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Stability Analysis for Parametric Vector Optimization Problems

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Preface

We study stability of minimal points and solutions to parametric (or perturbed) vector optimization problems in the framework of real topological vector spaces and, if necessary, normed spaces. Because of particular importance of finite-dimensional problems, called multicriteria optimization problems, which model various real-life phenomena, a special attention is paid to the finite-dimensional case. Since one can hardly expect the sets of minimal points and solutions to be singletons, set-valued mappings are natural tools for our studies.

Vector optimization problems can be stated as follows. Let X be a topological space and let Y be a topological vector space ordered by a closed convex pointed cone $\mathcal{K} \subset Y$. Vector optimization problem

$$\begin{array}{ll} \mathcal{K} - \min\ f_0(x) & \\ \text{subject to } x \in A_0\,, & \qquad (P_0) \end{array}$$

where $f : X \rightarrow Y$ is a mapping, and $A_0 \subset X$ is a subset of X, relies on finding the set $\mathrm{Min}(f_0, A_0, \mathcal{K}) = \{y \in f_0(A_0) \mid f_0(A_0) \cap (y - \mathcal{K}) = \{y\}\}$ called the **Pareto** or **minimal point** set of (P_0) , and the **solution set** $S(f_0, A_0, \mathcal{K}) = \{x \in A_0 \mid f_0(x) \in \mathrm{Min}(f_0, A_0, \mathcal{K})\}$. We often refer to problem (P_0) as the **original problem** or **unperturbed one**. The space X is the **argument** space and Y is the **outcome** space.

Let U be a topological space. We embed the problem (P_0) into a family (P_u) of vector optimization problems parametrised by a parameter $u \in U$,

$$\begin{array}{ll} \mathcal{K} - \min\ f(u,x) & \\ \text{subject to } x \in A(u)\,, & \qquad (P_u) \end{array}$$

where $f : U \times X \rightarrow Y$ is the parametrised objective function and $A : U \rightrightarrows Y$, is the feasible set multifunction, (P_0) corresponds to a parameter value u_0 . The performance multifunction $\mathcal{M} : U \rightrightarrows Y$, is defined as $\mathcal{M}(u) = \text{Min}(f(u,\cdot), A(u), \mathcal{K})$, and the solution multifunction $\mathcal{S} : U \rightrightarrows Y$, is given as $\mathcal{S}(u) = S(f(u,\cdot), A(u), \mathcal{K})$, and $f : U \times X \to Y$, $A(u) \subset X$.

Our aim is to study continuity properties of $\mathcal{M}$ and $\mathcal{S}$ as functions of the parameter u. Continuous behaviour of solutions as functions of parameters is of crucial importance in many aspects of the theory of vector optimization as well as in applications(correct formulation of the model and/or approximation) and numerical solution of the problem in question.
We investigate continuity in the sense of Hausdorff and Hőlder of the multivalued mappings of minimal points $\mathcal{M}(u)$ and solutions $\mathcal{S}(u)$ as functions of the parameter u under possibly weak assumptions. We attempt to avoid as much as possible compactness assumptions which are frequently over-used (see eg [83]).
It is a specific feature of vector optimization that the outcome space is equipped with a partial order generated by a cone the properties of which are important for stability analysis. In many spaces cones of nonnegative elements have empty interiors and because of this we derive stability results for cones with possibly empty interior. This kind of results are specific for vector optimization and do not have their counterpart in scalar optimization.
We introduce two new concepts: the notion of containment(with some variants for cones with empty interiors),[16], and the notion of strict minimality, [12].

The containment property (CP), defined in topological vector spaces, is introduced to study upper semicontinuities (in the sense of Hausdorff) of minimal points, [11, 16]. It is a variant of the domination property (DP), which appears frequently in the context of stability of solutions to parametric vector optimization problems. Although it is not a commonly adopted view point, the domination property may be accepted as a solution concept which generalizes the standard concept of a solution to scalar optimization problem. In consequence, the containment property (CP) may also be seen as a solution concept in vector optimization. To investigate more deeply this aspect we interpret the containment property as a generalization of the concept of the set of $\phi-$local solutions appearing in the

context of Lipschitz continuity of solutions to scalar optimization problems. Under mild assumptions the containment property imply that the set weakly minimal points equals the set of minimal points. This equality, in turn, is a typical ingradient of standard finite-dimensional sufficient conditions for upper semicontinuity of minimal points.
To study Hőlder upper continuity of minimal points we define the rate of containment of a set with respect to a cone, which is a real-valued function of a scalar argument, see [14, 15]. The rate of growth of this function influence decisively the rate of Hőlder continuity of minimal points, [15].

Strictly minimal points are introduced to study lower semicontinuities (lower Hausdorff, lower Hőlder) of minimal points [20, 13]. The definition of a strictly minimal point is given in topological vector spaces and it is a generalization of the notion of a super efficient point in the sense of Borwein and Zhuang defined in normed spaces. We discuss strict minimality in vector optimization by proving that it is a vector counterpart of the concept of $\phi-$ local solution to scalar optimization problem.

Theory of vector optimization may be considered as an abstract study of optimization problems with mappings taking values in the outcome space equipped with a partial order structure. As such, it contains many concepts and results which generalize and/or have their counterparts in scalar optimization. The very definition of the set of minimal points of vector optimization problem in the outcome space may serve as an example here. This is a counterpart of the optimal value of scalar optimization problem. Another example is the concept of well-posed optimization problem. In subsequent developments we often compare our results and considerations with the corresponding approaches in scalar optimization. For instance, we define several classes of well-posed vector optimization problems by generalizing the concept of scalar minimizing sequence and in these classes we investigate continuity of solutions. For scalar optimization problems, the existing approaches and results on well-posedness are extensively discussed in the monograph by Dontchev and Zolezzi [33].

Convergence and rates of convergence of solutions to perturbed optimization problems is one of crucial topics of stability analysis in optimization both from theoretical and numerical points of view. For scalar optimization it was investigated by many authors see eg., [72], [32], [47], [78], [55], [81], [59], [60], [82], [2], and many others. An exhaustive survey of current state of research is given in the recent monograph by Bonnans and Shapiro [26]. In vector optimization the results on Lipschitz continuity of solutions are not so numerous, and concern some classes of problems, for linear case see eg.,[28], [29], [30], for convex case see eg., [25], [31].

The organization of the material is as follows. In Chapter 2 we investigate upper Hausdorff continuity of the multivalued mapping M, $M(u) = \mathrm{Min}(\Gamma(u)|\mathcal{K})$ assigning to a given parameter value u from a topological space U the set of minimal points of the set $\Gamma(u) \subset Y$ with respect to cone $\mathcal{K} \subset Y$, where for any subset A of a topological vector space Y the set of minimal points is defined as $\mathrm{Min}(A|\mathcal{K}) = \{y \in A \mid A \cap (y - \mathcal{K}) = \{y\}\}$, and $\Gamma : U \rightrightarrows Y$, is a given multivalued mapping. The main tool which allows us to obtain the general result is the containment property (CP). Some infinite-dimensional examples are discussed. A special attention is paid to the containment property (CP) in finite-dimensional case, when $Y = R^m$.

In Chapter 3 we discuss upper Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the rate of containment δ which is a one-variable nondecreasing function, defined for a given set A and the order generating cone $\mathcal{K}$. The assumption of sufficiently fast growth rate of this function appears to be the crucial assumption for all upper Hőlder stability results of Chapter 3.

In Chapter 4 we apply the results obtained in Chapters 2 and 3 to derive conditions for upper Hausdorff and upper Hőlder stability of minimal points to parametric vector optimization problems by taking $\Gamma(u) = f(u, A(u))$. Moreover, we introduce the concept of $\Phi-$ strong solutions to vector optimization problem (P_0), which is a generalization of the concept of a $\phi-$local minimizer to scalar optimization problem, the latter being introduced by Attouch and

Wets [6].

In Chapter 5 we investigate the lower continuity and lower Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the notion of strict minimality mentioned above and the rate of strict minimality. In Section 5.5 we apply the results obtained in Chapter 5 to parametric vector optimization problems and we derive sufficient conditions for lower and lower Hőlder continuity of Pareto point multivalued mapping $\mathcal{M}$. An important tool here is the notion of $\Phi-$ strict solution to vector optimization problem introduced in Section 6.1. This notion can be interpreted as another possible generalization of the concept of $\phi-$local minimizer.

In Chapter 6 we propose several definitions of a well-posed vector optimization problem. All these definitions are based on properties of $\varepsilon-$solutions to vector optimization problems. For well-posed vector optimization problems we prove upper Hausdorff continuity of solution multivalued mapping S, $S(u) = S(f(u,\cdot), A(u), \mathcal{K})$.

5

Lower continuity of minimal points with respect to perturbations of the set.

The questions of lower semicontinuity of minimal points arise in investigation of some other problems, for instance, in vector variational inequalities, duality theory etc. The results obtained can be directly applied to stability of vector optimization problems.

In infinite-dimensional spaces, lower semicontinuity of minimal points was investigated by several authors, eg., by Attouch and Riahi [4], Penot and Sterna-Karwat [?], present author [?] and others, and in finite-dimensional spaces, by Gorokhovik and Rachkowski [38], Tanino, Nakayama, Sawaragi [83].

