Raport Badawczy

Research Report

RB/25/2001

Stability analysis for parametric vector optimization problems

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Warszawa 2001

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Preface

We study stability of minimal points and solutions to parametric (or perturbed) vector optimization problems in the framework of real topological vector spaces and, if necessary, normed spaces. Because of particular importance of finite-dimensional problems, called multicriteria optimization problems, which model various real-life phenomena, a special attention is paid to the finite-dimensional case. Since one can hardly expect the sets of minimal points and solutions to be singletons, set-valued mappings are natural tools for our studies.

Vector optimization problems can be stated as follows. Let X be a topological space and let Y be a topological vector space ordered by a closed convex pointed cone $\mathcal{K} \subset Y$. Vector optimization problem

$$\begin{array}{ll} \mathcal{K} - \min\ f_0(x) & \\ \text{subject to } x \in A_0\,, & (P_0) \end{array}$$

where $f : X \to Y$ is a mapping, and $A_0 \subset X$ is a subset of X, relies on finding the set $\mathrm{Min}(f_0, A_0, \mathcal{K}) = \{y \in f_0(A_0) \mid f_0(A_0) \cap (y - \mathcal{K}) = \{y\}\}$ called the **Pareto** or **minimal point** set of (P_0), and the **solution set** $S(f_0, A_0, \mathcal{K}) = \{x \in A_0 \mid f_0(x) \in \mathrm{Min}(f_0, A_0, \mathcal{K})\}$. We often refer to problem (P_0) as the **original problem** or **unperturbed one**. The space X is the **argument** space and Y is the **outcome** space.

Let U be a topological space. We embed the problem (P_0) into a family (P_u) of vector optimization problems parametrised by a parameter $u \in U$,

$$\begin{array}{ll} \mathcal{K} - \min\ f(u,x) & \\ \text{subject to } x \in A(u)\,, & (P_u) \end{array}$$

where $f : U \times X \to Y$ is the parametrised objective function and $A : U \rightrightarrows Y$, is the feasible set multifunction, (P_0) corresponds to a parameter value u_0. The performance multifunction $\mathcal{M} : U \rightrightarrows Y$, is defined as $\mathcal{M}(u) = \text{Min}(f(u,\cdot), A(u), \mathcal{K})$, and the solution multifunction $\mathcal{S} : U \rightrightarrows Y$, is given as $\mathcal{S}(u) = S(f(u,\cdot), A(u), \mathcal{K})$, and $f : U \times X \to Y$, $A(u) \subset X$.

Our aim is to study continuity properties of $\mathcal{M}$ and $\mathcal{S}$ as functions of the parameter u. Continuous behaviour of solutions as functions of parameters is of crucial importance in many aspects of the theory of vector optimization as well as in applications(correct formulation of the model and/or approximation) and numerical solution of the problem in question.
We investigate continuity in the sense of Hausdorff and Hőlder of the multivalued mappings of minimal points $\mathcal{M}(u)$ and solutions $\mathcal{S}(u)$ as functions of the parameter u under possibly weak assumptions. We attempt to avoid as much as possible compactness assumptions which are frequently over-used (see eg [83]).
It is a specific feature of vector optimization that the outcome space is equipped with a partial order generated by a cone the properties of which are important for stability analysis. In many spaces cones of nonnegative elements have empty interiors and because of this we derive stability results for cones with possibly empty interior. This kind of results are specific for vector optimization and do not have their counterpart in scalar optimization.
We introduce two new concepts: the notion of containment(with some variants for cones with empty interiors),[16], and the notion of strict minimality, [12].

The containment property (CP), defined in topological vector spaces, is introduced to study upper semicontinuities (in the sense of Hausdorff) of minimal points, [11, 16]. It is a variant of the domination property (DP), which appears frequently in the context of stability of solutions to parametric vector optimization problems. Although it is not a commonly adopted view point, the domination property may be accepted as a solution concept which generalizes the standard concept of a solution to scalar optimization problem. In consequence, the containment property (CP) may also be seen as a solution concept in vector optimization. To investigate more deeply this aspect we interpret the containment property as a generalization of the concept of the set of $\phi-$local solutions appearing in the

context of Lipschitz continuity of solutions to scalar optimization problems. Under mild assumptions the containment property imply that the set weakly minimal points equals the set of minimal points. This equality, in turn, is a typical ingradient of standard finite-dimensional sufficient conditions for upper semicontinuity of minimal points.
To study Hőlder upper continuity of minimal points we define the rate of containment of a set with respect to a cone, which is a real-valued function of a scalar argument, see [14, 15]. The rate of growth of this function influence decisively the rate of Hőlder continuity of minimal points, [15].

Strictly minimal points are introduced to study lower semicontinuities (lower Hausdorff, lower Hőlder) of minimal points [20, 13]. The definition of a strictly minimal point is given in topological vector spaces and it is a generalization of the notion of a super efficient point in the sense of Borwein and Zhuang defined in normed spaces. We discuss strict minimality in vector optimization by proving that it is a vector counterpart of the concept of $\phi-$ local solution to scalar optimization problem.

Theory of vector optimization may be considered as an abstract study of optimization problems with mappings taking values in the outcome space equipped with a partial order structure. As such, it contains many concepts and results which generalize and/or have their counterparts in scalar optimization. The very definition of the set of minimal points of vector optimization problem in the outcome space may serve as an example here. This is a counterpart of the optimal value of scalar optimization problem. Another example is the concept of well-posed optimization problem. In subsequent developments we often compare our results and considerations with the corresponding approaches in scalar optimization. For instance, we define several classes of well-posed vector optimization problems by generalizing the concept of scalar minimizing sequence and in these classes we investigate continuity of solutions. For scalar optimization problems, the existing approaches and results on well-posedness are extensively discussed in the monograph by Dontchev and Zolezzi [33].

Convergence and rates of convergence of solutions to perturbed optimization problems is one of crucial topics of stability analysis in optimization both from theoretical and numerical points of view. For scalar optimization it was investigated by many authors see eg., [72], [32], [47], [78], [55], [81], [59], [60], [82], [2], and many others. An exhaustive survey of current state of research is given in the recent monograph by Bonnans and Shapiro [26]. In vector optimization the results on Lipschitz continuity of solutions are not so numerous, and concern some classes of problems, for linear case see eg.,[28], [29], [30], for convex case see eg., [25], [31].

The organization of the material is as follows. In Chapter 2 we investigate upper Hausdorff continuity of the multivalued mapping M, $M(u) = \mathrm{Min}(\Gamma(u)|\mathcal{K})$ assigning to a given parameter value u from a topological space U the set of minimal points of the set $\Gamma(u) \subset Y$ with respect to cone $\mathcal{K} \subset Y$, where for any subset A of a topological vector space Y the set of minimal points is defined as $\mathrm{Min}(A|\mathcal{K}) = \{y \in A \mid A \cap (y - \mathcal{K}) = \{y\}\}$, and $\Gamma : U \rightrightarrows Y$, is a given multivalued mapping. The main tool which allows us to obtain the general result is the containment property (CP). Some infinite-dimensional examples are discussed. A special attention is paid to the containment property (CP) in finite-dimensional case, when $Y = R^m$.

In Chapter 3 we discuss upper Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the rate of containment δ which is a one-variable nondecreasing function, defined for a given set A and the order generating cone $\mathcal{K}$. The assumption of sufficiently fast growth rate of this function appears to be the crucial assumption for all upper Hőlder stability results of Chapter 3.

In Chapter 4 we apply the results obtained in Chapters 2 and 3 to derive conditions for upper Hausdorff and upper Hőlder stability of minimal points to parametric vector optimization problems by taking $\Gamma(u) = f(u, A(u))$. Moreover, we introduce the concept of $\Phi-$ strong solutions to vector optimization problem (P_0), which is a generalization of the concept of a $\phi-$local minimizer to scalar optimization problem, the latter being introduced by Attouch and

Wets [6].

In Chapter 5 we investigate the lower continuity and lower Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the notion of strict minimality mentioned above and the rate of strict minimality. In Section 5.5 we apply the results obtained in Chapter 5 to parametric vector optimization problems and we derive sufficient conditions for lower and lower Hőlder continuity of Pareto point multivalued mapping $\mathcal{M}$. An important tool here is the notion of $\Phi-$ strict solution to vector optimization problem introduced in Section 6.1. This notion can be interpreted as another possible generalization of the concept of $\phi-$local minimizer.

In Chapter 6 we propose several definitions of a well-posed vector optimization problem. All these definitions are based on properties of $\varepsilon-$solutions to vector optimization problems. For well-posed vector optimization problems we prove upper Hausdorff continuity of solution multivalued mapping S, $S(u) = S(f(u,\cdot), A(u), \mathcal{K})$.

