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Stability Analysis for Parametric Vector Optimization Problems

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Preface

We study stability of minimal points and solutions to parametric (or perturbed) vector optimization problems in the framework of real topological vector spaces and, if necessary, normed spaces. Because of particular importance of finite-dimensional problems, called multicriteria optimization problems, which model various real-life phenomena, a special attention is paid to the finite-dimensional case. Since one can hardly expect the sets of minimal points and solutions to be singletons, set-valued mappings are natural tools for our studies.

Vector optimization problems can be stated as follows. Let X be a topological space and let Y be a topological vector space ordered by a closed convex pointed cone $\mathcal{K} \subset Y$. Vector optimization problem

$$\begin{array}{ll} \mathcal{K} - \min \; f_0(x) & \\ \text{subject to } x \in A_0 \,, & \qquad (P_0) \end{array}$$

where $f : X \to Y$ is a mapping, and $A_0 \subset X$ is a subset of X, relies on finding the set $\mathrm{Min}(f_0, A_0, \mathcal{K}) = \{y \in f_0(A_0) \mid f_0(A_0) \cap (y - \mathcal{K}) = \{y\}\}$ called the **Pareto** or **minimal point** set of (P_0), and the **solution set** $S(f_0, A_0, \mathcal{K}) = \{x \in A_0 \mid f_0(x) \in \mathrm{Min}(f_0, A_0, \mathcal{K})\}$. We often refer to problem (P_0) as the **original problem** or **unperturbed one**. The space X is the **argument** space and Y is the **outcome** space.

Let U be a topological space. We embed the problem (P_0) into a family (P_u) of vector optimization problems parametrised by a parameter $u \in U$,

$$\begin{array}{ll} \mathcal{K} - \min \; f(u, x) & \\ \text{subject to } x \in A(u) \,, & \qquad (P_u) \end{array}$$

where $f : U \times X \to Y$ is the parametrised objective function and $A : U \rightrightarrows Y$, is the feasible set multifunction, (P_0) corresponds to a parameter value u_0. The performance multifunction $\mathcal{M} : U \rightrightarrows Y$, is defined as $\mathcal{M}(u) = \text{Min}(f(u,\cdot), A(u), \mathcal{K})$, and the solution multifunction $\mathcal{S} : U \rightrightarrows Y$, is given as $\mathcal{S}(u) = S(f(u,\cdot), A(u), \mathcal{K})$, and $f : U \times X \to Y$, $A(u) \subset X$.

Our aim is to study continuity properties of $\mathcal{M}$ and $\mathcal{S}$ as functions of the parameter u. Continuous behaviour of solutions as functions of parameters is of crucial importance in many aspects of the theory of vector optimization as well as in applications(correct formulation of the model and/or approximation) and numerical solution of the problem in question.
We investigate continuity in the sense of Hausdorff and Hőlder of the multivalued mappings of minimal points $\mathcal{M}(u)$ and solutions $\mathcal{S}(u)$ as functions of the parameter u under possibly weak assumptions. We attempt to avoid as much as possible compactness assumptions which are frequently over-used (see eg [83]).
It is a specific feature of vector optimization that the outcome space is equipped with a partial order generated by a cone the properties of which are important for stability analysis. In many spaces cones of nonnegative elements have empty interiors and because of this we derive stability results for cones with possibly empty interior. This kind of results are specific for vector optimization and do not have their counterpart in scalar optimization.
We introduce two new concepts: the notion of containment(with some variants for cones with empty interiors),[16], and the notion of strict minimality, [12].

The containment property (CP), defined in topological vector spaces, is introduced to study upper semicontinuities (in the sense of Hausdorff) of minimal points, [11, 16]. It is a variant of the domination property (DP), which appears frequently in the context of stability of solutions to parametric vector optimization problems. Although it is not a commonly adopted view point, the domination property may be accepted as a solution concept which generalizes the standard concept of a solution to scalar optimization problem. In consequence, the containment property (CP) may also be seen as a solution concept in vector optimization. To investigate more deeply this aspect we interpret the containment property as a generalization of the concept of the set of $\phi-$local solutions appearing in the

context of Lipschitz continuity of solutions to scalar optimization problems. Under mild assumptions the containment property imply that the set weakly minimal points equals the set of minimal points. This equality, in turn, is a typical ingradient of standard finite-dimensional sufficient conditions for upper semicontinuity of minimal points.
To study Hőlder upper continuity of minimal points we define the rate of containment of a set with respect to a cone, which is a real-valued function of a scalar argument, see [14, 15]. The rate of growth of this function influence decisively the rate of Hőlder continuity of minimal points, [15].