In finite-dimensional spaces, the key requirement which allows to prove lower semicontinuity of minimal points under perturbations is the density of properly minimal points in the set of minimal points (see eg [38]). The same remains true in any topological vector space with the notion of proper minimality suitably defined. In Section 5.2 we introduce the notion of strictly minimal points (Definition 5.2.1). In normed spaces, strict minimality generalizes the notion of super efficiency (see [27]). In Section 5.3 we give our main result (Theorem 5.3.1), where the key requirement is the density of strictly minimal points in $Min(A|\mathcal{K})$. Conditions ensuring this property are given in [?],[?].

5.1 Strong proper minimality

Let A be a subset of Y. The *domination property* (DP) holds for A if $A \subset Min(A|\mathcal{K}) + \mathcal{K}$. If a cone $\mathcal{K}_0 \subset Y$ is not pointed, then

$a_0 \in Min(A|\mathcal{K}_0)$ means that $(A - a_0) \cap (-\mathcal{K}_0) \subset \mathcal{K}_0$.

A point $a_0 \in A$ is *strongly properly minimal*, (see also [?]), $a_0 \in SPMin(A|\mathcal{K})$, if there exists a closed convex cone $\mathcal{K}_0$, $\mathcal{K}_0 \neq Y$, $\text{int}\mathcal{K}_0 \neq \emptyset$, $\mathcal{K} \setminus \{0\} \subset \text{int}\mathcal{K}_0$, such that for each $0-$neighbourhood W there exists a $0-$neighbourhood O

$$(\mathcal{K} \setminus W) + O \subset \mathcal{K}_0, \tag{24}$$

and $a_0 \in Min(A|\mathcal{K}_0)$.

Cone $\mathcal{K}$ has a base Θ if Θ is convex, $0 \notin \text{cl}\Theta$, where $\text{cl}(\cdot)$ stands for the closure, and $\mathcal{K} = \text{cone}(\Theta)$. For any $0-$neighbourhood V, we put

$$\mathcal{K}_d(V) = \text{cone}(\Theta + V).$$

Proposition 5.1.1 *Let $\mathcal{K}$ be a closed convex cone with a base Θ, and let $\mathcal{K}_0$ be a closed convex cone,$\mathcal{K}_0 \neq Y$, $\text{int}\mathcal{K}_0 \neq \emptyset$, $\mathcal{K} \setminus \{0\} \subset \text{int}\mathcal{K}_0$. If $\mathcal{K}_0$ satisfies 24, then*

$$\mathcal{K}_d(V) \subset \mathcal{K}_0 \tag{25}$$

for some $0-$neighbourhood V.

Proof. Since $0 \notin \text{cl}\Theta$, there exists a $0-$neighbourhood W such that $\Theta \cap W = \emptyset$. By 24, there exists a $0-$neighbourhood O such that $\Theta + O \subset \mathcal{K}_0$, or $\mathcal{K}_d(O) = \text{cone}(\Theta + O) \subset \mathcal{K}_0$.

□

Proposition 5.1.2 *Let $\mathcal{K}$ be a closed convex cone with a topologically bounded base Θ. For any $0-$neighbourhood V, cone $\mathcal{K}_d(V)$ satisfies condition 24, ie., for each $0-$neighbourhood W there exists a $0-$neighbourhood O such that*

$$(\mathcal{K} \setminus W) + O \subset \mathcal{K}_d(V). \tag{26}$$

Proof. Let W be a $0-$neighbourhood. Since Θ is topologically bounded, there exists $\bar{\lambda} > 0$ such that $\lambda\Theta \subset W$, for $0 \leq \lambda \leq \bar{\lambda}$, and for $x \in \mathcal{K} \setminus W$, we have $x = \lambda_x \theta_x$, where $\lambda_x > \bar{\lambda}$. Moreover, there exists a $0-$neighbourhood O such that $O \subset \bar{\lambda} V$. Hence

$$x + O \subset \lambda_x \theta_x + \bar{\lambda} V = \lambda_x(\theta_x + \frac{\bar{\lambda}}{\lambda_x} V) \subset \text{cone}(\Theta + V).$$

□

In Proposition 5.1.2, the boundedness of Θ is important as shows the example below.

Example 5.1.1 *Let* $Y = \ell^\infty$, *and* $\mathcal{K} = \ell^\infty_+$. *The functional* $f(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$ *has the property that* $f(x) > 0$ *for* $x \in \mathcal{K} \setminus \{0\}$. *Hence, the set*

$$\Theta = \{x \in \mathcal{K} \mid f(x) = 1\}$$

is a base of $\mathcal{K}$. Θ *is unbounded since the sequence* $\{x_k\} \subset \Theta$,

$$x_k = (0, ..., 0 \underbrace{2^k}_{k-th\ position}, 0, ...)$$

is unbounded and the condition 26 is not satisfied. To see this let us take a sequence $\{y_k\} \subset \mathcal{K} \setminus W$, $W = \{x \in \ell^\infty \ \ \sup_n |x_n| < 1\}$, *and* $\{q_k\}$, *where*

$$y_k = \frac{1}{k} x_k, \quad and \quad q_k = (0, ..., 0, \underbrace{\frac{1}{k}}_{k-th\ position}, 0, ...).$$

Now, $y_k + q_k \notin cone(\Theta + V)$, *for any* $0-$*neighbourhood* V *smaller than* $\bar{V} = \{x \in \ell^\infty \mid \sup_n |x_n| < 1\}$, *since*

$$z_k = y_k + q_k = \frac{1}{k} x_k + q_k = \frac{1}{k}[x_k + p_k],$$

where $p_k = (0, ..., 0, \underbrace{1}_{k-th\ position}, 0, ...)$. *The main drawback here is the fact that* y_k *has the representation* $y_k = \lambda_k \theta_k$ *with* $\{\lambda_k\}$ *tending to zero.*

Corollary 5.1 *Let* $\mathcal{K}$ *be a closed convex cone with a topologically bounded base* Θ *in a locally convex space* Y *and let* A *be a subset of* Y. *The following conditions are equivalent:*

(i) $a \in SPMin(A|\mathcal{K})$,

(ii) $a \in Min(A|cl\mathcal{K}_d(V))$, *where* V *is a convex* $0-$*neighbourhood.*

Proof. *(ii)* $\rightarrow$ *(i)*. If $a \in Min(A|\mathrm{cl}\mathcal{K}_d(V))$, then by Proposition 5.1.2, $\mathrm{cl}\mathcal{K}_d(V)$ satisfies condition 24, and hence $a \in SPMin(A|\mathcal{K})$.

(i) $\rightarrow$ *(ii)* Let $a \in SPMin(A|\mathcal{K})$. Then $a \in Min(A|\mathcal{K}_0)$, where $\mathcal{K}_0$ satisfies 24. By Proposition 5.1.1, there exists a $0-$neighbourhood V such that 25 holds, and hence $a \in Min(A|\mathrm{cl}\mathcal{K}_d(V))$.

□

Let us note that in any locally convex space, for all sufficiently small neighbourhoods V, $\mathcal{K}_d(V)$ are pointed, which may not be the case for $\mathrm{cl}\mathcal{K}_d(V)$.

5.2 Strict minimality.

Let $A \subset Y$ be a subset of a real Hausdorff topological vector space Y, and let $\mathcal{K}$ be a closed convex pointed cone.

Definition 5.2.1 *An element $a_0 \in A$ is strictly minimal (see also [?]), denoted by $a_0 \in SMin(A|\mathcal{K})$, if for any $0-$neighbourhood W there exists a $0-$neighbourhood O such that*

$$[(A \setminus (a_0 + W)) + O] \cap [a_0 - \mathcal{K}] = \emptyset. \qquad (27)$$

Equivalently

$$(A - a_0) \cap [O - \mathcal{K}] \subset W. \qquad (28)$$

Each strictly minimal point is clearly minimal. Moreover, the following proposition holds.

Proposition 5.2.1 *For any subset A of Y we have*

$$SPMin(A|\mathcal{K}) \subset SMin(A|\mathcal{K})$$

Proof. Let $a_0 \in SPMin(A|\mathcal{K})$ and W be any $0-$neighbourhood. By 24, there exists a $0-$neighbourhood O such that $[\mathcal{K} \setminus W] + O \subset \mathcal{K}_0$. Let W_1 be a $0-$neighbourhood such that $W_1 + W_1 \subset W$. By O_1 we denote a $0-$neighbourhood such that $[\mathcal{K} \setminus W_1] + O_1 \subset \mathcal{K}_0$.

We claim that $[(A \setminus (a_0 + W)) + O_1 \cap W_1] \cap [a_0 - \mathcal{K}] = \emptyset$. Otherwise, it would be $(a - a_0) + q = -k$, for some $a \in A \setminus$

(a_0+W), $q \in W_1 \cap O_1$, and $k \in \mathcal{K}$. We would have $k \in \mathcal{K} \setminus W_1$ since otherwise $a - a_0 \in W$. But then, by 24, it would be $-k - q = a - a_0 \in -\mathcal{K}_0$, which is a contradiction with the minimality of a_0 with respect to $\mathcal{K}_0$. This proves that $a_0 \in SMin(A|\mathcal{K})$.