3

Hőlder continuity of minimal points under perturbations of the set

In this chapter we derive criteria for upper Lipschitz/Hőlder and Lipschitz/Hőlder continuities of the set of minimal points $Min(A|\mathcal{K})$ with respect to a convex closed pointed cone $\mathcal{K} \subset Y$ of a given subset $A \subset Y$ of a normed space Y when A is subjected to perturbations. In general, there exist many ways of dealing with perturbations whenever they appear. We express perturbations by multivalued mappings Γ, defined on a space of perturbations U, with $\Gamma(u_0) = A$, and consider the family of problems (P_u) of finding $Min(\Gamma(u)|\mathcal{K})$. Upper Hőlder property at u_0 ensures that the distance of a solution of perturbed problem (P_u) to the set of solutions of unperturbed problem (P_{u_0}) can be estimated via the distance of perturbations $\|u - u_0\|$ raised to some power q. Hence, upper Hőlder property is of interest when it is impossible or too difficult to deal with the original problem and one wants to know the magnitude of the error made by accepting a solution of perturbed problem as a solution of the original problem. For instance, numerical representation of problems lead to perturbations due to finite precision. The upper Lipschitz property (upper Hőlder property with $q = 1$) has already appeared in investigation of stability of different problems, see eg., [69, 70, 71].
In Sections 3.1, 3.2 we consider problems with int$\mathcal{K} \neq \emptyset$. In Section 3.1 we introduce two functions measuring depart from minimality; the invidual rate of containment μ defined for $y \in Y$, and the rate

of containmment δ of a subset $A \subset Y$ with respect to $\mathcal{K}$, which is a real-valued function of one real variable and measures the depart from minimality on A as a function of the distance from the minimal point set. In Section 3.2 we investigate upper Hőlder continuity of $Min(\Gamma(u)|\mathcal{K})$ at a given point u_0. The main requirement we impose is that for small arguments the rate of containment δ is a sufficiently fast growing function.
In Sections 3.3, 3.4 we do not make any assumption concerning the interior of $\mathcal{K}$. In Section 3.3 we define the dual rate of containment d and in Section 3.4 we investigate upper Hőlder continuity of $Min(\Gamma(u)|\mathcal{K})$ at a given point u_0. The main requirement we impose is that for small arguments the rate of containment d is a sufficiently fast growing function.

Let $U = (U, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be normed spaces. We say that a multivalued mapping $F : U \rightrightarrows Y$, is:

upper Lipschitz at u_0, (compare eg. [69, 70, 71]), if there exist a neighbourhood U_0 of u_0 and a positive constant L such that

$$F(u) \subset F(u_0) + L \cdot \|u - u_0\| B \quad \text{for} \quad u \in U_0,$$

lower Lipschitz at u_0, if there exist a neighbourhood U_0 of u_0 and a positive constant L such that

$$F(u_0) \subset F(u) + L \cdot \|u - u_0\| B \quad \text{for} \quad u \in U_0,$$

Lipschitz continuous at u_0 with constant L if it is upper and lower Lipschitz continuous at u_0 with constant L,

upper Hőlder of order q at u_0 with constant L if there exists a neighbourhood U_0 such that

$$F(u) \subset Fu_0) + L\|u - u_0\|^q \cdot B \quad \text{for } u \in U_0,$$

lower Hőlder of order q at u_0 with constant L if there exists a neighbourhood U_0 such that

$$F(u_0) \subset F(u) + L\|u - u_0\|^q \cdot B \quad \text{for } u \in U_0,$$

Hőlder at u_0 of order ℓ with constant L if it is upper and lower Hőlder of order ℓ at u_0 with constant L.

$F : U \rightrightarrows Y$, is **locally Lipschitz** around u_0, [9], if there exist a neighbourhood $U_0 \subset \text{dom}F$ of u_0 and a positive constant ℓ such that

$$F(u_1) \subset F(u_2) + \ell \cdot \|u_1 - u_2\| B \quad \text{for} \quad u_1, u_2 \in U .$$

Let $y_0 \in F(u_0)$. We say that $F : U \rightrightarrows Y$, is **pseudo-Lipschitz,** [9], around $(y_0, u_0) \in \text{graph}F$, if there exist a neighbourhood $U_0 \subset \text{dom}F$ of u_0, a neighbourhood V_0 of y_0, and a positive constant ℓ such that

$$F(u_1) \cap V_0 \subset F(u_2) + \ell \cdot \|u_1 - u_2\| B .$$

3.1 Rate of containment

Let $Y = (Y, \|\cdot\|)$ be a normed space and let $\mathcal{K}$ be a closed convex pointed cone in Y. By $B(a,r)$ we denote the open ball of radius r and centre a, $B(0,1) = B$. For any subset A of Y and any $y \in Y$ we have $d(y, A) = \inf_{a \in A} \|y - a\|$ and $B(A, \varepsilon) = \{y \in Y \mid d(y, A) < \varepsilon\}$. For any $\varepsilon > 0$ denote

$$A(\varepsilon) = A \setminus B(Min(A|\mathcal{K})), \varepsilon) .$$

In $\varepsilon - \delta$ setting the containment property (CP) holds for a subset $A \subset Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$A(\varepsilon) + B(0, \delta) \subset Min(A|\mathcal{K}) + \mathcal{K}. \tag{14}$$

Definition 3.1.1 *Let $\mathcal{K} \subset Y$ be a convex cone in Y. A function $cont : \mathcal{K} \to R_+$, defined as*

$$cont(k) = \sup\{r \mid k + rB \subset \mathcal{K}\}$$

is called the **primal cone containment function.**

The function $cont$ is positively homogeneous, ie., $cont(\lambda k) = \lambda cont(k)$, $\lambda \geq 0$, and suplinear, ie., $cont(k_1 + k_2) \geq cont(k_1) + cont(k_2)$.

Definition 3.1.2 (Rate of containment) *The function $\mu : Y \to R$ defined as*

$$\mu(y) = \sup_{\eta \in Min(A|\mathcal{K}) \cap (y - \mathcal{K})} cont(y - \eta) \tag{15}$$

is the **rate of containment of** y **with respect to** A *and* $\mathcal{K}$. *The* **rate of containment of subset** $A \subset Y$ **with respect to cone** $\mathcal{K}$ *is the function* $\delta : R_+ \to R \cup \{+\infty, -\infty\}$ *defined as*

$$\delta(\varepsilon) = \inf_{y \in A(\varepsilon)} \mu(y).$$

We have $\{y \in Y \mid \mu(y) > -\infty\} = Min(A|\mathcal{K}) + \mathcal{K}$. For $y \in Min(A|\mathcal{K})$, it is $\mu(y) = 0$. If int$\mathcal{K} \neq \emptyset$ and $y \in [Min(A|\mathcal{K}) + \mathcal{K}]$, we have $\mu(y) \geq 0$, and moreover, $\mu(y) = 0$ if and only if $y \in WMin(A|\mathcal{K})$ (see Proposition 3.1.5 below). The value $\mu(y)$ gives the maximal radius r such that $k + rB \subset \mathcal{K}$, over all $k \in y - [Min(A|\mathcal{K}) \cap (y - \mathcal{K})] \subset \mathcal{K}]$. In this sense $\mu(y)$ measures the depart from minimality of an element y. If int$\mathcal{K} = \emptyset$, then $\mu(y) = 0$ for any $y \in Min(A|\mathcal{K}) + \mathcal{K}$. In turn, $\delta(\varepsilon) \geq 0$, $\varepsilon > 0$, if and only if $A \subset Min(A|\mathcal{K}) + \mathcal{K}$, ie., (DP) holds for A (see Proposition 3.1.2 below). For any subset $A \subset Y$ of Y for which (DP) holds, and any $\varepsilon > 0$, the value $\delta(\varepsilon)$ gives the minimal depart from minimality over all elements of A whose distance from the minimal point set $Min(A|\mathcal{K})$ is not smaller than ε.

In the example below we calculate $\mu(y)$ for y from the closed unit ball.

Example 3.1.1 *Let* $Y = R^2$, *and* $A = clB$, *and* $\mathcal{K} = R^2_+ = \{(y_1, y_2) \in R^2 \mid y_1 \geq 0 \ y_2 \geq 0\}$. *Clearly,* (DP) *and* (CP) *holds for* A, *and*

$$Min(A|\mathcal{K}) = \{(\eta_1, \eta_2) \in A \mid \eta_2 = -\sqrt{1 - \eta_1^2} \quad -1 \leq \eta_1 \leq 0\}.$$

Put $Min(A|\mathcal{K})_y = Min(A|\mathcal{K}) \cap (y - \mathcal{K})$. *For any representation of* $(0,0)$ *in the form* $(0,0) = \eta + k_\eta$, *where* $\eta \in Min(A|\mathcal{K})$, $k_\eta \in \mathcal{K}$, *we have* $\eta = (\eta_1, \eta_2) \in Min(A|\mathcal{K})_{(0,0)} = Min(A|\mathcal{K})$,

$$cont(k_\eta) = \min\{-\eta_1, \sqrt{1 - \eta_1^2}\} = \begin{cases} \sqrt{1 - \eta_1^2} & \textit{for} \quad -1 \leq \eta_1 \leq -1/\sqrt{2} \\ -\eta_1 & \textit{for} \quad -1/\sqrt{2} \leq \eta_1 \leq 0 \end{cases}$$

and $\mu((0,0)) = \sup_{\{-1 \leq \eta_1 \leq 0\}} cont(k_\eta) = 1/\sqrt{2}$. *For* $y \in A$, $y = (y_1, y_2)$, $y_2 \geq 0$,

$$Min(A|\mathcal{K})_{(y_1, y_2)} = \{(\eta_1, \eta_2) \mid \eta_2 = -\sqrt{1 - \eta_1^2}, \ -1 \leq \eta_1 \leq \min\{0, y_1\}\},$$

and

$$\mu(y) = \max_{\{-1\leq\eta_1\leq\min\{0,y_1\}\}} cont(k_\eta) = \max_{\{-1\leq\eta_1\leq\min\{0,y_1\}\}} \min\{y_1-\eta_1, y_2+\sqrt{1-\eta_1}\}$$

For $y \in A$, $y = (y_1, y_2)$, $y_2 < 0$,

$$Min(A|\mathcal{K})_{(y_1,y_2)} = \{(\eta_1,\eta_2) \mid \eta_2 = -\sqrt{1-\eta_1^2},\ -\sqrt{1-y_2^2} \leq \eta_1 \leq \min\{0, y_1\}\},$$

and

$$\mu(y) = \max_{\{-\sqrt{1-y_2^2}\leq\eta_1\leq\min\{0,y_1\}\}} cont(k_\eta) = \max_{\{-\sqrt{1-y_2^2}\leq\eta_1\leq\min\{0,y_1\}\}} \min\{y_1-\eta_1, y_2+\sqrt{1-\eta_1^2}\}.$$

Now we investigate the relationship between (CP) property and the rate of containment δ. We start with the following technical lemma.