Strictly minimal points are introduced to study lower semicontinuities (lower Hausdorff, lower Hőlder) of minimal points [20, 13]. The definition of a strictly minimal point is given in topological vector spaces and it is a generalization of the notion of a super efficient point in the sense of Borwein and Zhuang defined in normed spaces. We discuss strict minimality in vector optimization by proving that it is a vector counterpart of the concept of $\phi-$ local solution to scalar optimization problem.

Theory of vector optimization may be considered as an abstract study of optimization problems with mappings taking values in the outcome space equipped with a partial order structure. As such, it contains many concepts and results which generalize and/or have their counterparts in scalar optimization. The very definition of the set of minimal points of vector optimization problem in the outcome space may serve as an example here. This is a counterpart of the optimal value of scalar optimization problem. Another example is the concept of well-posed optimization problem. In subsequent developments we often compare our results and considerations with the corresponding approaches in scalar optimization. For instance, we define several classes of well-posed vector optimization problems by generalizing the concept of scalar minimizing sequence and in these classes we investigate continuity of solutions. For scalar optimization problems, the existing approaches and results on well-posedness are extensively discussed in the monograph by Dontchev and Zolezzi [33].

Convergence and rates of convergence of solutions to perturbed optimization problems is one of crucial topics of stability analysis in optimization both from theoretical and numerical points of view. For scalar optimization it was investigated by many authors see eg., [72], [32], [47], [78], [55], [81], [59], [60], [82], [2], and many others. An exhaustive survey of current state of research is given in the recent monograph by Bonnans and Shapiro [26]. In vector optimization the results on Lipschitz continuity of solutions are not so numerous, and concern some classes of problems, for linear case see eg.,[28], [29], [30], for convex case see eg., [25], [31].

The organization of the material is as follows. In Chapter 2 we investigate upper Hausdorff continuity of the multivalued mapping M, $M(u) = \mathrm{Min}(\Gamma(u)|\mathcal{K})$ assigning to a given parameter value u from a topological space U the set of minimal points of the set $\Gamma(u) \subset Y$ with respect to cone $\mathcal{K} \subset Y$, where for any subset A of a topological vector space Y the set of minimal points is defined as $\mathrm{Min}(A|\mathcal{K}) = \{y \in A \mid A \cap (y - \mathcal{K}) = \{y\}\}$, and $\Gamma : U \rightrightarrows Y$, is a given multivalued mapping. The main tool which allows us to obtain the general result is the containment property (CP). Some infinite-dimensional examples are discussed. A special attention is paid to the containment property (CP) in finite-dimensional case, when $Y = R^m$.

In Chapter 3 we discuss upper Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the rate of containment δ which is a one-variable nondecreasing function, defined for a given set A and the order generating cone $\mathcal{K}$. The assumption of sufficiently fast growth rate of this function appears to be the crucial assumption for all upper Hőlder stability results of Chapter 3.

In Chapter 4 we apply the results obtained in Chapters 2 and 3 to derive conditions for upper Hausdorff and upper Hőlder stability of minimal points to parametric vector optimization problems by taking $\Gamma(u) = f(u, A(u))$. Moreover, we introduce the concept of $\Phi-$ strong solutions to vector optimization problem (P_0), which is a generalization of the concept of a $\phi-$local minimizer to scalar optimization problem, the latter being introduced by Attouch and

Wets [6].

In Chapter 5 we investigate the lower continuity and lower Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the notion of strict minimality mentioned above and the rate of strict minimality. In Section 5.5 we apply the results obtained in Chapter 5 to parametric vector optimization problems and we derive sufficient conditions for lower and lower Hőlder continuity of Pareto point multivalued mapping $\mathcal{M}$. An important tool here is the notion of $\Phi-$ strict solution to vector optimization problem introduced in Section 6.1. This notion can be interpreted as another possible generalization of the concept of $\phi-$local minimizer.

In Chapter 6 we propose several definitions of a well-posed vector optimization problem. All these definitions are based on properties of $\varepsilon-$solutions to vector optimization problems. For well-posed vector optimization problems we prove upper Hausdorff continuity of solution multivalued mapping S, $S(u) = S(f(u,\cdot), A(u), \mathcal{K})$.

2

Upper Hausdorff continuity of minimal points with respect to perturbations of the set

In this chapter we study upper Hausdorff continuity of the set-valued mapping $M : U \rightrightarrows Y$, called **the minimal point multifunction,**

$$M(u) = Min(\Gamma(u)|\mathcal{K}),$$

where $\Gamma : U \rightrightarrows Y$ is a given set-valued mapping. Let us note that in parametric vector optimization problems of the form

$$\mathcal{K} - \min f(u,x)$$
$$\text{subject to } x \in A(u)$$

the performance multifunction $\mathcal{M} : U \rightrightarrows Y$, is given by

$$\mathcal{M}(u) = Min(f(u,\cdot), A(u), \mathcal{K}) = Min(f(u, A(u))|\mathcal{K}).$$

Hence, $\mathcal{M}(u) = M(u)$, with $\Gamma(u) = f(u, A(u))$. The motivation for studying upper Hausdorff continuity of Pareto point multivalued mapping $\mathcal{M}$ is that this type of continuity is a standard ingredient of stability results of solutions multivalued mapping $\mathcal{S}$. This aspect will be discussed in detail in chapter 7.