□

A characterisation of strict minimality via section mapping $S : Y \to Y$, $S(y) = A \cap (x - \mathcal{K})$, is given in Th.2 and Corollaries 1 and 2 of [?].

Proposition 5.2.2 *If $\mathcal{K}$ is normal, then $0 \in SMin(\mathcal{K}|\mathcal{K})$.*

Proof. Since $\mathcal{K}$ is normal, for each 0−neighbourhood W, there exists a 0−neighbourhood O such that $[O+\mathcal{K}] \cap [O-\mathcal{K}] \subset W$, and thus $\mathcal{K} \cap [O - \mathcal{K}] \subset W$.

□

The following proposition gives a characterisation of strict minimality in terms of nets.

Proposition 5.2.3 *Let $A \subset Y$ be a subset of the space Y and $a_0 \in Min(A|\mathcal{K})$. The following are equivalent:*

(i) $a_0 \in SMin(A|\mathcal{K})$,

(ii) for any nets $\{x_\alpha\}$, $\{y_\alpha\}$ such that $\{x_\alpha\} \subset A$, $y_\alpha \in x_\alpha + \mathcal{K}$, and $y_\alpha \to a_0$, it must be $x_\alpha \to a_0$.

Proof. Suppose on the contrary that there exist two nets $\{x_\alpha\}$, $\{y_\alpha\}$ such that $\{x_\alpha\} \subset A$, $y_\alpha \to a_0$, $x_\alpha \leq_K y_\alpha$, and x_α does not tend to a_0. This means that there exists a 0−neighbourhood $\bar{W}$ such that for a certain subnet $\{x_\beta\} \subset \{x_\alpha\}$ we have $x_\beta - a_0 \notin \bar{W}$. On the other hand, $y_\beta = x_\beta + c_\beta$, for some $c_\beta \in \mathcal{K}$, or

$$x_\beta - a_0 = y_\beta - a_0 - c_\beta .$$

Since $\{y_\beta\}$ tends to a_0, for each 0−neighbourhood V we have $y_\beta - a_0 \in V$ for $\beta \geq \beta_v$.

Hence, $\{x_{\beta_v}\}$ forms a subnet of $\{x_\beta\}$ and $x_{\beta_v} - a_0 \in [A - a_0] \cap [V - \mathcal{K}]$, but $x_{\beta_v} - a_0 \notin \bar{W}$, which contradicts the strict minimality of a_0.

Suppose now that $a_0 \notin SMin(A|\mathcal{K})$. There exists a 0−neighbourhood $\bar{W}$ such that for each 0−neighbourhood V one can find $x_v \in A$, $q_v \in V$, $c_v \in \mathcal{K}$ such that

$$x_v - a_0 = q_v - c_v\,,$$

where q_v tends to zero and $x_v - a_0 \notin \bar{W}$. Moreover, $x_v + c_v = q_v + a_0$, ie, $x_v \leq_K y_v = q_v + a_0$, and $\{y_v\}$ tends to a_0 but $\{x_v\}$ does not. This contradicts *(ii)*.

□

By Propositions 5.2.2, 5.2.3 and Proposition 1.3 of [?] we get the following corollary.

Corollary 5.2 *$\mathcal{K}$ is normal if and only if $0 \in SMin(\mathcal{K}|\mathcal{K})$.*

Example 5.2.1 1. *Let $Y = \ell^\infty$, and $\mathcal{K} = \ell^\infty_+$ be a natural ordering cone, $\mathcal{K} = \{x = (x_n) \in \ell^\infty \mid x_n \geq 0 \;\; n \in N\}$. Let*

$$A = \{x \in \ell^\infty \mid \|x\|_\infty \leq 1\}.$$

We have $x_0 = (-1, -1..., -1, ...) \in Min(A|\mathcal{K})$ and $x_0 \in SMin(A|\mathcal{K})$. To see the latter we need to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in (A - x_0) \cap (Q - \mathcal{K})$, where $Q = \{q \in \ell^\infty \mid \|q\|_\infty < \delta\}$, we have $\|y\|_\infty < \varepsilon$. Indeed, let $y = x - x_0 = q - k$, where $x \in A$, $q \in Q$, $k \in \mathcal{K}$. Since $\|x_0 + q - k\|_\infty \leq 1$ we have $k^n \leq q^n$ for all $n \in N$, and consequently

$$|q^n - k^n| \leq q^n + k^n \leq 2q^n\,,$$

which means that it is enough to take $\delta = \varepsilon/2$.

2. *As previously, let $Y = \ell^\infty$, and let $\mathcal{K} = \ell^\infty_+$ be a natural ordering cone. Let*

$$A = \{x \in \ell^\infty \mid f(x) = 0\}$$

where f is a continuous linear functional, $f(x) = \sum_{n=1}^\infty \frac{x_n}{2^n}$. The set A is a subspace and $Min(A|\mathcal{K}) = A$. But $SMin(A|\mathcal{K}) = \emptyset$.

First we show that $0 \notin SMin(A|\mathcal{K})$. Consider the sequence $\{x_k\} \subset A$ defined as

$$x_k = (\frac{1}{k}, 0, ...0, \underbrace{\frac{-2^{k-1}}{k}}_{k-th\ position}, 0, ...).$$

We have $x_k = q_k - c_k$, where

$$q_k = (\frac{1}{k}, 0, ...) \quad c_k = (0, ..., 0, \underbrace{\frac{2^{k-1}}{k}}_{k-th\ position}, 0, ...) \in \mathcal{K},$$

and $\|q_k\|_\infty = \frac{1}{k}$, $\|x_k\|_\infty = \frac{2^{k-1}}{k} \geq 1$. According to Proposition 5.2.3, $0 \notin SMin(A|\mathcal{K})$. To see that for any $a \in A$, $a \notin SMin(A|\mathcal{K})$, consider the sequence $\{z_k\} \subset A$, $z_k = x_k + a$. Now, it is enough to observe that $z_k - a = q_k - c_k$, and to apply Proposition 5.2.3.

5.3 Main results

Let Y be a real Hausdorff topological vector space and let $\Gamma : U \to Y$ be a multivalued mapping defined on a topological space U. By M we denote the multivalued mapping, $M : U \to Y$, $M(u) = Min(\Gamma(u)|\mathcal{K})$.

Theorem 5.3.1 *Let $\mathcal{K}$ be a closed convex pointed cone in Y, and $u_0 \in dom\Gamma$. Assume that*

$$Min(\Gamma(u_0)|\mathcal{K}) \subset cl(SMin(\Gamma(u_0)|\mathcal{K})), \tag{29}$$

and (DP) holds for all $\Gamma(u)$ in a certain neighbourhood U_0 of u_0. If Γ is $\mathcal{K}-$l.c. and u.H.c. at u_0, then M is l.c. at (u_0).

Proof.

Let us note first that since $\Gamma(u_0) \neq \emptyset$ and Γ is $\mathcal{K}-$l.c. at u_0, it must be $\Gamma(u) \neq \emptyset$ in some neighbourhood $\tilde{U}$ of u_0, and, by (DP), $Min(\Gamma(u)|\mathcal{K}) \neq \emptyset$, for $u \in \tilde{U} \cap U_0$.

Take any $y_0 \in Min(\Gamma(u_0)|\mathcal{K})$. We show that M is l.c. at (y_0, u_0). Let W be a $0-$neighbourhood, and let W_1, W_2 be

0−neighbourhoods such that $W_1+W_1 \subset W$ and $W_2+W_2 \subset W_1$. By (29), there exists $y_1 \in SMin(\Gamma(u_0)|\mathcal{K})$, $y_1 \in y_0 + W_2$. By strict minimality of y_1, there exists a 0−neighbourhood O such that

$$[(\Gamma(u_0) \setminus (y_1 + W_2)) + O] \cap (y_1 - \mathcal{K}) = \emptyset .$$

Therefore,

$$[(\Gamma(u_0) \setminus (y_1 + W_2)) + O_1] \cap (y_1 + O_1 - \mathcal{K}) = \emptyset , \qquad (30)$$

for any 0−neighbourhood O_1 such that $O_1 + O_1 \subset O$.
On the other hand,

$$\Gamma(u_0) + O_1 \cap W_2 \subset [(\Gamma(u_0) \setminus (y_1 + W_2)) + O_1 \cap W_2] \cup (y_1 + W_1) .$$

There exists a neighbourhood U_1 of u_0 such that for all $u \in U_1$

$$\begin{aligned} \Gamma(u) &\subset \Gamma(u_0) + O_1 \cap W_2 \qquad (31) \\ &\subset [(\Gamma(u_0) \setminus (y_1 + W_2)) + O_1 \cap W_2] \cup (y_1 + W_1) . \end{aligned}$$

Moreover, there exists a neighbourhood U_2 of u_0 such that for $u \in U_2$

$$(y_1 + O_1 \cap W_2 - \mathcal{K}) \cap \Gamma(u) \neq \emptyset ,$$

i.e., for each $u \in U_2$ there exists y_u, $y_u \in \Gamma(u) \cap (y_1 + O_1 \cap W_2 - \mathcal{K})$, and consequently

$$y_u - \mathcal{K} \subset y_1 + O_1 \cap W_2 - \mathcal{K} .$$

Now, by 30,

$$(y_u - \mathcal{K}) \cap [(\Gamma(u_0) \setminus (y_1 + W_2)) + O_1 \cap W_2] = \emptyset .$$

By 31, for $u \in U_1 \cap U_2$

$$(y_u - \mathcal{K}) \cap \Gamma(u) \subset y_1 + W_1 \subset y_0 + W .$$

By (DP), for each $u \in U_0 \cap U_1 \cap U_2$ there exists $\eta_u \in Min(\Gamma(u)|\mathcal{K}) = M(u)$ such that

$$\eta_u \in (y_u - \mathcal{K}) \cap \Gamma(u) \subset y_0 + W ,$$

which completes the proof.