Lemma 3.1 *Let Y be a locally convex vector space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Suppose that an element $y \in Y$ has a representations in the form $y = a_\alpha + k_\alpha$, If at least one of the conditions holds:*

(i) $\mathcal{A}$ is weakly compact,

(ii) $\mathcal{A}$ is bounded and weakly closed, and $\mathcal{K}$ has a weakly compact base,

then y can be represented in the form $y = a_0 + k_0$, with $a_0 \in \mathcal{A}$, $k_0 \in \mathcal{K}$, beeing limit points of some subnets contained in $\{a_\alpha\}$ and $\{k_\alpha\}$, respectively.

Proposition 3.1.1 *Let $Y = (Y, \|\cdot\|)$ be a normed space. Let $\mathcal{K}$ be a closed convex pointed cone in Y and let $A \subset Y$ be a subset of Y. Under one of the following conditions:*

(i) $Min(A|\mathcal{K})$ is weakly compact,

(ii) $Min(A|\mathcal{K})$ is bounded, weakly closed, and $\mathcal{K}$ has a weakly compact base,

for any $y \in Min(A|\mathcal{K})+\mathcal{K}$, there exists a representation $y = \eta_y+k_y$, with $\eta_y \in Min(A|\mathcal{K})$, and $k_y + \mu(y)B \subset \mathcal{K}$.

Proof. Let $y \in Min(A|\mathcal{K}) + \mathcal{K}$. For any $\alpha > 0$, one can find a representation $y = \eta_\alpha + k_\alpha$, $\eta_\alpha \in Min(A|\mathcal{K})$, $k_\alpha \in \mathcal{K}$, $k_\alpha + cont(k_\alpha)B \subset \mathcal{K}$, satisfying

$$cont(k_\alpha) \leq \mu(y) \quad \text{and} \quad cont(k_\alpha) > \mu(y) - \alpha\,.$$

We start by proving that under any of the conditions (i), (ii), y can be represented in the form

$$y = \eta_0 + k_0\,, \tag{16}$$

where $\eta_0 \in Min(A|\mathcal{K})$, $k_0 \in \mathcal{K}$, $\eta_0 = \lim_\alpha \eta_\alpha$, $k_0 = \lim_\alpha k_\alpha$. In the case (i), since $Min(A|\mathcal{K})$ is weakly compact, there exists a weakly convergent subnet of the net $\{\eta_\alpha\}$. Without loss of generality we can assume that the net $\{\eta_\alpha\}$ weakly converges to some $\eta_0 \in Min(A|\mathcal{K})$. Since $\mathcal{K}$ is closed and convex, the net $\{k_\alpha\}$, where $k_\alpha = y - \eta_\alpha$, converges weakly to $k_0 \in \mathcal{K}$, and $y = \eta_0 + k_0$.
To prove (16) in case (ii) suppose that Θ is a weakly compact base of $\mathcal{K}$, $k_\alpha = \lambda_\alpha \theta_\alpha$, $\lambda_\alpha \geq 0$, and $\{\theta_\alpha\} \subset \Theta$ contains a weakly convergent subnet. Without loss of generality we can assume that $\{\theta_\alpha\}$ converges to $\theta_0 \in \Theta$. Since $Min(A|\mathcal{K})$ is bounded and $\|\theta\| \geq M_0$ for all $\theta \in \Theta$ we get

$$M_1 \geq \|y - \eta_\alpha\| = \lambda_\alpha \|\theta_\alpha\| \geq M_0 \lambda_\alpha\,,$$

for some positive constants M_0, M_1. This implies that $\{\lambda_\alpha\}$ is bounded, and thus the net $\{k_\alpha\}$ contains a convergent subnet, ie., we can assume that $\{k_\alpha\}$ weakly converges to some $k_0 = \lambda_0 \theta_0 \in \mathcal{K}$. In consequence, $\eta_\alpha = y - k_\alpha$ converges weakly to some $\eta_0 \in Min(A|\mathcal{K})$ and we get a representation $y = \eta_0 + k_0$.

To complete the proof we show that $k_0 + \mu(y)B \subset \mathcal{K}$. On the contrary, if it were $k_0 + \mu(y)b \notin \mathcal{K}$, for some $b_0 \in B$, by separation arguments it would be

$$f(k_0 + \mu(y)b_0) < 0 < f(k) \quad \text{for } k \in \mathcal{K}\,,$$

for some $f \in \mathcal{K}^*$, $\mathcal{K}^* = \{f \in Y^* \mid f(k) \geq 0\}$. By the weak convergence of $\{k_\alpha\}$ to k_0, and $\{(cont(k_\alpha) - \mu(y))b_0\}$ to zero we would have

$$f(k_\alpha + cont(k_\alpha)b_0) = f(k_0 + \mu(y)b_0) + f(k_\alpha - k_0) + f([cont(k_\alpha) - \mu(y)]b_0) < 0\,,$$

which would contradict the fact that $k_\alpha + cont(k_\alpha)B \subset \mathcal{K}$. □

Let

$$\text{dom}\delta = \{\varepsilon \in R_+ \mid \delta(\varepsilon) < +\infty\} = \{\varepsilon \in R_+ \mid A(\varepsilon) \neq \emptyset\}$$

be the domain of δ. The following properties of the rate of containment are direct consequences of the definition.

1. The rate of containment $\delta : R_+ \to R$ is nondecreasing. Indeed, let $\varepsilon_1, \varepsilon_2 \in \text{dom}\delta$, $\varepsilon_1 > \varepsilon_2 > 0$. Then $A(\varepsilon_1) \subset A(\varepsilon_2)$, and consequently $\delta(\varepsilon_1) = \inf_{x \in A(\varepsilon_1)} \mu(x) \geq \delta(\varepsilon_2)$.

2. Assume that there exists at least one $\eta \in Min(A|\mathcal{K})$ which is not an isolated point of A, and (DP) holds for A. Suppose that one of the conditions holds:

   (i) $Min(A|\mathcal{K})$ is weakly compact,

   (ii) $Min(A|\mathcal{K})$ is bounded and weakly closed, and $\mathcal{K}$ kas a weakly compact base.

   Then $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$. Indeed, suppose that

   $$+\infty > \delta(\varepsilon_n) = \inf_y \in A(\varepsilon_n)\mu(y) > c$$

   for some $\varepsilon_n \to 0$ and $c > 0$. If so, then $\mu(y) > c$ for all n and $y \in A(\varepsilon_n)$. Moreover, one can choose $y_n \to \eta \in Min(A|\mathcal{K})$, $y_n \in A(\varepsilon_n)$, and there exists a representation $y_n = \eta_n + k_n$, $\eta_n \in Min(A|\mathcal{K})$, $k_n + cB \subset \mathcal{K}$. In the same way as in Proposition 3.1.1, in view of (i) and (ii) we can prove that $\eta = \eta_0 + k_0$, where $\lim_n \eta_n = \eta_0 \in Min(A|\mathcal{K})$, $\lim_n k_n = k_0 \in \mathcal{K}$. Consequently, it must be $k_0 = 0$, but on the other hand, $k_0 + c/2B \subset \mathcal{K}$ which is a contradiction. This proves the assertion.

3. If $\text{int}\mathcal{K} = \emptyset$, then, for any $\varepsilon \in \text{dom}\delta$, $\delta(\varepsilon) = 0$ if and only if (DP) holds for A, and $\delta(\varepsilon) = -\infty$ if and only if (DP) does not hold for A.

Proposition 3.1.2 *Let $A \subset Y$ be a nonempty subset of Y with $Min(A|\mathcal{K})$ nonempty and closed. The following are equivalent:*

(i) (DP) holds for A

(ii) $\delta(\varepsilon) \geq 0$ *for all* $\varepsilon \in dom\delta$.

Proof. *(ii)* $\to$ *(i)*. Suppose that (DP) does not hold, ie., there exists $y \in A$ which cannot be represented in the form $y = \eta + k$, with $\eta \in Min(A|\mathcal{K})$, and $k \in \mathcal{K}$. Hence, $\mu(y) = -\infty$. By closedness of $Min(A|\mathcal{K})$, $y \in A(\varepsilon)$, for some $\varepsilon > 0$. Consequently, $\delta(\varepsilon) = -\infty$, contradictory to *(ii)*.
(i) $\to$ *(ii)*. By (DP), for each $y \in A$ we have $y = \eta + k$, $\eta \in Min(A|\mathcal{K})$, $k \in \mathcal{K}$. Hence, $\mu(y) \geq 0$, and *(ii)* follows. □

Proposition 3.1.3 *Let* Y *be a normed space and let* $\mathcal{K}$ *be a closed convex pointed cone with nonempty interior. Let* A *be a subset of* Y. *The following are equivalent:*

(i) (CP) *holds for* A,

(ii) $\delta(\varepsilon) > 0$ *for each* $\varepsilon \in dom\delta$.