In the present chapter we derive sufficient conditions for upper Hausdorff continuity of the minimal point multivalued mapping M corresponding to a given multivalued mapping Γ. The main Theorems of this chapter are Theorems 2.2.1, 2.4.1, and 2.4.2. In Chapter 4, by applying these theorems to the mapping $\Gamma(u) = f(u, A(u))$ we derive sufficient conditions for upper continuity of Pareto point multivalued mapping $\mathcal{M}$. The main tool for our results of this chapter is the containment property defined below.

2.1 Containment property

Let $A \subset Y$ be a subset of a Hausdorff topological vector space Y, and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. For any 0−neighbourhood W, denote

$$A(W) = A \setminus (Min(A|\mathcal{K}) + W).$$

Definition 2.1.1 ([11]) (Containment property) *We say that the* **containment property** *(CP) holds for A if for every 0−neighbourhood W there exists a 0−neighbourhood O such that*

$$A(W) + O \subset Min(A|\mathcal{K}) + \mathcal{K}. \quad (1)$$

Note that a 0−neighbourhood O satisfying (1) is chosen independently of $y \in A(W)$. If $A \neq \emptyset$ and (CP) holds, $\text{Min}(A|\mathcal{K}) \neq \emptyset$. Recall that the domination property (DP) holds for A if $A \subset \text{Min}(A|\mathcal{K}) + \mathcal{K}$. In general, neither (CP) implies (DP) nor conversely. Even for compact sets (CP) may not hold. To see this, consider in R^2 the set $A = \{(y_1, y_2) \mid 0 \leq y_1 \leq 1 \;\; 0 \leq y_2 \leq 1\}$. For the cone $\mathcal{K} = R^2_+ = \{(y_1, y_2) \mid y_1 \geq 0 \;\; y_2 \geq 0\}$, (CP) does not hold. Here $\text{Min}(A|\mathcal{K}) \neq \text{WMin}(A|\mathcal{K})$.

However, we have the following proposition.

Proposition 2.1.1 *If (CP) holds for A, then $A \subset clMin(A|\mathcal{K}) + \mathcal{K}$.*

Proof. Let $x \in A$. If $x \in \text{clMin}(A|\mathcal{K})$, the inclusion holds. If $x \notin \text{clMin}(A|\mathcal{K})$, by (CP), there exists a 0−neighbouhood O such that $x + O \subset \text{Min}(A|\mathcal{K}) + \mathcal{K}$ which yields the result.

□

Proposition 2.1.2 *Let $int\mathcal{K} \neq \emptyset$ and let $A \subset Y$ be a subset of Y. If (CP) holds for A, then $WMin(A|\mathcal{K}) \subset clMin(A|\mathcal{K})$.*

Proof. On the contrary, suppose that there is $y \in \text{WMin}(A|\mathcal{K}) \setminus \text{clMin}(A|\mathcal{K})$. Hence, $(y - \text{int}\mathcal{K}) \cap A = \emptyset$, and

$$(y - \text{int}\mathcal{K}) \cap (\text{Min}(A|\mathcal{K}) + \mathcal{K}) = \emptyset. \quad (*)$$

Since $y \notin \mathrm{clMin}(A|\mathcal{K})$, and Y is Hausdorff, by (CP), there exists $0-$neighbourhood O in Y such that

$$y + O \subset \mathrm{Min}(A|\mathcal{K}) + \mathcal{K},$$

and consequently $(y - \mathrm{int}\mathcal{K}) \cap (\mathrm{Min}(A|\mathcal{K}) + \mathcal{K}) \neq \emptyset$, contradictory to $(*)$.

□

By Theorem 1.1 of [57],p.136, $\mathrm{WMin}(A|\mathcal{K})$ is closed whenever A is closed.This implies that $\mathrm{clMin}(A|\mathcal{K}) \subset \mathrm{WMin}(A|\mathcal{K})$. Hence, by Proposition 2.1.2 we obtain the following corollary.