□

Note that in the proof we use $\mathcal{K}$−lower continuity of Γ only in the vicinity of y_0.

In view of Proposition 5.2.1, by Theorem 5.3.1, we can formulate the following result which generalizes Theorem 3.1 of [?].

Theorem 5.3.2 *Let $\mathcal{K}$ be a closed convex pointed cone in Y, $u_0 \in dom\Gamma$. If*

$$Min(\Gamma(u_0)|\mathcal{K}) \subset clSPMin(\Gamma(u_0)|\mathcal{K}), \qquad (32)$$

Γ is u.H.c. and $\mathcal{K}$−l.c. at u_0, and (DP) holds for all $\Gamma(u)$ in some neighbourhood of u_0, then M is l.c. at u_0.

Sufficient conditions for lower continuity of minimal points can also be given by assuming that 0 is a strictly minimal point of $\mathcal{K}$, which, by Corollary 5.2, amounts to saying that $\mathcal{K}$ is normal. We have the following result.

Theorem 5.3.3 *Let $\mathcal{K}$ be a closed convex normal cone in Y. Assume that $\Gamma(u_0)$ is closed, $clMin(\Gamma(u_0)|\mathcal{K})$ is compact, and (DP) holds for all $\Gamma(u)$ in a certain neighbourhood U_0 of u_0.*

If Γ is $\mathcal{K}$−l.c. and u.H.c. at u_0, then M is l.c. at u_0.

Proof. Let $y_0 \in Min(\Gamma(u_0)|\mathcal{K})$. We start by showing that, under our assumptions, for any 0−neighbourhood W there exists a 0−neighbourhood V such that

$$[((Min(\Gamma(u_0)|\mathcal{K}) + \mathcal{K}) \setminus (y_0 + W)) + V] \cap (y_0 - \mathcal{K}] = \emptyset. \qquad (33)$$

To see this, suppose on the contrary that there exists some W such that, for any 0−neighbourhood V, we have

$$y_0 - k_v = \eta_v + k_v^1 + q_v = z_v + q_v,$$

where k_v, $k_v^1 \in \mathcal{K}$, $\eta_v \in Min(\Gamma(u_0)|\mathcal{K})$, $z_v = \eta_v + k_v^1 \notin y_0 + W$, and the net q_v tends to 0. Since cl$Min(\Gamma(u_0)|\mathcal{K})$ is compact, the net $\{\eta_v \,|\, v \in V\}$ contains a converging subnet. Without loss of generality we may assume that the net itself converges to a certain $\eta \in \Gamma(u_0)$. Consequently,

$$y_0 - \eta = \lim k_v + k_v^1, \qquad (34)$$

and, since $\mathcal{K}$ is closed, $y_0 - \eta \in \mathcal{K}$ which implies that $y_0 = \eta$. By 34, $\lim k_v + k_v^1 = 0$, and, since $\mathcal{K}$ is normal, by Proposition 1.3 p.62 of [?], $\{k_v\}$ and $\{k_v^1\}$ both tend to zero. By taking any 0–neighbourhood W_1 such that $W_1 + W_1 \subset W$, one can find a 0–neighbourhood V_0 such that for all $V \subset V_0$ we have $\eta_v + k_v^1 \subset \eta + W_1 + W_1 \subset y_0 + W$, contradictory to the assumption that $\eta_v + k_v^1 \not\subset y_0 + W$. This proves 33.

Let W_1 be a 0–neighbourhood such that $W_1 + W_1 \subset W$. By 33, there exists a 0–neighbourhood V_1 such that for any 0–neighbourhood V_2, $V_2 + V_2 \subset V_1$, we have

$$[((Min(\Gamma(u_0)|\mathcal{K}) + \mathcal{K}) \setminus (y_0 + W_1)) + V_2] \cap [(y_0 + V_2) - \mathcal{K}] = \emptyset .$$

On the other hand, since (DP) holds for $\Gamma(u_0)$,

$$\Gamma(u_0) + V_2 \cap W_1 \subset [((Min(\Gamma(u_0)|\mathcal{K}) + \mathcal{K}) \setminus (y_0 + W_1)) + V_2 \cap W_1] \cup (y_0 + W) .$$

There exists a neighbourhood U_1 of u_0 such that

$$\Gamma(u) \subset [((Min(\Gamma(u_0)|\mathcal{K}) + \mathcal{K}) \setminus (y_0 + W_1)) + V_2 \cap W_1] \cup (y_0 + W) . \tag{35}$$

for $u \in U_1$. Moreover, there exists a neighbourhood U_2 of u_0 such that

$$(y_0 + V_2 \cap W_1 - \mathcal{K}) \cap \Gamma(u) \neq \emptyset ,$$

for $u \in U_2$. Hence, for $u \in U_2$ there exists $y_u \in \Gamma(u) \cap (y_0 + V_2 \cap W_1 - \mathcal{K})$, and

$$y_u - \mathcal{K} \subset y_0 + V_2 \cap W_1 - \mathcal{K} .$$

Since, $y_u \in V_2 \cap W_1 \subset V_2$, by 33,

$$(y_u - \mathcal{K}) \cap [((Min(\Gamma(u_0)|\mathcal{K}) + \mathcal{K}) \setminus (y_0 + W_1)) + V_2 \cap W_1] = \emptyset .$$

By 35, and by (DP), for $u \in U_0 \cap U_1 \cap U_2$, there exists $\eta_u \in Min(\Gamma(u)|\mathcal{K})$ such that

$$\eta_u \in (y_u - \mathcal{K}) \cap \Gamma(u) \subset (y_0 + W) . \tag{36}$$

This completes the proof.

□

5.4 Lower semicontinuity of minimal points in normed spaces

Let Y be a real linear normed space with the closed unit ball B. We start with the following proposition.

Definition 5.4.1 ([48],[49]) *We say that cone $\mathcal{K} \subset Y$ allows plastering if there exists another closed convex pointed cone $\mathcal{K}_0$ such that for each $k \in \mathcal{K}$*

$$k + b\|k\|B \subset \mathcal{K}_0,$$

where $b > 0$ is independent of k.

Proposition 5.4.1 *The following are equivalent:*

(i) there exists a closed convex pointed cone $\mathcal{K}_0$ satisfying condition (24),

(ii) $\mathcal{K}$ allows plastering $\mathcal{K}_0$,

(iii) $\mathcal{K}$ has a bounded base.

Proof. $(i) \leftrightarrow (ii)$. If $\mathcal{K}$ allows plastering $\mathcal{K}_0$, then for any $\varepsilon > 0$ and $k \in \mathcal{K}$, $\|k\| \geq \varepsilon$, we have $k + b\varepsilon B \subset \mathcal{K}_0$ and $\mathcal{K}_0$ satisfies condition (24).

Suppose now that $\mathcal{K}_0$ satisfies condition (24). There exists $b > 0$ such that for $k_0 \in \mathcal{K}$, $\|k_0\| \geq 1$, we have

$$k_0 + bB \subset \mathcal{K}_0. \tag{37}$$

Take any $k \in \mathcal{K}$. By 37, $k_0 + bB \subset \mathcal{K}_0$, where $k_0 = k/\|k\| \in \mathcal{K}$. Consequently, $k + b\|k\|B \subset \mathcal{K}_0$, which means that $\mathcal{K}$ allows plastering $\mathcal{K}_0$.

$(ii) \rightarrow (iii)$. Suppose that cone $\mathcal{K}$ allows plastering $\mathcal{K}_0$. This means that there exists a linear continuous functional $f \in \mathcal{K}_0^+$ which is strictly uniformly positive on $\mathcal{K}$, ie.,

$$f(x) \geq b\|x\| \quad \text{for} \quad x \in \mathcal{K}.$$

The set $\Theta = \{x \in \mathcal{K} \mid f(x) = 1\}$ is clearly bounded closed and convex, $0 \notin \Theta$, and $\mathcal{K} = cone(\Theta)$.

$(iii) \rightarrow (ii)$. For the proof of this part see Krasnoselskii [48].