Proof. *(i)* $\to$ *(ii)*. By Proposition 2.1.4, for any $\varepsilon \in \mathrm{dom}\delta$, and $y \in A(\varepsilon) \neq \emptyset$, there exists $\kappa > 0$ such that

$$y = \eta + k, \quad \eta \in Min(A|\mathcal{K}), \quad k + \kappa B \subset \mathcal{K}.$$

Consequently, $\mu(y) \geq \kappa$, and $\delta(\varepsilon) \geq \kappa > 0$.
(ii) $\to$ *(i)*. Let $\varepsilon \in \mathrm{dom}\delta$, and $\delta(\varepsilon) = c > 0$. Then $\mu(y) \geq c$, for each $y \in A(\varepsilon)$, which means that $y = \eta_y + k_y$, where $\eta_y \in Min(A|\mathcal{K})$, $k_y + c/2 \cdot B \subset \mathcal{K}$. Thus, (CP) holds. □

Proposition 3.1.4 *Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in a normed space* $Y = (Y, \|\cdot\|)$ *and let* $int\mathcal{K} \neq \emptyset$. *Let* $A \subset Y$ *be a nonempty subset of* Y *and let* (CP) *holds for* A. *Under one of the following conditions:*

(i) $Min(A|\mathcal{K})$ *is weakly compact,*

(ii) $Min(A|\mathcal{K})$ *is bounded and weakly closed, and* $\mathcal{K}$ *has a weakly compact base,*

for any $\varepsilon > 0$

(1) $A(\varepsilon) + \delta(\varepsilon)B \subset Min(A|\mathcal{K}) + \mathcal{K}$,

(2) each $y \in A(\varepsilon)$ *can be represented in the form* $y = \eta_y + k_y$ *, where* $\eta_y \in Min(A|\mathcal{K})$ *,* $k_y + \delta(\varepsilon) \cdot B \subset \mathcal{K}$ *.*

Proof. (2) follows directly from Proposition 3.1.1. (1) follows from (2). □

Proposition 3.1.5 *Let* Y *be a normed space and let* $\mathcal{K} \subset Y$ *be a closed convex cone in* Y*, with nonempty interior. Let* $A \subset Y$ *be a subset of* Y *and* (DP) *holds for* A*. The following are equivalent:*

(i) $\mu(y) = 0$,

(ii) $y \in WMin(A|\mathcal{K})$.

Proof. $(i) \to (ii)$. By (i), any representation of y in the form $y = \eta + k$, $\eta \in Min(A|\mathcal{K})$, $k \in \mathcal{K}$, satisfies $k \in \partial\mathcal{K}$ which means that $A \cap [y - \text{int}\mathcal{K}] = \emptyset$, ie., $y \in WMin(A|\mathcal{K})$.
$(ii) \to (i)$. If it were $\mu(y) \geq \alpha > 0$, it would be $y = \eta + k$, $\eta \in Min(A|\mathcal{K})$, $k + \alpha B \subset \mathcal{K}$, which would imply that $y \notin WMin(A|\mathcal{K})$. □

3.2 Upper Hőlder continuity of minimal points for cones with nonempty interior.

Let $U = (U, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be normed spaces and let $\Gamma : U \rightrightarrows Y$ be a multivalued mapping.
In this section we prove sufficient conditions for upper Hőlder continuity of the minimal point set-valued mapping $M : U \rightrightarrows Y$,

$$M(u) = Min(\Gamma(u)|\mathcal{K}).$$

At the beginning of this Chapter we indicate some situations where upper Hőlder continuity has a natural significance. One more example of such situation comes from parametric vector optimization. Theorem 6.4 of [11] and Theorem 6.2 of [12] reveal the importance of upper type continuities of the performance multivalued mapping $\mathcal{P}$ for continuity of solutions to parametric vector optimization problems.
In the theorem below we give sufficient conditions for upper Hőlder continuity of minimal point multifunction M.

Theorem 3.2.1 *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $int\mathcal{K} \neq \emptyset$. Let $\Gamma : U \rightrightarrows Y$, be a set-valued mapping which is upper Hőlder continuous with ℓ_1 and constant L_1 and lower Hőlder continuous with order ℓ_2 and constant L_2 at u_0. Suppose that one of the following conditions hold:*

(i) $Min(\Gamma(u_0)|\mathcal{K})$ is weakly compact,

(ii) $Min(\Gamma(u_0)|\mathcal{K})$ is bounded and weakly closed and $\mathcal{K}$ has a weakly compact base.

If the rate of containment δ of $\Gamma(u_0)$, satisfies the condition $\delta(\varepsilon) \geq c \cdot \varepsilon^p$, with $c > 0$, then

$$M(u) \subset M(u_0) + (L_1 + \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}})\|u - u_0\|^{\min\{\ell_1, \frac{\min\{\ell_1,\ell_2\}}{p}\}} \cdot B.$$

for all u in some neighbourhood of u_0.

Proof. By the upper Hőlder continuity of Γ,

$$\begin{aligned}\Gamma(u) &\subset \Gamma(u_0) + L_1\|u-u_0\|^{\ell_1} \cdot B \\ &\subset [M(u_0) + L_1 \cdot \|u-u_0\|^{\ell_1} \cdot B + \left(\tfrac{L_1+L_2}{c}\right)^{\frac{1}{p}} \cdot \|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B] \cup \\ &\cup [(\Gamma(u_0) \setminus (M(u_0) + \left(\tfrac{L_1+L_2}{c}\right)^{\frac{1}{p}} \|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)) + L_1 \cdot \|u-u_0\|^{\ell_1} \cdot B],\end{aligned}$$

for u in a neighbourhood U_0 of u_0. By the lower Hőlder continuity of Γ, there exists a neighbourhood U_1 of u_0 such that $\Gamma(u_0) \subset \Gamma(u) + L_2\|u-u_0\|^{\ell_2}B$, for $u \in U_1$.
We claim that for $u \in U_0 \cap U_1$

$$M(u) \cap [(\Gamma(u_0) \setminus (M(u_0) + \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} \|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B) + L_1\|u-u_0\|^{\ell_1} \cdot B] = \emptyset. \quad (*)$$

Indeed, $y = \gamma + b_1$, where $\gamma \in \Gamma(u_0) \setminus (M(u_0) + \left(\frac{L_1+L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)$, $b_1 \in L_1\|u-u_0\|^{\ell_1} \cdot B$.
By Proposition 3.1.4, any $z \in \Gamma(u_0) \setminus [M(u_0) + \varepsilon \cdot B]$, $\varepsilon > 0$, can be represented in the form $z = \eta_z + k_z$, $\eta_z \in Min(\Gamma(u_0)|\mathcal{K})$,

$k_z + \delta(\varepsilon) \cdot B \subset \mathcal{K}$. Hence,

$$\gamma = \eta_\gamma + k_\gamma\,,\ \eta_\gamma \in M(u_0)\,,\ k_\gamma + \delta\Big(\Big(\frac{L_1+L_2}{c}\Big)^{\frac{1}{p}} \|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}}\Big)\cdot B \subset \mathcal{K}.$$

By the lower Hőlder continuity of Γ, $\eta_\gamma = \gamma_1 + b_2$, $\gamma_1 \in \Gamma(u)$, $b_2 \in L_2\|u-u_0\|^{\ell_2} \cdot B$, and consequently, since $\delta(\varepsilon) \geq c \cdot \varepsilon^p$,

$$\begin{aligned} y - \gamma_1 &= \gamma + b_1 - \eta_\gamma + b_2 = \eta_\gamma + k_\gamma + b_1 - \eta_\gamma + b_2 \\ &\subset k_\gamma + (L_1 + L_2)\|u-u_0\|^{\min\{\ell_1,\ell_2\}} \cdot B \\ &\subset k_\gamma + \delta\big(\big(\tfrac{L_1+L_2}{c}\big)^{\frac{1}{p}} \|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}}\big) \cdot B \subset \mathcal{K}. \end{aligned}$$

By this, (*) follows. Hence,

$$M(u) \subset M(u_0) + L_1 \cdot \|u-u_0\|^{\ell_1} \cdot B + \big(\tfrac{L_1+L_2}{c}\big)^{\frac{1}{p}} \cdot \|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B$$

$$\subset M(u_0) + \big(L_1 + \big(\tfrac{L_1+L_2}{c}\big)^{\frac{1}{p}}\big)\|u-u_0\|^{\min\{\ell_1, \frac{\min\{\ell_1,\ell_2\}}{p}\}} \cdot B\,,$$

for $u \in U_0 \cap U_1$, which completes the proof.

□

Corollary 3.1 *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $int\mathcal{K} \neq \emptyset$. Let $\Gamma : U \rightrightarrows Y$ be a Hőlder set-valued mapping of order ℓ at u_0 with constant L. Suppose that one of the conditions hold:*

(i) $Min(\Gamma(u_0)|\mathcal{K})$ is weakly compact,

(ii) $Min(\Gamma(u_0)|\mathcal{K})$ is bounded and weakly closed and $\mathcal{K}$ has a weakly compact base.

If the rate of containment of $\Gamma(u_0)$ satisfies the condition $\delta(\varepsilon) \geq c \cdot \varepsilon^p$, with $p > 1$, and $c > 0$, then minimal point multivalued mapping M is upper Hőlder at u_0 with constant $\Big(L + \big(\frac{2L}{c}\big)^{\frac{1}{p}}\Big)$ and order $\frac{\ell}{p}$.

Corollary 3.2 *Let $Y = (Y, \|\cdot\|)$, $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in a normed space Y, $int\mathcal{K} \neq \emptyset$. Let $\Gamma : U \rightrightarrows Y$ be a Lipschitz set-valued mapping at u_0 with constant L. Suppose that one of the conditions holds:*

(i) $Min(\Gamma(u_0)|\mathcal{K})$ *is weakly compact,*

(ii) $Min(\Gamma(u_0)|\mathcal{K})$ *is bounded and weakly closed, and* $\mathcal{K}$ *has a weakly compact base.*

If the rate of containment of $\Gamma(u_0)$ *satisfies the condition* $\delta(\varepsilon) \geq c \cdot \varepsilon$, *where* $c > 0$, *then* M *is upper Lipschitz at* u_0 *with constant* $\frac{(2+c)L}{c}$.

3.3 Rate of weak containment.

Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$ with the dual $\mathcal{K}^* \subset Y^*$. Let Θ^* be a base of $\mathcal{K}^*$.

