vector optimization theory, SIAM J.Control and Optimization, 28(1990), 158-172

[35] Fu W., On the density of proper efficient points, Proc. Amer.Math.Soc. 124(1996), 1213-1217

[36] Giles John R., Convex Analysis with Applications to Differentiation of Convex Functions, Pitman Publishing INC, Boston, London, Melbourne 1982

[37] X.H. Gong, Density of the set of positive proper minimal points in the set of minimal points, J.Optim.Th.Appl., **86**, 609-630

[38] V.V. Gorokhovik, N.N. Rachkovski, On stability of vector optimization problems, (Russian) Wesci ANB **2**, (1990), 3-8

[39] A. Guerraggio, E. Molho, A. Zaffaroni, On the notion of proper efficiency in vector optimization, J.Optim.Th.Appl. **82**, (1994), 1-21

[40] Henig M., The domination property in multicriteria optimization, JOTA,114(1986), 7-16

[41] M.I. Henig, Proper efficiency with respect to cones, J.Optim.Theory Appl. **36**, (1982), 387-407

[42] Holmes R.B., Geometric Functional Analysis, Springer Verlag, New York - Heidelberg-Berlin 1975

[43] Hyers D.H., Isac G.,Rassias T.M., Nonlinear Analysis and Applications, World Scientific Publishing, Singapore 1997

[44] Jahn J., Mathematical Vector Optimization in Partially Ordered Linear Spaces, Peter Lang Frankfurt am Main 1986

[45] J. Jahn, *A generalization of a theorem of Arrow-Barankin-Blackwell*, SIAM J.Cont.Optim. **26**, (1988), 995-1005

[46] Jameson G., Ordered Linear Spaces, Springer Verlag, Berlin-Heidelberg-New York 1970

[47] Janin R., Gauvin J., Lipschitz dependence of the optimal solutions to elementary convex programs, Proceedings of the 2nd Catalan Days on Applied Mathematics, Presses University, Perpignan 1995

[48] M.A. Krasnoselskii, Positive solutions to operator equations (Russian), "Fiz.Mat. Giz.", Moscow, 1962

[49] M.A. Krasnoselskii, E.A. Lifschitz, A.W. Sobolev, Positive linear systems, (Russian), "Izd. Nauka" Moscow, 1985

[50] Krein M.G. Rutman, Linear operators leaving invariant a cone in Banach spaces, Uspechi Metematiczeskich Nauk, 1948

[51] Kuratowski K., Topology, Academic Press, New York, Polish Scientific Publishers, Warsaw 1966

[52] Kuratowski K.(1966) - Topology, Academic Press, New York, Polish Scientific Publishers, Warsaw

[53] Kutateladze S.S.(1976) - Convex ϵ-programming, Doklady ANSSR(249), no.6,pp.1048-1059,[Soviet Mathematics Doklady 20(1979) no.2]

[54] Kurcyusz S., Matematyczne podstawy optymalizacji, PWN, Warszawa 1982

[55] Li-Wu, Error bounds for piecewise convex quadratic programs and applications, SIAM Journal on Control and Optimization, 33(1995), pp1510-1529

[56] Loridan P.(1984) - ϵ-solutions in vector minimization problems, Journal of Optimization Theory and Applications 43 ,no.2 pp.265-276

[57] Luc D.T. Theory of Vector Optimization, Springer Verlag, Berlin-Heidelberg-New York 1989

[58] Luc D.T.(1990) - Recession cones and the domination property in vector optimization, Mathematical Programming(49),pp.113-122

[59] Mordukhovich B., Sensitivity analysis for constraints and variational systems by means of set-valued differentiation, Optimization 31(1994), 13-43

[60] Mordukhovich B., Shao Yong Heng, Differential characterisations of convering, metric regularity and Lipschitzian properties of multifunctions between Banach spaces, Nonlinear Analysis, Theory, Methods, and Applications, 25(1995), 1401-1424

[61] Namioka I., Partially ordered linear topological spaces, Memoirs Amer. Math.Soc. 24(1957)

[62] Nikodem K., Continuity of $\mathcal{K}-$convex set-valued functions, Bulletin of the Polish Academy of Sciences, Mathematics, 34(1986), 393-399

[63] Penot J-P. Sterna A., Parametrized multicriteria optimization; order continuity of optimal multifunctions, JMAA, 144(1986), 1-15

[64] Penot J-P. Sterna A., Parametrized multicriteria optimization; continuity and closedness of optimal multifunctions, JMAA, 120(1986), 150-168

[65] Peressini A.L. Ordered Topological Vector Spaces, Harper and Row, New York-Evanston-London, 1967

[66] Petschke M., On a theorem of Arrow,Barankin and Blackwell, SIAM J. on Control and Optimization, 28(1990), 395-401

[67] R. Phelps, Support cones in Banach spaces, Advances in Mathematics **13**, (1994), 1-19

[68] Robertson A.P. and Robertson W.J., Topological Vector Spaces, Cambridge University Press 1964

[69] Robinson S.M., Generalized equations and their solutions, Part I: Basic theory, Mathematical Programming Study 10(1979),128-141

[70] Robinson S.m., A characterisation of stability in linear programming, Operations Research 25(1977), 435-447

[71] Robinson S.m., Stability of systems of inequalitites, part II, differentiable nonlinear systems, SIAM J.Numerical Analysis 13(1976), 497-513

[72] Rockafellar R.T., Lipschitzian properties of multifunctions, Nonlinear Analysis, Theory, Methods and Applications, vol.9(1985), 867-885

[73] Rockafellar T., Wets R.J.-B., Variational systems, an introduction, in: Multifunctions and Integrands. Stochastic Analysis, Approximation and Optimization, Proceedings, Catania 1983, ed. by G.Salinetti, Springer Verlag, Berlin 1984

[74] Rolewicz S., On paraconvex multifunctions, Operations Research Verfahren, 31(1978), 540-546

[75] Rolewicz S., On $\gamma-$paraconvex multifunctions, Mathematica Japonica, 24(1979), 293-300

[76] Rolewicz S., On optimal problems described by graph $\gamma-$paraconvex multifunctions, in: Functional Differential Systems and Related Topics, ed. M.Kisielewicz, Proceedings of the First International Conference held at Błażejewko, 19-26 May 1979

[77] Rolewicz S., On graph $\gamma-$paraconvex multifunctions, Proc.Conf. Special Topics of Applied Analysis, Bonn 1979, 213-217, North Holland, Amsterdam-New York-Oxford, 1980

[78] Rolewicz S. Pallaschke D., Foundations of Mathematical Optimization,

[79] Rudin W., Functional Analysis, Mc Graw Hill Book Company, New York 1973

[80] Schaefer H.H. Topological Vector Spaces, Springer Verlag, New York, Heidelberg, Berlin 1971

[81] Yen N.D., Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint, Mathematics of OR, 20(1995), 695-705

[82] Yang X,Q., Directional derivatives for set-valued mappings and applications, Mathematical Methods of OR, 48(1998), 273-283

[83] Sawaragi Y., Nakayama H., Tanino T., Theory of Multiobjective Optimization, Academic Press 1985

[84] Truong Xuan Duc Ha, On the existence of efficient points in locally convex spaces, J. Global Optim., **4**, (1994), 267-278

[85] D. Zhuang, *Density results for proper efficiencies*, SIAM J. on Cont.Optim, **32**, (1994), 51-58

[86] Aśić M.D., Dugośija D.(1986)- Uniform convergence and Pareto optimality, optimization 17, no.6,pp.723-729