□

A point $a_0 \in A$ is said to be *super efficient* [27], $a_0 \in SE(A|\mathcal{K})$, if there exists a number M such that

$$\mathrm{cl}(\mathrm{cone}(A - a_0)) \cap (B - \mathcal{K}) \subset MB.$$

Proposition 5.4.2 *For any subset A of Y we have*

$$SE(A|\mathcal{K}) \subset SMin(A|\mathcal{K}).$$

Proof. Suppose on the contrary that $a_0 \notin SMin(A|\mathcal{K})$. By 28, there exists $\varepsilon_0 > 0$ such that for each n

$$[(A - a_0) \setminus \varepsilon_0 B] \cap [1/nB - \mathcal{K}] \neq \emptyset,$$

and one can choose $a_n \in A$, $\|a_n - a_0\| \geq \varepsilon_0$, such that $a_n - a_0 = 1/n(b_n - k_n)$. Consequently, $n(a_n - a_0) = b_n - k_n$ and $\|n(a_n - a_0)\| \to +\infty$, which means that $a_0 \notin SE(A|\mathcal{K})$.

□

Theorem 5.4.1 *Suppose that $\mathcal{K}$ has a bounded base Θ. Then*

$$SPMin(A|\mathcal{K}) = SE(A|\mathcal{K}).$$

Proof. If $a_0 \in SPMin(A|\mathcal{K})$, then, by Proposition 5.1.1, there exists $\varepsilon > 0$ such that

$$(A - a_0) \cap (-\mathcal{K}_d(\varepsilon)) = \{0\},$$

where, as previously, $\mathcal{K}_d(\varepsilon) = \mathrm{cone}(\Theta + \varepsilon B)$. Thus, $\mathrm{cone}(A - a_0) \cap (\varepsilon B - \Theta) = \emptyset$. Now, by the same arguments as those used in the proof of Proposition 3.4[27], we conclude that $a_0 \in SE(A|\mathcal{K})$.

Suppose now that $a_0 \notin SPMin(A|\mathcal{K})$. By Proposition 5.1.1, for any $\varepsilon > 0$

$$(A - a_0) \cap [-\mathrm{cone}(\Theta + \varepsilon B)] \neq \emptyset.$$

Equivalently, $\mathrm{cone}(A - a_0) \cap (-\Theta + \varepsilon B) \neq \emptyset$. By the same arguments as those used in the proof of Theorem 4.1 [39], $a_0 \notin SE(A|\mathcal{K})$, which completes the proof.

□

In view of the above results, in normed spaces we have the following variant of Theorem 5.3.2.

Theorem 5.4.2 *Let Y be a real linear normed space and $\mathcal{K} \subset Y$ a closed convex pointed cone in Y. Let $u_0 \in dom\Gamma$. Suppose that*

$$Min(\Gamma(u_0)|\mathcal{K}) \subset cl(SE(\Gamma(u_0)|\mathcal{K})), \tag{38}$$

and (DP) holds for all $\Gamma(u)$ in a certain neighbourhood U_0 of u_0. If Γ is $\mathcal{K}-l.c.$ at (y_0, u_0) and u.H.c. at u_0, then M is l.c. at (y_0, u_0).

Proof. By Proposition 5.4.2, each super efficient point is strictly minimal, and by Theorem 5.3.1, the assertion follows.

□

Let $\mathcal{K}$ be a Bishop-Phelps cone, i.e.,

$$\mathcal{K}_\alpha = \{y \in Y \mid f(y) \geq \alpha \|y\| \|f\|\},$$

where f is a linear continuous functional on Y and $0 < \alpha < 1$. This is a closed convex pointed cone. If it is nontrivial, then $\mathcal{K}_\alpha$ has a bounded base Θ

$$\Theta = \{z \in \mathcal{K} \mid f(z) = 1\}.$$

The following characterisation holds.

Proposition 5.4.3 *Let Y be a real linear normed space, A a nonempty subset of Y and $a_0 \in Min(A|\mathcal{K}_\alpha)$. If there exists $\beta < \alpha$ such that $a_0 \in Min(A|\mathcal{K}_\beta)$, then $a_0 \in SPMin(A|\mathcal{K}_\alpha)$.*

Proof. By Proposition 5.4.1, cone $\mathcal{K}_\beta$ satisfies condition 24. Moreover, for $z \in \mathcal{K}_\alpha$, $\|z\| \geq \varepsilon$, we have

$$\begin{aligned} f(z+o) = f(z)+f(o) &\geq \alpha\|f\| \cdot \|z\| + f(o) \\ &\geq \alpha\|z+o\| \cdot \|f\| - \alpha\|f\| \cdot \|o\| - \|f\| \cdot \|o\| \\ &\geq \|f\| \cdot \|z+o\|[\alpha - \tfrac{(\alpha+1)\|o\|}{\varepsilon - \|o\|}]. \end{aligned}$$

To have $\alpha - \frac{(\alpha+1)\|o\|}{\varepsilon - \|o\|} > \beta$ we choose

$$\|o\| < \frac{(\alpha - \beta)\varepsilon}{2\alpha + 1 - \beta}.$$

□

From Proposition 5.4.3 it follows that $\mathcal{K}_\alpha$ allows plastering $\mathcal{K}_\beta$, $\beta < \alpha$, with $b = (\alpha - \beta)/(2\alpha + 1 - \beta)$.

For Bishop-Phelps cones, the following well-known result [67], gives sufficient conditions for the domination property to hold.

Theorem 5.4.3 *Let Y be a Banach space and A a nonempty closed subset of Y. If $\inf f(A) > -\infty$, then for any $a \in A$ there exists $a_0 \in A$ such that $a_0 \in a - \mathcal{K}_\alpha$ and a_0 is minimal.*

By Theorem 5.4.3 and Proposition 5.4.3 we obtain the following stability result.

Theorem 5.4.4 *Let Y be a Banach space and $\Gamma(u_0) \neq \emptyset$. Assume that there exists a neighbourhood U_0 of u_0 such that all the sets $\Gamma(u)$ are closed and $\inf_{y \in \Gamma(u)} f(y) > -\infty$,*

If

$$Min(\Gamma(u_0)|\mathcal{K}_\alpha) \subset cl(\bigcup_{\beta<\alpha} Min(\Gamma(u_0)|\mathcal{K}_\beta)), \tag{39}$$

Γ is $\mathcal{K}$−l.c. and u.H.c. at u_0, then $\mathcal{M}$ is l.c. at u_0.

Proof. Follows from Theorem 5.4.3, Theorem 5.4.3, and Theorem 5.3.2. □

Theorem 5.4.4 can be viewed as a variant of the stability result proved by Attouch and Riahi [4].

Conditions 29 of Theorem 5.3.1, 32 of Theorem 5.3.2 and 38 of Theorem 5.4.2 are density type requirements. Density property has been investigated on different levels of generality and for different notions of proper minimality (e.g., [27], [?], [?], [44]). Here we use the result of Borwein and Zhuang [27].

We say that a subset A of Y is $\mathcal{K}$-lower bounded if there is a constant $M > 0$ such that

$$A \subset MB + \mathcal{K}.$$

A subset A is $\mathcal{K}-$lower bounded if either it is topologically bounded, i.e., $A \subset MB$ for some positive constant $M > 0$, or there exists an element m such that $a - m \in \mathcal{K}$ for all $a \in A$.

Theorem 5.4.5 (Borwein, Zhuang [27]) *Let Y be a Banach space, $\mathcal{K}$ an ordering cone and A a nonempty subset of Y. Assume that $\mathcal{K}$ has a closed and bounded base Θ. If either of the following conditions is satisfied, then $SE(A|\mathcal{K})$ is norm-dense in the nonempty set $Min(A|\mathcal{K})$:*

(i) A is weakly compact;

(ii) A is weakly closed and $\mathcal{K}-$lower bounded while Θ is weakly compact.

For convex sets condition (ii) can be rewritten in the form

(ii)' A is convex and closed and $\mathcal{K}-$lower bounded while Θ is weakly compact.

In view of this result we can rewrite Theorem 5.4.2 in the following form.

Theorem 5.4.6 *Let Y be a Banach space. Suppose that $\mathcal{K}$ posseses a weakly compact base, $\Gamma(u_0)$ is closed and convex, $Min(\Gamma(u_0)|\mathcal{K})$ is bounded, and (DP) holds for all $\Gamma(u)$ in a certain neighbourhood of u_0.*

If Γ is $\mathcal{K}-$l.c. and u.H.c. at u_0, then M is l.c. at u_0.

Proof. It is enough to observe that if $Min(\Gamma(u_0)|\mathcal{K})$ is topologically bounded and (DP) holds for $\Gamma(u_0)$, then $\Gamma(u_0)$ is $\mathcal{K}-$bounded. Thus, by Theorem 5.4.5, $Min(\Gamma(u_0)|\mathcal{K}) \subset \mathrm{cl}(SE(\Gamma(u_0)|\mathcal{K}))$. Now, the assertion follows from Theorem 5.4.2.