Definition 3.3.1 *A function* $dcont_{\theta^*} : \mathcal{K} \to R_+$, *defined as*

$$dcont_{\theta^*}(k) = \inf\{\theta^*(k) \mid \theta^* \in \Theta^*\}$$

is called the Θ^***−dual cone containment function.**

If it is clear from the context which base Θ^* is used, we omit the index θ^* and we apply the simplified notation *dcont*. The terminology "primal cone containment function" and "dual cone containment function" is motivated by the fact that the fomulae defining these functions form a pair of dual optimization problems (see also Section 2.4, Example 3.3.1).

By formula(10) of Section 2.4,

$$\sup\{r \mid k - r \cdot y_0 \in \mathcal{K}\} = \inf\{\theta^*(k) \mid \theta^*(y_0) = 1, \quad \theta^* \in \mathcal{K}^*\}.$$

This proves that $cont(k) \leq dcont_{\Theta^*}(k)$ for each $k \in \mathcal{K}$.

In Theorem 1.1.2 and Proposition 1.1.4 we have shown that when int$\mathcal{K} \neq \emptyset$ (i.e., $cor\mathcal{K} \neq \emptyset$) cone $\mathcal{K}^*$ has a base. By similar arguments, $\mathcal{K}^*$ has a base whenever $\mathcal{K}^i = \{y \in Y \mid f(y) > 0 \quad \text{for all } f \in \mathcal{K} \setminus \{0\}\} \neq \emptyset$. Indeed, if $y_0 \in \mathcal{K}^i$, the set

$$\Theta^* = \{f \in \mathcal{K}^* \mid f(y_0) = 1\} \tag{17}$$

is a base of $\mathcal{K}^*$.

Proposition 3.3.1 $\mathcal{K}^*$ *has a base if and only if* $\mathcal{K}^i \neq \emptyset$.

Proof. We need only to show the "only if" part. Since any base Θ^* of $\mathcal{K}^*$ is convex, $0 \notin w - * - cl(\Theta^*$, which means that there exists an $y_0 \in Y$ with the property that $\theta^*(y_0) > \kappa > 0$ for each $\theta^* \in \Theta^*$, which entails that $y_0 \in \mathcal{K}^i$. □

Let $A \subset Y$ be a subset of Y. As defined in Section 2.3, Definition 2.3.1, the weak containment property, (WCP), holds for A if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $y \in A(\varepsilon)$ there exists $\eta_y \in Min(A|\mathcal{K})$ satisfying

$$\theta^*(y - \eta_y) > \delta$$

for each $\theta^* \in \Theta^*$.

Definition 3.3.2 (Weak containment rate) *The* **rate of weak containment of a set** A **with respect to** $\mathcal{K}$ *is the function* $d : R_+ \to R$ *defined as*

$$d(\varepsilon) = \inf_{y \in A(\varepsilon)} \nu(y)$$

where $\nu : Y \to R$ *is the* **dual rate of containment of** y **with respect to** A **and** $\mathcal{K}$

$$\nu(y) = \sup_{\eta \in Min(A|\mathcal{K}) \cap (y - \mathcal{K})} \inf_{\theta^* \in \Theta^*} \theta^*(y - \eta).$$

Denote $Min(A|\mathcal{K})_y = Min(A|\mathcal{K}) \cap (y - \mathcal{K})$, for any $y \in Min(A|\mathcal{K}) + \mathcal{K}$. If $y \in Min(A|\mathcal{K}) + \mathcal{K}$, then $\nu(y) \geq 0$.
By using the function $dcont_{\Theta^*}$ defined in Definition 3.3.1, the rate of weak containment can be rewritten as follows

$$d(\varepsilon) = \inf_{y \in A(\varepsilon)} \sup_{\eta \in Min(A|\mathcal{K})_y} dcont_{\Theta^*}(y - \eta).$$

Proposition 3.3.2 *Let* $(Y, \|\cdot\|)$ *be a normed space and let* $A \subset Y$ *be a subset of* Y. *Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in* Y *and let* $\mathcal{K}^* \subset Y^*$ *be its dual cone with a base* Θ^*.
For any $y \in Min(A|\mathcal{K}) + \mathcal{K}$, *if* $Min(A|\mathcal{K})_y$ *is weakly compact, then there exist* $\eta_y \in Min(A|\mathcal{K})$ *such that*

$$\nu(y) = \inf_{\theta^* \in \Theta^*} \theta^*(y - \eta_y).$$

Proof. Let $y \in Min(A|\mathcal{K}) + \mathcal{K}$. We have $\inf_{\theta^* \in \Theta^*} \theta^*(y - \eta) \leq \nu(y)$, for each $\eta \in Min(A|\mathcal{K})_y$, and for any $\rho > 0$, there exists $\eta_\rho \in Min(A|\mathcal{K})_y$ such that for any $\theta^* \in \Theta^*$

$$\theta^*(y - \eta_\rho) \geq \inf_{\theta^* \in \Theta^*} \theta^*(y - \eta_\rho) > \nu(y) - \rho .$$

Since $Min(A|\mathcal{K})_y$ is weakly compact, the net $\{\eta_\rho\}$ contains a weakly convergent subnet and without loss of generality we can assume that the net $\{\eta_\rho\}$ converges weakly to $\eta_y \in Min(A|\mathcal{K})_y$. Since $\mathcal{K}$ is weakly closed, the net $\{k_\rho = y - \eta_\rho\}$ tends to some $k_y \in \mathcal{K}$, and $y = \eta_y + k_y$. Thus,

$$\inf_{\theta^* \in \Theta^*} \theta^*(y - \eta_y) \geq \nu(y) ,$$

which completes the proof. □

Proposition 3.3.3 *Let $(Y, \|\cdot\|)$ be a normed space and let $A \subset Y$ be a subset of Y. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y and let $\mathcal{K}^*$ be its dual with a base Θ^*. The following conditions are equivalent:*

(i) (WCP) holds for A,

(ii) $d(\varepsilon) > 0$ for each $\varepsilon > 0$.

Proof. $(i) \to (ii)$. Take any $\varepsilon > 0$ and $y \in A(\varepsilon)$. By (WCP), there exist $\delta > 0$ and $\eta_y \in Min(A|\mathcal{K})$ such that

$$\inf_{\theta^* \in \Theta^*} \theta^*(y - \eta_y) \geq \delta .$$

Hence

$$\nu(y) = \sup_{\eta \in Min(A|\mathcal{K})_y} \inf_{\theta^* \in \Theta^*} \theta^*(y - \eta) \geq \delta ,$$

and $d(\varepsilon) = \inf_{y \in A(\varepsilon)} \nu(y) \geq \delta > 0$.

$(ii) \to (i)$. Let $d(\varepsilon) = \alpha > 0$. For each $y \in A(\varepsilon)$

$$\sup_{\eta \in Min(A|\mathcal{K})_y} \inf_{\theta^* \in \Theta^*} \theta^*(y - \eta) \geq \alpha ,$$

and consequently,

$$\inf_{\theta^* \in \Theta^*} \theta^*(y - \eta_y) > \alpha/2 ,$$

for some $\eta_y \in Min(A|\mathcal{K})_y$, ie., (WCP) holds.

□

Proposition 3.3.4 *Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in a topological vector space Y with $\mathcal{K}^i \neq \emptyset$. If Θ_1^* and Θ_2^* are any two bases of the form (17), with $y_1, y_2 \in \mathcal{K}^i$ such that $y_2 \in (ry_1 + \mathcal{K})$, then there exists a positive real number β with*

$$dcont_{\Theta_1^*}(k) \geq \beta \cdot dcont_{\Theta_2^*}(k).$$

Proof. Let Θ_1^*, Θ_2^* be any two bases of the form (17), ie., for $y_1, y_2 \in \mathcal{K}^i$ we have

$$\begin{aligned} \Theta_1^* &= \{\theta_1^* \in \mathcal{K}^* \mid \theta_1^*(y_1) = 1\} \\ \Theta_2^* &= \{\theta_2^* \in \mathcal{K}^* \mid \theta_2^*(y_2) = 1\}. \end{aligned}$$

For any $k \in \mathcal{K}$, and $\theta_1^* \in \Theta_1^*$, there exists $\bar{\theta}_2^* \in \Theta_2^*$ such that

$$\theta_1^*(k) = \theta_1^*(y_2)\bar{\theta}_2^*(k),$$

where $\theta_1^*(y_2) > 0$. Hence,

$$\theta_1^*(k) \geq \theta_1^*(y_2) \inf_{\bar{\theta}_2^* \in \Theta_2^*} \bar{\theta}_2^*(k) \geq \theta_1^*(y_2) \inf_{\theta_2^* \in \Theta_2^*} \theta_2^*(k),$$

and

$$\inf_{\theta_1^* \in \Theta_1^*} \theta_1^*(k) \geq \inf_{\theta_1^* \in \Theta_1^*} \theta_1^*(y_2) \inf_{\theta_2^* \in \Theta_2^*} \theta_2^*(k), \tag{18}$$

Since $y_2 \in r \cdot y_1 + \mathcal{K}$, by (10), $\beta = \inf_{\theta_1^* \in \Theta_1^*} \theta_1^*(y_2) > 0$, and by (18),

$$dcont_{\Theta_1^*} \geq \beta \cdot dcont_{\Theta_2^*}.$$

□

Example 3.3.1 Cone containment functions in finite-dimensional case

In Definitions 3.1.1, 3.3.1 we have defined two cone containment functions for a closed convex cone $\mathcal{K}$ in Y having the dual $\mathcal{K}^* \subset Y^*$ with a base Θ^*. Namely,

the **primal cone containment function** $cont : \mathcal{K} \to R_+$,

$$cont(k) = \sup\{r \mid k + rB \subset \mathcal{K}\},$$

and
the Θ^***–dual cone containment function,** $dcont_{\Theta^*} : \mathcal{K} \to R_+$,

$$dcont_{\Theta^*}(k) = \inf\{\theta^*(k) \mid \theta^* \in \Theta^* \ \}.$$

1. $Y = (R^n, \|\cdot\|_\infty)$, $\mathcal{K} = R^n_+$, $B = \{y \in R^n \mid \|y\|_\infty \leq 1\}$.
The function cont has the form

$$\begin{aligned} cont(k) = \ & \max\ r \\ & \textit{subject to} \\ & k_1 - r \geq 0 \\ & k_2 - r \geq 0 \\ & \dots \\ & k_n - r \geq 0 \end{aligned}$$

This is a linear programming problem. On the other hand, the function dcont has the form

$$\begin{aligned} dcont(k) = \ & \min\ c_1k_1 + \dots + c_nk_n \\ & \textit{subject to} \\ & c_1 + \dots + c_n = 1 \\ & c_1 \geq 0 \\ & \dots \\ & c_n \geq 0 \end{aligned}$$

3.4 Upper Hölder continuity of minimal points for cones with possibly empty interior.

Now we prove the main theorem of this section.

Theorem 3.4.1 *Let* $Y = (Y, \|\cdot\|)$ *and* $U = (U, \|\cdot\|)$ *be normed spaces. Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in* Y*, and let* $\mathcal{K}^*$ *be its dual with an equicontinuous base* Θ^**. Let* $\Gamma : U \rightrightarrows Y$*, be a set-valued mapping which is upper Hőlder of order* ℓ_1 *with constant* L_1 *and lower Hőlder of order* ℓ_2 *with constant* L_2 *at* u_0.

If

(i) the dual rate of containment d *of* $\Gamma(u_0)$*, satisfies the condition* $d(\varepsilon) \geq c \cdot \varepsilon^p$*, with* $c > 0$,

(ii) $Min(\Gamma(u_0)|\mathcal{K})$ is weakly compact,

then

$$M(u) \subset M(u_0) + (L_1 + \left(2\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}})\|u - u_0\|^{\min\{\ell_1, \frac{\min\{\ell_1,\ell_2\}}{p}\}} \cdot B .$$

for all u in some neighbourhood of u_0.

Proof. In this proof we follow the same reasoning as in the proof of Theorem 3.2.1. Using the same notation we only need to show that under our assumptions, for $u \in U_0 \cap U_1$

$$M(u) \cap [(\Gamma(u_0) \setminus (M(u_0) + \left(2\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)) + L_1\|u-u_0\|^{\ell_1} \cdot B] = \emptyset . \quad (*)$$

To this aim take any

$$y \in \Gamma(u) \cap [(\Gamma(u_0) \setminus (M(u_0) + \left(2\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)) + L_1\|u-u_0\|^{\ell_1} \cdot B] ,$$

for $u \in U_0 \cap U_1$. We have $y = \gamma + b_1$, where $\gamma \in \Gamma(u_0) \setminus (M(u_0) + \left(2\frac{L_1+L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)$, $b_1 \in L_1\|u - u_0\|^{\ell_1} \cdot B$.

Since Θ^* is equicontinuous we can assume that $\theta^*(B) \leq 1$, for each $\theta^* \in \Theta^*$. Hence, for each $b \in L_1\|u - u_0\|^{\ell_1} \cdot B$ we have

$$-L_1\|u - u_0\|^{\ell_1} \leq \theta^*(b) \leq L_1\|u - u_0\|^{\ell_1} .$$

By Proposition 3.3.2, there exists $\eta_\gamma \in Min(\Gamma(u_0)|\mathcal{K})$ satisfying

$$\theta^*(\gamma - \eta_\gamma) \geq \nu(\gamma) = \inf_{\theta^* \in \Theta^*} \theta^*(\gamma - \eta_\gamma) \geq d(\varepsilon) \geq c \cdot \varepsilon^p$$

for each $\theta^* \in \Theta^*$. By the lower Hőlder continuity of Γ, $\eta_\gamma = \gamma_1 + b_2$, $\gamma_1 \in \Gamma(u)$, $b_2 \in L_2\|u - u_0\|^{\ell_2} \cdot B$. Finally

$$\begin{aligned}
\theta^*(y-\gamma_1) &= \theta^*(y - \gamma) + \theta^*(\gamma - \eta_\gamma) + \theta^*(\eta_\gamma - \gamma_1) \\
&\geq -L_1\|u - u_0\|^{\ell_1} - L_2\|u - u_0\|^{\ell_2} + d(\left(2\tfrac{L_1+L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}}) \\
&\geq -(L_1 + L_2)\|u - u_0\|^{\min\{\ell_1,\ell_2\}} + 2(L_1 + L_2)\|u - u_0\|^{\min\{\ell_1,\ell_2\}} > 0 .
\end{aligned}$$

Consequently, $f(y - \gamma_1) \geq 0$ for any $f \in \mathcal{K}^*$. By Lemma 8.6 of [54], $y - \gamma_1 \in \mathcal{K}$, which proves $(*)$ and completes the proof.

□

3.5 Rate of containment for convex sets

By definition, for any subset $A \subset Y$,

$$\delta(\varepsilon) = \inf_{y \in A(\varepsilon)} \mu(y),$$

where $A(\varepsilon) = \{y \in A \mid d(y, Min(A|\mathcal{K})) \geq \varepsilon\}$.

Proposition 3.5.1 *Let $A \subset Y$ be a convex subset of Y. Under one of the following conditions:*

(i) *$Min(A|\mathcal{K})$ is weakly compact,*

(ii) *$Min(A|\mathcal{K})$ is weakly closed and bounded, and $\mathcal{K}$ has a weakly compact base,*

we have

$$\delta(\varepsilon) = \inf_{y \in E(\varepsilon)} \mu(y), \tag{19}$$

where $E(\varepsilon) = \{y \in A \mid d(y, Min(A|\mathcal{K})) = \varepsilon\}$.

Proof. We have

$$\delta(\varepsilon) \leq \inf_{y \in E(\varepsilon)} \mu(y).$$

If it were $\delta(\varepsilon) < \inf_{y \in E(\varepsilon)} \mu(y) = e_0$, then $\mu(\bar{y}) < e_0$, for some $\bar{y} \in A$, $d(\bar{y}, Min(A|\mathcal{K})) > \varepsilon$. In view of Proposition 3.1.1, $\bar{y} = \eta_y + k_y$, $k_y + \mu(\bar{y}) \cdot B \subset \mathcal{K}$.
On the segment $\langle \eta_y, y \rangle \subset A$ one could find a point z belonging to $E(\varepsilon)$, $z = \lambda \eta_y + (1-\lambda) y$. Hence, $z = \eta_y + (1-\lambda) k_y = \eta_y + k_z$, $k_z = (1-\lambda) k_y$, $k_z + (1-\lambda)\mu(\bar{y}) B \subset \mathcal{K}$, and $\mu(\bar{y}) \geq (1-\lambda)\mu(\bar{y}) = \mu(z) \geq e_0$, contrary to the choice of $\bar{y}$.

□

A convex subset $A \subset Y$ of a space Y is **strictly convex** if each boundary point is extremal, ie., cannot be represented as a convex combination of any two other different points of A.

Proposition 3.5.2 *Let A be a strictly convex subset of Y, and $y \in A \cap (Min(A|\mathcal{K}) + \mathcal{K})$. If $\mu(y)$ is attained, there is exactly one representation $y = \eta + k$, $\eta \in Min(A|\mathcal{K})$, $k + B(0, \mu(y)) \in \mathcal{K}$.*

Proof. Suppose that there are two different representations $y = \eta_1 + k_1$, $y = \eta_2 + k_2$, $\eta_1, \eta_2 \in Min(A|\mathcal{K})$, $k_1 + B(0, \mu(y)) \in \mathcal{K}$, $k_2 + B(0, \mu(y)) \in \mathcal{K}$. Since $(y - \mathcal{K}) \cap A$ is convex, for any $0 \leq \lambda \leq 1$, $\lambda\eta_1 + (1-\lambda)\eta_2 \in (y - \mathcal{K}) \cap A$. By the strict convexity of A, there exists $k_3 \neq 0$, $k_3 + B(0, \alpha) \subset \mathcal{K}$, $\alpha > 0$, such that $1/2\eta_1 + 1/2\eta_2 = \eta_3 + k_3$, $\eta_3 \in Min(A|\mathcal{K})$. Indeed, if there would be no such k_3, it would be $1/2\eta_1 + 1/2\eta_2 \in Min(A|\mathcal{K})$, hence $1/2\eta_1 + 1/2\eta_2$ would be a boundary point which is impossible. Consequently, we have $y - \eta_3 = 1/2k_1 + 1/2k_2 + k_3$, and

$$1/2k_1 + 1/2k_2 + k_3 + B(0, \mu(y) + \alpha) \subset \mathcal{K},$$

contrary to the definition of $\mu(y)$.

□

Proposition 3.5.3 *Let A be a convex subset of $(Y, \|\cdot\|)$ and let (DP) hold for A. If the norm $\|\cdot\|$ is $\mathcal{K}-$monotonic, then the distance function $d(x, Min(A|\mathcal{K}))$ is convex.*

Proof. Let us take any x_1, x_2 such that $\|x_1 - \eta_1\| = d(x_1, Min(A|\mathcal{K})) = \varepsilon_1$, $\|x_2 - \eta_2\| = d(x_2, Min(A|\mathcal{K})) = \varepsilon_2$, and $0 \leq \lambda \leq 1$. Let $\eta(\lambda) = (1-\lambda)\eta_1 + \lambda \cdot \eta_2$, $x(\lambda) = (1-\lambda)x_1 + \lambda \cdot x_2$. We have

$$\|x(\lambda) - \eta(\lambda)\| \leq (1-\lambda)\varepsilon_1 + \lambda \cdot \varepsilon_2 .$$

By the convexity of A, $\eta(\lambda) \in A$, and by (DP), $\eta(\lambda) = \eta_3 + k$, where $\eta_3 \in Min(A|\mathcal{K})$, $k \in \mathcal{K}$. Now, by the $\mathcal{K}-$monotonicity of the norm

$$\|x(\lambda) - \eta_3\| \leq \|x(\lambda) - \eta(\lambda)\| ,$$

which proves the convexity of $d(x, Min(A|\mathcal{K}))$.