□

Theorem 5.4.7 *Let Y be a Banach space. Suppose that $\mathcal{K}$ has a bounded base and all the sets $\Gamma(u)$ in a certain neighbourhood U_0 of u_0 are weakly compact. If Γ is u.H.c. and $\mathcal{K}-$l.c. at u_0, then $\mathcal{M}$ is l.c. at u_0.*

5.5 Modulus of minimality.

Definition 5.5.1 ([12],[20]) (Strict minimality) *We say that* $x \in Min(A|\mathcal{K})$ *is* **strictly minimal point,** $x \in SM(A|\mathcal{K})$, *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$[A \setminus B(x,\varepsilon)] \cap [(x + \delta B(0,1)) - \mathcal{K}] = \emptyset .$$

Definition 5.5.2 (Modulus of minimality) *The modulus of minimality of* A, $SMin(A|\mathcal{K}) \neq \emptyset$, *is the function* $m : R_+ \to R_+$, *defined as*

$$m(\varepsilon) = \inf_{x \in SM(A|\mathcal{K})} \nu(\varepsilon, x) \tag{40}$$

where $\nu : R_+ \times SM(A|\mathcal{K}) \to R_+$, *is the modulus of minimality of* $x \in SMin(A|\mathcal{K})$ *defined as*

$$\nu(\varepsilon, x) = \sup_{\substack{\delta : \\ (A \setminus B(x,\varepsilon)) \\ \cap [x + \delta B(0,1) - \mathcal{K}] = \emptyset .}} \delta \tag{41}$$

Equivalently,

$$[(A \setminus B(x,\varepsilon)) + \delta B(0,1)] \cap [x - \mathcal{K}] = \emptyset . \tag{42}$$

5.6 Lower Hőlder continuity of minimal points

Theorem 5.6.1 *Let* Y *be a normed space and let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone. Assume that* $\Gamma : U \to Y$ *is a set-valued mapping defined on a normed space* U. *If*

(i) $M(u_0) \subset cl(SM(u_0))$, *and for any* $\varepsilon > 0$, *for all* $z \in SM(u_0)$, $m(\varepsilon) > 0$ *and*

$$[(\Gamma(u_0) \setminus (z + \varepsilon \cdot B)) + m(\varepsilon) \cdot B] \cap (z - \mathcal{K}) = \emptyset ,$$

(ii) DP *holds for all* $\Gamma(u)$ *in some neighbourhood* U_1 *of* u_0,

(iii) Γ *is Hausdorff continuous at* u_0, *ie., for each* $\varepsilon > 0$ *there exists a neighbourhood* U_2 *of* u_0 *such that*

$$\Gamma(u) \subset \Gamma(u_0) + \varepsilon \cdot B ,$$

and

$$\Gamma(u_0) \subset \Gamma(u) + \varepsilon \cdot B\,,$$

for $u \in U_2$,

then M *is lower Hausdorff semicontinuous at* u_0, *ie., for each* $\varepsilon > 0$

$$M(u_0) \subset M(u) + \varepsilon \cdot B\,,$$

for $u \in U_1 \cap U_2$.

Proof. Let us take any $\varepsilon > 0$, and $y \in M(u_0)$. By (i), there exists $y_1 \in SM(u_0)$, such that $y_1 \in y + \frac{1}{4}\varepsilon \cdot B$, and

$$[(\Gamma(u_0) \setminus (y_1 + \frac{1}{2}\varepsilon \cdot B)) + m(\frac{1}{2}\varepsilon) \cdot B] \cap (y_1 - \mathcal{K}) = \emptyset\,.$$

Hence

$$[(\Gamma(u_0) \setminus (y_1 + \frac{1}{2}\varepsilon \cdot B)) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B] \cap (y_1 + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B - \mathcal{K}) = \emptyset\,. \tag{43}$$

I. Consider first the case , where $m(\varepsilon) \leq \frac{1}{2}\varepsilon$. By the upper Hausdorff semicontinuity of Γ, for $u \in U_2$,

$$\begin{aligned}\Gamma(u) &\subset \Gamma(u_0) + \tfrac{1}{2}m(\tfrac{1}{2}\varepsilon) \cdot B \\ &\subset [(\Gamma(u_0) \setminus (y_1 + \tfrac{1}{2}\varepsilon \cdot B)) + \tfrac{1}{2}m(\tfrac{1}{2}\varepsilon) \cdot B] \cup [y_1 + (\tfrac{1}{2}m(\tfrac{1}{2}\varepsilon) + \tfrac{1}{2}\varepsilon) \cdot B]\,,\end{aligned} \tag{44}$$

and by the lower Hausdorff semicontinuity of Γ, for $u \in U_1$ there exists $y_2 \in \Gamma(u)$ such that

$$y_2 \in y_1 + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B\,,$$

and,

$$y_2 - \mathcal{K} \subset y_1 + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B - \mathcal{K}\,.$$

By (43),

$$(y_2 - \mathcal{K}) \cap [(\Gamma(u) \setminus (y_1 + \frac{1}{2}\varepsilon \cdot B)) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B] = \emptyset\,.$$

Now, by (44), for $u \in U_1$,

$$(y_2 - \mathcal{K}) \cap \Gamma(u) \subset y_1 + (\frac{1}{2}m(\frac{1}{2}\varepsilon) + \frac{1}{2}\varepsilon) \cdot B\,.$$

Since (DP) holds for $\Gamma(u)$, for $u \in U_1 \cap U_2$ there exists $\eta_2 \in M(u)$ such that

$$\eta_2 \subset (y_2 - \mathcal{K}) \cap \Gamma(u) \subset y_1 + (\frac{1}{2}m(\frac{1}{2}\varepsilon) + \frac{1}{2}\varepsilon) \cdot B\,,$$

and since $m(\varepsilon) \leq \frac{1}{2}\varepsilon$,

$$\eta_2 \in y_1 + \frac{3}{4}\varepsilon \cdot B \subset y + \varepsilon \cdot B\,.$$

This means that for $u \in U_1 \cap U_2$

$$M(u_0) \subset M(u) + \varepsilon \cdot B\,,$$

which completes the proof in the case I.

Consider now the case II, where $m(\varepsilon) > \frac{1}{2}\varepsilon$. By the upper Hausdorff semicontinuity of Γ, for $u \in U_2$,

$$\begin{aligned}\Gamma(u) &\subset \Gamma(u_0) + \tfrac{1}{8}\varepsilon \cdot B \\ &\subset [(\Gamma(u_0) \setminus (y + \tfrac{1}{2}\varepsilon \cdot B)) + \tfrac{1}{8}\varepsilon \cdot B] \cup [y_1 + (\tfrac{1}{8}\varepsilon + \tfrac{1}{2}\varepsilon) \cdot B]\,, \end{aligned} \tag{45}$$

and by the lower Hausdorff semicontinuity of Γ, there exists $y_2 \in \Gamma(u)$, $u \in U_2$, such that

$$y_2 \in y_1 + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B\,.$$

In consequence

$$y_2 - \mathcal{K} \subset y_1 + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B - \mathcal{K}\,,$$

and by (43),

$$(y_2 - \mathcal{K}) \cap [(\Gamma(u_0) \setminus (y_1 + \frac{1}{2}\varepsilon \cdot B)) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B] = \emptyset\,.$$

Since $\frac{1}{2}m(\frac{1}{2}\varepsilon) > \frac{1}{8}\varepsilon$ the latter implies that

$$(y_2 - \mathcal{K}) \cap [(\Gamma(u_0) \setminus (y_1 + \frac{1}{2}\varepsilon \cdot B)) + \frac{1}{8}\varepsilon \cdot B] = \emptyset\,.$$

Now, by (45),

$$(y_2 - \mathcal{K}) \cap \Gamma(u) \subset y_1 + \frac{5}{8}\varepsilon \cdot B\,.$$

Since (DP) holds for $\Gamma(u)$, $u \in U_1$, there exists $\eta_2 \in M(u)$, $u \in U_1 \cap U_2$ such that

$$\eta_2 \in (y_2 - \mathcal{K}) \cap \Gamma(u) \subset y_1 + \frac{5}{8}\varepsilon \cdot B\,,$$

and

$$\eta_2 \in y + \frac{7}{8}\varepsilon \cdot B \subset y + \varepsilon \cdot B\,.$$

This means that for $u \in U_1 \cap U_2$

$$M(u_0) \subset M(u) + \varepsilon \cdot B\,,$$

which completes the proof.

□

Theorem 5.6.2 *Let Y be a normed space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone. Assume that $\Gamma : U \to Y$ is a set-valued mapping defined on a normed space U. If*

(i) *$M(u) \subset cl(SM(u))$, for any $u \in U_1$ and for any $\varepsilon > 0$, for all $z \in SM(u)$, $m(\varepsilon) = \inf_{u \in U_1} m(\varepsilon, u) > 0$, where $m(\cdot, u)$ is the modulus of minimality of the set $\Gamma(u)$, and*

$$[(\Gamma(u) \setminus (z + \varepsilon \cdot B)) + m(\varepsilon) \cdot B] \cap (z - \mathcal{K}) = \emptyset\,,$$

(ii) *DP holds for $\Gamma(u_0)$*

(iii) *Γ is Hausdorff continuous at u_0, ie., for each $\varepsilon > 0$ there exists a neighbourhood U_2 of u_0 such that*

$$\Gamma(u) \subset \Gamma(u_0) + \varepsilon \cdot B\,,$$

and

$$\Gamma(u_0) \subset \Gamma(u) + \varepsilon \cdot B\,,$$

for $u \in U_2$,

then M is upper Hausdorff semicontinuous at u_0, ie., for each $\varepsilon > 0$ there exists a neighbourhood U_3 of u_0 such that

$$M(u) \subset M(u_0) + \varepsilon \cdot B,$$

for $u \in U_3$.

Proof. Let us take any $\varepsilon > 0$, and $y(u) \in M(u)$, $u \in U_1 \cap U_2$. By (i), there exists $y_1(u) \in SM(u)$, such that $y_1(u) \in y_1 + \frac{1}{4}\varepsilon \cdot B$, and

$$[(\Gamma(u) \setminus (y_1(u) + \frac{1}{2}\varepsilon \cdot B)) + m(\frac{1}{2}\varepsilon) \cdot B] \cap (y_1(u) - \mathcal{K}) = \emptyset.$$

Hence

$$[(\Gamma(u)\setminus(y_1(u)+\frac{1}{2}\varepsilon\cdot B))+\frac{1}{2}m(\frac{1}{2}\varepsilon)\cdot B]\cap(y_1(u)+\frac{1}{2}m(\frac{1}{2}\varepsilon)\cdot B-\mathcal{K}) = \emptyset. \tag{46}$$

Consider first the case where $m(\varepsilon) \leq \frac{1}{2}\varepsilon$. By the lower Hausdorff semicontinuity of Γ, for $u \in U_2$,

$$\begin{aligned}\Gamma(u_0) &\subset \Gamma(u) + \tfrac{1}{2}m(\tfrac{1}{2}\varepsilon) \cdot B \\ &\subset [(\Gamma(u) \setminus (y_1(u) + \tfrac{1}{2}\varepsilon \cdot B)) + \tfrac{1}{2}m(\tfrac{1}{2}\varepsilon) \cdot B] \cup [y_1(u) + (\tfrac{1}{2}m(\tfrac{1}{2}\varepsilon) + \tfrac{1}{2}\varepsilon) \cdot B],\end{aligned} \tag{47}$$

and by the upper Hausdorff semicontinuity of Γ, there exists $y_2(u_0) \in \Gamma(u_0)$ such that

$$y_2(u_0) \in y_1(u) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B,$$

for any $u \in U_2$, and,

$$y_2(u_0) - \mathcal{K} \subset y_1(u) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B - \mathcal{K}.$$

By (46),

$$(y_2(u_0) - \mathcal{K}) \cap [(\Gamma(u) \setminus (y_1(u) + \frac{1}{2}\varepsilon \cdot B)) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B] = \emptyset.$$

Now, by (47), for $u \in U_1 \cap U_2$,

$$(y_2(u_0) - \mathcal{K}) \cap \Gamma(u_0) \subset y_1(u) + (\frac{1}{2}m(\frac{1}{2}\varepsilon) + \frac{1}{2}\varepsilon) \cdot B.$$

Since (DP) holds for $\Gamma(u_0)$, for $u \in U_1 \cap U_2$, there exists $\eta_2(u_0) \in M(u_0)$ such that

$$\eta_2(u_0) \subset (y_2(u_0) - \mathcal{K}) \cap \Gamma(u_0) \subset y_1(u) + (\frac{1}{2}m(\frac{1}{2}\varepsilon) + \frac{1}{2}\varepsilon) \cdot B,$$

and since $m(\varepsilon) \leq \frac{1}{2}\varepsilon$,

$$\eta_2(u_0) \in y_1(u) + \frac{3}{4}\varepsilon \cdot B \subset y(u) + \varepsilon \cdot B.$$

This means that for $u \in U_1 \cap U_2$

$$M(u) \subset M(u_0) + \varepsilon \cdot B,$$

which completes the proof in the case I.

Consider now the case II, where $m(\varepsilon) > \frac{1}{2}\varepsilon$. By the lower Hausdorff semicontinuity of Γ, for $u \in U_1$,

$$\begin{aligned}\Gamma(u_0) &\subset \Gamma(u) + \tfrac{1}{8}\varepsilon \cdot B \\ &\subset [(\Gamma(u) \setminus (y_1(u) + \tfrac{1}{2}\varepsilon \cdot B)) + \tfrac{1}{8}\varepsilon \cdot B] \cup [y_1(u) + (\tfrac{1}{8}\varepsilon + \tfrac{1}{2}\varepsilon) \cdot B],\end{aligned} \tag{48}$$

and by the upper Hausdorff semicontinuity of Γ, there exists $y_2(u_0) \in \Gamma(u_0)$, $u \in U_1$, such that

$$y_2(u_0) \in y_1(u) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B.$$

In consequence

$$y_2(u_0) - \mathcal{K} \subset y_1(u) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B - \mathcal{K},$$

and by (46),

$$(y_2(u_0) - \mathcal{K}) \cap [(\Gamma(u) \setminus (y_1(u) + \frac{1}{2}\varepsilon \cdot B)) + \frac{1}{2}m(\frac{1}{2}\varepsilon) \cdot B] = \emptyset.$$

Since $\frac{1}{2}m(\frac{1}{2}\varepsilon) > \frac{1}{8}\varepsilon$ the latter implies that

$$(y_2(u_0) - \mathcal{K}) \cap [(\Gamma(u) \setminus (y_1(u) + \frac{1}{2}\varepsilon \cdot B)) + \frac{1}{8}\varepsilon \cdot B] = \emptyset.$$

Now, by (48),

$$(y_2(u_0) - \mathcal{K}) \cap \Gamma(u_0) \subset y_1(u) + \frac{5}{8}\varepsilon \cdot B .$$

Since (DP) holds for $\Gamma(u_0)$, there exists $\eta_2(u_0) \in M(u_0)$, such that

$$\eta_2(u_0) \in (y_2(u_0) - \mathcal{K}) \cap \Gamma(u_0) \subset y_1(u) + \frac{5}{8}\varepsilon \cdot B ,$$

and

$$\eta_2(u_0) \in y(u) + \frac{7}{8}\varepsilon \cdot B \subset y(u) + \varepsilon \cdot B .$$

This means that for $u \in U_1 \cap U_2$

$$M(u) \subset M(u_0) + \varepsilon \cdot B ,$$

which completes the proof.

□

Theorem 5.6.3 *Let Y be a normed space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone. Assume that $\Gamma : U \to Y$ is a set-valued mapping defined on a normed space U. If*

(i) $M(u) \subset cl(SM(u))$, *in some neighbourhood U_2 of u_0 and for any $\varepsilon > 0$, for all $z \in SM(u)$, $m(\varepsilon) = \inf_{u \in U_2} m(\varepsilon, u) > 0$, where $m(\cdot, u)$ is the modulus of minimality of $\Gamma(u)$,*

$$[(\Gamma(u) \setminus (z + \varepsilon \cdot B)) + m(\varepsilon) \cdot B] \cap (z - \mathcal{K}) = \emptyset ,$$

where $m(\varepsilon) \geq 2k\varepsilon$, where $k > 0$,

(ii) *DP holds for all $\Gamma(u)$ in some neighbourhood U_0 of u_0,*

(iii) *Γ is locally Lipschitz at u_0, ie.,*

$$\Gamma(u_1) \subset \Gamma(u_2) + L\|u_1 - u_2\| \cdot B$$

for u_1, u_2 in a neighbourhood U_1 of u_0, *then M is locally Lipschitz at u_0, ie., for each $u_1,\ u_2 \in U_0 \cap U_1$*

$$M(u_1) \subset M(u_2) + (1 + \frac{2}{k})L \cdot B\,.$$

Proof. Let $u_1,\ u_2 \in U_0 \cap U_1 \cap U_2$. Let $y \in M(u_1)$. By (i), there exists $y_1 \in SM(u_1)$, such that $y_1 \in y + \frac{1}{k}L\|u_1 - u_2\| \cdot B$. Since $y_1 \in SM(u_1)$,

$$[(\Gamma(u_1)\backslash(y_1+\frac{1}{k}L\|u_1-u_2\|\cdot B))+m(\frac{1}{k}L\|u_1-u_2\|)\cdot B]\cap(y_1-\mathcal{K}) = \emptyset\,,$$

and hence

$$[\Gamma(u_1)\backslash(y_1+\frac{1}{k}L\|u_1-u_2\|\cdot B)+\frac{1}{2}m(\frac{1}{k}L\|u_1-u_2\|)\cdot B]\cap(y_1+\frac{1}{2}m(\frac{1}{k}L\|u_1-u_2\|)\cdot B-\mathcal{K}) = \tag{49}$$

By local Lipschitz continuity of Γ,

$$\begin{aligned}\Gamma(u_2) &\subset \Gamma(u_1) + L\|u_1 - u_2\| \cdot B \\ &\subset [(\Gamma(u_1) \setminus (y_1 + \tfrac{1}{k}L\|u_1 - u_2\| \cdot B)) + L\|u_1 - u_2\| \cdot B] \cup [y_1 + (1 + \tfrac{1}{k})L\|u_1 - \end{aligned} \tag{50}$$

and since $y_1 \in \Gamma(u_1)$, there exists $y_2 \in \Gamma(u_2)$ such that

$$y_2 \in y_1 + L\|u_1 - u_2\| \cdot B\,,$$

and, since $L\|u_1 - u_2\| \leq \frac{1}{2}m(\frac{1}{k}L\|u_1 - u_2\|)$,

$$y_2-\mathcal{K} \subset y_1+L\|u_1-u_2\|\cdot B-\mathcal{K} \subset y_1+\frac{1}{2}m(\frac{1}{k}L\|u_1-u_2\|)\cdot B-\mathcal{K}\,.$$