□

Proposition 3.5.4 *Let $A \subset Y$ be a subset of Y. Under one of the following conditions*

(i) *$Min(A|\mathcal{K})$ is weakly compact,*

(ii) *Min(A|$\mathcal{K}$) is weakly closed and bounded, and $\mathcal{K}$ has a weakly compact base,*

for any $y \in Min(A|\mathcal{K}) + \mathcal{K}$, and $0 \le \beta \le 1$, we have

$$\mu(y(\beta)) = \beta\mu(y),$$

where $y = \eta_y + k_y$, $\eta_y \in Min(A|\mathcal{K})$, $k_y + B(0, \mu(y)) \in \mathcal{K}$, and $y(\beta) = \eta_y + \beta \cdot k_y$.

Proof. Let $y \in Min(A|\mathcal{K}) + \mathcal{K}$. By Proposition 3.1.1, a representation $y = \eta_y + k_y$, $\eta_y \in Min(A|\mathcal{K})$, $k_y + B(0, \mu(y)) \subset \mathcal{K}$ exists. Since $\beta k_y + B(0, \beta \cdot \mu(y)) \subset \mathcal{K}$, we have $\mu(y(\beta)) \ge \beta\mu(y)$. If it were $\mu(y(\beta)) > \beta\mu(y)$, then it would exist a representation $y(\beta) = \eta_1 + k_1$, such that $k_1 + B(0, \alpha) \subset \mathcal{K}$, and $\alpha > \beta\mu(y)$. Then, since $0 \le \beta \le 1$,

$$k_2 = y - \eta_1 = y - y(\beta) + y(\beta) - \eta_1 = (1 - \beta)k_y + k_1 \in \mathcal{K},$$

and

$$(1 - \beta)k_y + k_1 + B(0, (1 - \beta)\mu(y) + \alpha) \subset \mathcal{K},$$

$(1 - \beta)\mu(y) + \alpha > \mu(y)$, which would contradict the definition of $\mu(y)$.

□

Proposition 3.5.5 *Let A be a convex subset of Y. Suppose that one of the following conditions hold:*

(i) *Min(A|$\mathcal{K}$) is weakly compact,*

(ii) *Min(A|$\mathcal{K}$) is bounded and weakly closed and $\mathcal{K}$ has a weakly compact base.*

The function μ is concave on $Min(A|\mathcal{K}) + \mathcal{K}$.

Proof. Let $y_1, y_2 \in Min(A|\mathcal{K}) + \mathcal{K}$, and $0 \le \lambda \le 1$. By Proposition 3.1.1, there exist $\eta_1, \eta_2 \in Min(A|\mathcal{K})$ such that $y_1 = \eta_1 + k_1$, $y_2 = \eta_2 + k_2$, $k_1 + \mu(x_1)B \subset \mathcal{K}$, $k_2 + \mu(x_2)B \subset \mathcal{K}$. Since A is convex, $Min(A|\mathcal{K}) + \mathcal{K}$ is convex, and

$$y(\lambda) = \lambda y_1 + (1-\lambda)y_2 = \lambda\eta_1 + (1-\lambda)\eta_2 + \lambda k_1 + (1-\lambda)k_2 \in Min(A|\mathcal{K}) + \mathcal{K},$$

where $\lambda k_1 + (1-\lambda)k_2 + B(0, \lambda\mu(y_1) + (1-\lambda)\mu(y_2)) \subset \mathcal{K}$.
Hence, $\eta(\lambda) = \lambda\eta_1 + (1-\lambda)\eta_2 \in Min(A|\mathcal{K}) + \mathcal{K}$, $\eta(\lambda) = \eta_3^\lambda + k_3^\lambda$, where $\eta_3^\lambda \in Min(A|\mathcal{K})$, $k_3^\lambda + \mu(\eta(\lambda))B \subset \mathcal{K}$.
Finally,

$$y(\lambda) = \eta_3^\lambda + k_3^\lambda + \lambda k_1 + (1-\lambda)k_2\,,$$

and $k_3^\lambda + \lambda k_1 + (1-\lambda)k_2 + B(0, \mu(\eta(\lambda)) + \lambda\mu(y_1) + (1-\lambda)\mu(y_2)) \subset \mathcal{K}$, ie., $\mu(y(\lambda)) \geq \lambda\mu(y_1) + (1-\lambda)\mu(y_2)$.

□

Corollary 3.3 *Under assumptions of Proposition 3.5.5 the function μ is locally Lipschitz and weakly upper semicontinuous on $Min(A|\mathcal{K}) + int\mathcal{K}$.*

Proof. See Theorem 10 of [36].

□

Proposition 3.5.6 *Under assumptions of Proposition 3.5.5 the rate of containment δ is continuous.*

Proof. In view of standard theorems on continuity of the marginal function (see [24],[8]) it is enough to show that the set-valued mapping $A : R_+ \rightrightarrows Y$,

$$A(\varepsilon) = \{y \in A \mid d(y, Min(A|\mathcal{K})) \geq \varepsilon\}$$

is lower and upper Hausdorff semicontinuous.
For $\varepsilon > \varepsilon_0$ we have $A(\varepsilon) \subset A(\varepsilon_0)$. To prove the upper Hausdorff semicontinuity we need to show that for any $\alpha > 0$ there exists $\delta > 0$ such that for $\delta < \varepsilon < \varepsilon_0$

$$A(\varepsilon) \subset A(\varepsilon_0) + \alpha B\,.$$

□

Now we are in a position to prove the convexity result for the function δ.

Theorem 3.5.1 *Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$, $int\mathcal{K} \neq \emptyset$. Let $A \subset Y$ be a convex subset of Y and one of the following conditions hold:*

(i) $Min(A|\mathcal{K})$ is weakly compact,

(ii) $Min(A|\mathcal{K})$ is bounded and weakly closed and $\mathcal{K}$ has a weakly compact base.

If (DP) holds for A, then δ is quasiconvex.

Proof. Since A is convex, by Proposition 3.5.1,

$$\delta(\varepsilon) = \inf_{y \in A(\varepsilon)} \mu(y) = \inf_{y \in E(\varepsilon)} \mu(y).$$

Let $\varepsilon_1, \ \varepsilon_2 \in \mathrm{dom}\delta$, $\varepsilon_2 < \varepsilon_1$. For any $\alpha > 0$ we choose $y_1^\alpha \in E(\varepsilon_1)$ such that

$$\mu(y_1^\alpha) < \delta(\varepsilon_1) + \alpha. \tag{20}$$

In view of Proposition 3.1.1, by (i), or (ii), there exists a representation realizing the rate of containment of y_1^α,

$$y_1^\alpha = \eta^\alpha + k^\alpha \in A,$$

where $\eta^\alpha \in Min(A|\mathcal{K})$, $k^\alpha + B(0, \mu(y_1^\alpha)) \subset \mathcal{K}$, $\|k^\alpha\| \geq \varepsilon_1$.

Let $0 \leq \lambda \leq 1$. Since the distance function $d(\cdot, Min(A|\mathcal{K}))$ is continuous, there exists $0 \leq \kappa(\lambda) \leq 1$ such that $d(y^\alpha(\lambda), Min(A|\mathcal{K})) = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$, where $y^\alpha(\lambda) = \eta^\alpha + \kappa(\lambda)k^\alpha \in A$. By Proposition 3.5.4, $\mu(y^\alpha(\lambda)) = \kappa(\lambda)\mu(y_1^\alpha)$.

Let $\varepsilon(\lambda) = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$. We have

$$\begin{aligned} \delta(\varepsilon(\lambda)) &= \inf\nolimits_{y \in E(\varepsilon(\lambda))} \mu(y) \\ &\leq \mu(y^\alpha(\lambda)) = \kappa(\lambda)\mu(y_1^\alpha) \\ &\leq \delta(\varepsilon_1) + \alpha. \end{aligned}$$

Since $\alpha > 0$ is arbitrary, by noting that $\delta(\varepsilon_1) \geq \delta(\varepsilon_2)$ we get

$$\delta(\varepsilon(\lambda)) \leq \max\{\delta(\varepsilon_1), \ \delta(\varepsilon_2)\}.$$

□

A subset A of a linear space Y is **starshaped** at $a \in A$ if $(1-\alpha)a+\alpha \cdot y \in A$ whenever $y \in A$ and $\alpha \in]0,1]$. A subset A of a linear space Y is **starshaped** if it starshaped at 0, ie., if $\alpha \cdot y \in A$ whenever $y \in A$ and $\alpha \in]0,1]$. A function $f : Y \to R$ is called **starshaped** if its epigraph

$$Epi(f) = \{(y,r) \in Y \times R \mid f(y) \geq r\}$$

is starshaped, ie., $f(\alpha v) \leq \alpha f(v)$ for $\alpha \in [0,1]$, and $v \in \mathrm{dom} f$.

We have the following proposition.

Proposition 3.5.7 *Let $\mathcal{K} \subset Y$ be a closed convex cone in a normed space Y, int$\mathcal{K} \neq \emptyset$. If $A \subset Y$ is a starshaped subset of Y and (DP) holds for A, then, under one of the following conditions:*

(i) *$Min(A|\mathcal{K})$ is weakly compact,*

(ii) *$Min(A|\mathcal{K})$ is bounded and weakly closed and $\mathcal{K}$ has a weakly compact base,*

for any $y \in A$ and $\beta \in]0,1]$, we have

$$\mu(\beta \cdot y) \geq \beta\mu(y),$$

ie., μ is starshaped for $y \in A$.