By (49),

$$(y_2-\mathcal{K})\cap[\Gamma(u_1)\backslash(y_1+\frac{1}{k}L\|u_1-u_2\|\cdot B)+\frac{1}{2}m(\frac{1}{k}L\|u_1-u_2\|)\cdot B] = \emptyset\,,$$

and since $L\|u_1 - u_2\| \leq \frac{1}{k}m(\frac{1}{k}L\|u_1 - u_2\|)$,

$$(y_2 - \mathcal{K}) \cap [\Gamma(u_1) \setminus (y_1 + \frac{1}{k}L\|u_1 - u_2\| \cdot B) + L\|u_1 - u_2\| \cdot B] = \emptyset\,.$$

Now, by (50),

$$(y_2 - \mathcal{K}) \cap \Gamma(u_2) \subset y_1 + (1 + \frac{1}{k})L\|u_1 - u_2\| \cdot B\,.$$

Since (DP) holds for $\Gamma(u_2)$, there exists $\eta_2 \in M(u_2)$ such that

$$\eta_2 \subset (y_2 - \mathcal{K}) \cap \Gamma(u_2) \subset y_1 + (1+\frac{1}{k})L\|u_1 - u_2\| \cdot B \subset y + (1+\frac{2}{k})L\|u_1 - u_2\| .$$

This means that for $u_1, u_2 \in U_0 \cap U_1$

$$M(u_1) \subset M(u_2) + (1 + \frac{2}{k})L\|u_1 - u_2\|) \cdot B ,$$

which completes the proof.

□

Theorem 5.6.4 *Let Y be a normed space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone. Assume that $\Gamma : U \to Y$ is a set-valued mapping defined on a normed space U. If*

(i) $M(u_0) \subset cl(SM(u_0))$,

$$[(\Gamma(u_0) \setminus (z + \varepsilon \cdot B)) + m(\varepsilon) \cdot B] \cap (z - \mathcal{K}) = \emptyset ,$$

where $m(\varepsilon)$ is the modulus of minimality of $\Gamma(u_0)$, $m(\varepsilon) \geq k\varepsilon$, $k > 0$,

(ii) *DP holds for all $\Gamma(u)$ in some neighbourhood U_0 of u_0,*

(iii) *Γ is upper and lower Lipschitz at u_0, ie.,*

$$\Gamma(u) \subset \Gamma(u_0) + L\|u - u_0\| \cdot B$$

$$\Gamma(u_0) \subset \Gamma(u) + \frac{1}{2}L\|u - u_0\| \cdot B$$

for u in a neighbourhood U_1 of u_0,

then M is lower Lipschitz at u_0, ie., for $u \in U_0 \cap U_1$

$$M(u_0) \subset M(u) + (1 + \frac{2}{k})L\|u - u_0\| .$$

Proof. Let $u \in U_0 \cap U_1$. Let $y \in M(u_0)$. By (i), there exists $y_1 \in SM(u_0)$, such that $y_1 \in y + \frac{1}{k}L\|u - u_0\| \cdot B$. Since $y_1 \in SM(u_0)$,

$$[(\Gamma(u_0) \setminus (y_1 + \frac{1}{k}L\|u - u_0\| \cdot B)) + m(\frac{1}{k}L\|u - u_0\|) \cdot B] \cap (y_1 - \mathcal{K}) = \emptyset ,$$

and hence

$$[\Gamma(u_0)\backslash(y_1+\frac{1}{k}L\|u_1-u_2\|\cdot B)+\frac{1}{2}m(\frac{1}{k}L\|u_1-u_2\|)\cdot B]\cap(y_1+\frac{1}{2}m(\frac{1}{k}L\|u_1-u_2\|)\cdot B-\mathcal{K}) = \tag{51}$$

By the upper Lipschitz continuity of Γ,

$$\begin{aligned}\Gamma(u) &\subset \Gamma(u_0) + L\|u-u_0\|\cdot B\\ &\subset [(\Gamma(u_0)\setminus(y_1+\tfrac{1}{k}L\|u-u_0\|\cdot B)) + L\|u-u_0\|\cdot B]\cup[y_1+(\tfrac{1}{k}+1)L\|u-u_0\|\end{aligned}\tag{52}$$

and since $y_1\in\Gamma(u_0)$, by the lower Lipschitz continuity, there exists $y_2\in\Gamma(u)$ such that

$$y_2\in y_1+\frac{1}{2}L\|u-u_0\|\cdot B\,,$$

and, since $\frac{1}{2}L\|u-u_0\|\le\frac{1}{2}m(\frac{1}{k}L\|u-u_0\|)$,

$$y_2-\mathcal{K}\subset y_1+\frac{1}{2}L\|u-u_0\|\cdot B-\mathcal{K}\subset y_1+\frac{1}{2}m(\frac{1}{k}L\|u-u_0\|)\cdot B-\mathcal{K}\,.$$

By (51),

$$(y_2-\mathcal{K})\cap[\Gamma(u_0)\backslash(y_1+\frac{1}{k}L\|u-u_0\|\cdot B)+\frac{1}{2}m(\frac{1}{k}L\|u-u_0\|)\cdot B]=\emptyset\,,$$

and since $L\|u-u_0\|\le m(\frac{1}{k}L\|u-u_0\|)$,

$$(y_2-\mathcal{K})\cap[\Gamma(u_0)\setminus(y_1+\frac{1}{k}L\|u-u_0\|\cdot B)+L\|u-u_0\|\cdot B]=\emptyset\,.$$

Now, by (52),

$$(y_2-\mathcal{K})\cap\Gamma(u)\subset y_1+(1+\frac{1}{k})L\|u-u_0\|\cdot B\,.$$

Since (DP) holds for $\Gamma(u)$, there exists $\eta_2\in M(u)$ such that

$$\eta_2\subset(y_2-\mathcal{K})\cap\Gamma(u)\subset y_1+(1+\frac{1}{k})L\|u-u_1\|\cdot B\subset y+(1+\frac{2}{k})L\|u-u_0\|\,.$$

This means that for $u\in U_0\cap U_1$

$$M(u_0)\subset M(u)+\frac{k+2}{k}L|u-u_0\|\cdot B\,,$$

which completes the proof.

□

5.7 Lower continuity of minimal points in vector optimization

Definition 5.7.1 *The solution $x_0 \in S(f, A, \mathcal{K})$ is called $\phi-$***strict** *if there is no $x \in S(f, A, \mathcal{K})$, $x \neq x_0$, such that*

$$f(x_0)-f(x) \in \phi(\|x_0-x\|)\cdot B+\mathcal{K}, \text{ or } f(x)-f(x_0)-\phi(\|x_0-x\|)\cdot B \in -\mathcal{K},$$

where $\phi : R_+ \to R_+$ is a nondecreasing admissible function.

Proposition 5.7.1 *Let $\mathcal{K} \subset Y$ be a closed convex cone in a normed space Y, and $int\mathcal{K} \neq \emptyset$. Let $f : X \to Y$ be a Lipschitz mapping defined on a normed space X, and let $A \subset X$ be a subset of X. If $x_0 \in S(f, A, \mathcal{K})$ is $\phi-$strict, then $f(x_0)$ is a strictly minimal element of $f(A)$.*

Proof. Suppose that $x_0 \in S(f, A, \mathcal{K})$ is $\phi-$strong. Because of the symmetry of balls in Y, there is no $x \in A$, $x \neq x_0$, such that

$$f(x) - f(x_0) + \phi(\|x - s_x\|) \cdot B \subset -\mathcal{K}.$$

Since $\|f(x) - f(s_x)\| < L\|x - s_x\|$ and ϕ is nondecreasing $\phi(\frac{1}{L}\|f(x) - f(s_x)\|) \leq \phi(\|x - s_x\|)$ and

$$f(x)-f(x_0)+\phi(\frac{1}{L}\|f(x)-f(x_0)\|)\cdot B \subset f(x)-f(x_0)+\phi(\|x-s_x\|)\cdot B \subset -\mathcal{K}.$$

Take $\varepsilon > 0$ and put $W = \{f(x) \mid \|f(x) - f(x_0)\| < L\varepsilon\}$. For any any $x \in A \setminus W$ we have $\|f(x) - f(x_0)\| \geq L\varepsilon$. Since ϕ is nondecreasing, $\phi(\|f(x) - f(x_0)\|) \geq \phi(L\varepsilon)$, and hence, there is no $x \in A \setminus W$, $x \neq x_0$, such that

$$f(x)-f(x_0)+\phi(\varepsilon)\cdot B \subset f(x)-f(s_x)+\phi(\frac{1}{L}\|f(x)-f(s_x)\|)\cdot B \subset -\mathcal{K},$$

which means that

$$[f(A) - W] \cap [\phi(\varepsilon) \cdot B - \mathcal{K}] = \emptyset,$$

ie., $f(x_0)$ is strictly minimal.

□

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