Proof. Take any $y \in A$ and $\beta \in]0,1]$. By Proposition 3.5.4, y can be represented in the form

$$y = \eta_y + k_y\,, \qquad (*)$$

where $\eta_y \in Min(A|\mathcal{K})$, $k_y + \mu(y)B \subset \mathcal{K}$. Since A is starshaped, $\beta y \in A$, and by $(*)$, $\beta y = \beta\eta_y + \beta k_y$.
Since $\eta_y \in A$, we have $\beta\eta_y \in A$ and, by (DP), there exists $\eta_1 \in Min(A|\mathcal{K})$ and $k_1 \in \mathcal{K}$ such that $\beta\eta_y = \eta_1 + k_1$. Finally,

$$\beta y = \eta_1 + k_1 + \beta k_y\,,$$

and $\mu(\beta y) \geq \beta\mu(y)$.

□

3.6 Hőlder continuity of minimal points for cones with nonempty interior

We say that a multivalued mapping $\Gamma : U \rightrightarrows Y$, is **locally Lipschitz** around u_0 if there exists a constant $L > 0$ such that

$$\Gamma(u_1) \subset \Gamma(u_2) + L\|u_1 - u_2\| \cdot B$$

for u_1, u_2 from a neighbourhood U_0 of u_0.

We say that a multivalued mapping $\Gamma : U \rightrightarrows Y$, is **locally Hőlder** around u_0 of order ℓ if there exists a constant $L > 0$ such that

$$\Gamma(u_1) \subset \Gamma(u_2) + L\|u_1 - u_2\|^{\ell} \cdot B$$

for u_1, u_2 from a neighbourhood U_0 of u_0.

Let $\Sigma : U \rightrightarrows Y$, be a set-valued mapping defined on a normed space U. By $\delta(\cdot, u)$ we denote the rate of containment of the set $\Sigma(u)$ with respect to $\mathcal{K}$.

Definition 3.6.1 *We say that (CP)* **holds uniformly** *for Σ on a subset $A \subset U$ if, for any $\varepsilon > 0$,*

$$\delta(\varepsilon) = \inf_{u \in A} \delta(\varepsilon, u) > 0 .$$

Definition 3.6.2 *We say that (CP)* **holds uniformly** *for Σ around u_0 if there exists a neighbourhood U_0 of u_0 such that for any $\varepsilon > 0$,*

$$\delta(\varepsilon) = \inf_{u \in U_0} \delta(\varepsilon, u) > 0 .$$

In both cases $\delta(\varepsilon)$ is called **the uniform rate of containment.**

Proposition 3.6.1 *Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$, $int\mathcal{K} \neq \emptyset$. Let $\Sigma : U \rightrightarrows Y$, be a set-valued mapping defined on a normed space U. Suppose that one of the conditions hold:*

(i) *$Min(\Sigma(u)|\mathcal{K})$ are weakly compact for all u from some neighbourhood U_1 of u_0,*

(ii) *$Min(\Sigma(u)|\mathcal{K})$ are bounded and weakly closed for all u from some neighbourhood U_1 of u_0, and $\mathcal{K}$ has a weakly compact base.*

If (CP) holds uniformly for Σ *around* u_0, *for any* $\varepsilon > 0$, *each* $y \in \Sigma(u) \setminus [Min(\Sigma(u)|\mathcal{K}) + \varepsilon \cdot B]$ *can be represented in the form*

$$y = \eta_y + k_y\,, \quad \eta_y \in Min(\Sigma(u)|\mathcal{K})\,, \quad k_y + \delta(\varepsilon) \cdot B \subset \mathcal{K}\,,$$

for all u *from some neighbourhood of* u_0.

Proof. Observe that since $\delta(\varepsilon, u) > 0$ for all u in a neighbourhood U_0 of u_0, by Proposition 3.1.3, (CP) holds for $\Sigma(u)$, $u \in U_0$.
Let $\varepsilon > 0$ and $u \in U_0 \cap U_1$. Take any $y \in \Sigma(u) \setminus [Min(\Sigma(u)|\mathcal{K}) + \varepsilon \cdot B]$.
By Proposition 3.1.4,

$$y = \eta_y + k_y\,, \quad \eta_y \in Min(\Sigma(u)|\mathcal{K})\,, \quad k_y + \mu(y, \Sigma(u)) \cdot B \subset \mathcal{K}\,,$$

and thus

$$k_y + \delta(\varepsilon) \cdot B \subset k_y + \delta(\varepsilon, u) \subset k_y + \mu(y, \Sigma(u)) \cdot B \subset \mathcal{K}\,.$$

□

m

Theorem 3.6.1 *Let* $\mathcal{K} \subset Y$ *be a closed convex pointed based cone in a normed space* $(Y, \|\cdot\|)$, $int\mathcal{K} \neq \emptyset$. *Let* $\Gamma : U \to Y$ *be a set-valued mapping defined on a normed space* U. *Suppose that one of the conditions holds:*

(i) $Min(\Gamma(u)|\mathcal{K})$ *are weakly compact for all* u *from some neighbourhood* U_1 *of* u_0,

(ii) $Min(\Gamma(u)|\mathcal{K})$ *are bounded and weakly closed for all* u *from some neighbourhood* U_1 *of* u_0, *and* $\mathcal{K}$ *has a weakly compact base.*

If

(1) Γ *is locally Hölder at* u_0, *of order* ℓ *with constant* L *ie.,*

$$\Gamma(u_1) \subset \Gamma(u_2) + L\|u_1 - u_2\|^\ell \cdot B\,,$$

for any u_1, u_2 *from some neighbourhood* U_0 *of* u_0,

(2) *the uniform rate of containment* δ *satisfies the condition* $\delta(\varepsilon) \geq c \cdot \varepsilon^p$, *where* $c > 0$,

then

$$M(u_1) \subset M(u_2) + (L + \frac{2}{c}L^{\frac{1}{p}})\|u_1 - u_2\|^{\frac{\ell}{p}} \cdot B\,,$$

for all u_1, u_2 *in some neighbourhood of* u_0.

Proof. Let

$$\delta(\varepsilon) = \inf_{u \in U_1} \delta(\varepsilon, u)\,,$$

and $u_1,\ u_2 \in U_0 \cap U_1$. By (1),

$$\begin{aligned}\Gamma(u_1) &\subset \Gamma(u_2) + L\|u_1 - u_2\|^{\ell} \cdot B \\ &\subset [M(u_2) + L\|u_1 - u_2\|^{\ell} \cdot B + \tfrac{\sqrt[p]{2}}{c}L^{\frac{1}{p}}\|u_1 - u_2\|^{\frac{\ell}{p}} \cdot B] \cup \\ &\cup [\Gamma(u_2) \setminus (M(u_2) + \tfrac{\sqrt[p]{2}}{c}L^{\frac{1}{p}}\|u_1 - u_2\|^{\frac{\ell}{p}} \cdot B)] + L \cdot \|u_1 - u_2\|^{\ell} \cdot B\,.\end{aligned}$$

Let us take any

$$y \in \Gamma(u_1) \cap [(\Gamma(u_2) \setminus [M(u_2) + \frac{\sqrt[p]{2}}{c}L^{\frac{1}{p}}\|u_1 - u_2\|^{\frac{\ell}{p}} \cdot B]) + L\|u_1 - u_2\|^{\ell} \cdot B]\,.$$

We have $y = \gamma + b_1$, where $\gamma \in \Gamma(u_2) \setminus (M(u_2) + \frac{\sqrt[p]{2}}{c}L^{\frac{1}{p}}\|u_1 - u_2\|^{\frac{\ell}{p}} \cdot B)$, and $b_1 \in L\|u_1 - u_2\|^{\ell} \cdot B$. Furthermore, by Proposition 3.6.1,

$$\gamma = \eta_\gamma + k_\gamma\,, \text{ where } \eta_\gamma \in M(u_2)\,,\ k_\gamma + \delta(\frac{\sqrt[p]{2}}{c}L^{\frac{1}{p}}\|u_1 - u_2\|^{\frac{\ell}{p}}) \cdot B \subset \mathcal{K}\,.$$

Again, by (1),

$$\eta_\gamma = \gamma_1 + b_2\,, \text{ where } \gamma_1 \in \Gamma(u_1)\,,\ b_2 \in L\|u_1 - u_2\|^{\ell} \cdot B\,,$$

and consequently, since $\delta(\varepsilon) \geq c \cdot \varepsilon^p$,

$$\begin{aligned}y - \gamma_1 &= \gamma + b_1 - \eta_\gamma + b_2 = \eta_\gamma + k_\gamma + b_1 - \eta_\gamma + b_2 \\ &\subset k_\gamma + 2L\|u_1 - u_2\|^{\ell} \cdot B \subset k_\gamma + \delta(\tfrac{\sqrt[p]{2}}{c}L^{\frac{1}{p}}\|u_1 - u_2\|^{\frac{\ell}{p}}) \cdot B \\ &\subset K\,,\end{aligned} \tag{21}$$

By (21), $y \notin M(u_1)$ for any $y \in \Gamma(u_1) \cap [\Gamma(u_2) \setminus (M(u_2) + \frac{2}{c}L^{\frac{1}{p}}\|u_1 - u_2\|^{\frac{\ell}{p}} \cdot B) + L\|u_1 - u_2\|^{\ell} \cdot B]$. Hence,

$$M(u_1) \subset M(u_2) + (L + \frac{2}{c}L^{\frac{1}{p}})\|u_1 - u_2\|^{\frac{\ell}{p}} \cdot B\,,$$

for $u \in U_0 \cap U_1$.

□

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