Corollary 2.1 *Let $\mathrm{int}\mathcal{K} \neq \emptyset$. Let $A \subset Y$ be a nonempty and closed subset of Y. If (CP) holds for A, we have $WMin(A|\mathcal{K}) = cl\ Min(A|\mathcal{K})$.*

Corollary 2.2 *Let $\mathrm{int}\mathcal{K} \neq \emptyset$. Let $A \subset Y$ be a nonempty and closed subset of Y. If (CP) holds for A and $Min(A|\mathcal{K}) = WMin(A|\mathcal{K})$, then (DP) holds for A.*

Proposition 2.1.3 *Let $\mathrm{int}\mathcal{K} \neq \emptyset$. Let $A \subset Y$ be a nonempty compact subset of Y. The following conditions are equivalent:*

(i) (CP) holds for A,

(ii) (DP) holds for A and $Min(A|\mathcal{K}) = WMin(A|\mathcal{K})$.

The following proposition gives an equivalent formulation of the containment property (CP) for cones with nonempty interior.

Proposition 2.1.4 *Let $\mathcal{K}$ be a closed convex pointed cone in Y, $\mathrm{int}\mathcal{K} \neq \emptyset$, and $A \subset Y$ a subset of Y. The following are equivalent:*

(i) (CP) holds for A

(ii) for each $0-$neighbourhood W there exists a $0-$neighbourhood O such that each $y \in A(W)$ can be represented as

$$y = \eta_y + k_y, \quad \text{where} \quad \eta_y \in Min(A|\mathcal{K}), \text{ and } k_y + O \subset \mathcal{K}. \quad (2)$$

Proof. $(i) \to (ii)$. For any $0-$neighbourhood O, define

$$\mathcal{K}_O = \{k \in \mathcal{K} \mid k + O \subset \mathcal{K}\}.$$

Clearly, we have $\text{int}\mathcal{K} = \bigcup_{O \in \mathcal{N}} \mathcal{K}_O$. We show that for any $0-$neighbourhood Q there exists a $0-$neighbourhood O such that

$$(\text{Min}(A|\mathcal{K}) + \mathcal{K})_Q \subset \text{Min}(A|\mathcal{K}) + \mathcal{K}_O, \tag{3}$$

where $(\text{Min}(A|\mathcal{K}) + \mathcal{K})_Q = \{y \in Y \mid y + Q \subset \text{Min}(A|\mathcal{K}) + \mathcal{K}\}$. Indeed, let $a \in (\text{Min}(A|\mathcal{K}) + \mathcal{K})_Q$, ie., $a + Q \subset \text{Min}(A|\mathcal{K}) + \mathcal{K}$. Since $0 \in \text{cl}(-\mathcal{K})$, for any $0-$neighbourhood Q there exists a $0-$neighbourhood O such that $Q \cap (-\mathcal{K}_O) \neq \emptyset$. Thus there exists $q \in Q \cap (-\mathcal{K}_O)$ such that $a + q \in \text{Min}(A|\mathcal{K}) + \mathcal{K}$, and consequently $a \in \text{Min}(A|\mathcal{K}) + \mathcal{K}_O$.
Suppose now that (CP) holds for A, ie., for each $0-$neighbourhood W there exists a $0-$neighbourhood Q such that for any $y \in A(W)$

$$y \in (\text{Min}(A|\mathcal{K}) + \mathcal{K})_Q,$$

and by (3), for some $0-$neighbourhood O, $y \in \text{Min}(A|\mathcal{K}) + \mathcal{K}_O$.
$(ii) \to (i)$. Obvious.

□

2.1.1 Containment property in finite-dimensional case.

Let $\mathcal{K} \subset R^m$ be a closed convex and pointed cone in the $m-$dimensional space R^m with the norm $\|\cdot\|$.
It was shown by Petschke [66] that a pointed closed convex cone $\mathcal{K}$ in R^m admits a compact base Θ. Hence, we have $m \leq \|\theta\| \leq M$ for any $\theta \in \Theta$.
Let $A \subset R^m$. If A is convex and closed, then $Min(A|\mathcal{K})$ need not be closed (see Arrow, Barankin, Blackwell [3]). Hence, even for convex sets in finite-dimensional case (CP) does not imply (DP). We start with a result establishing the relation between the domination property (DP) and the containment property (CP).

Theorem 2.1.1 *Let $\mathcal{K}$ be a closed convex and pointed cone in R^m, $int\mathcal{K} \neq \emptyset$. Let A be a closed convex subset of R^m. Assume that $clMin(A|\mathcal{K})$ is compact. If* *(i)* $clMin(A|\mathcal{K}) = WMin(A|\mathcal{K})$,

(ii) (DP) *holds for* A,

then (CP) *holds for* A.

Proof. The set $\mathrm{cl}Min(A|\mathcal{K}) + \mathcal{K}$ is closed and convex, since $\mathrm{cl}Min(A|\mathcal{K})$ is compact, and $A + \mathcal{K} = \mathrm{cl}Min(A|\mathcal{K}) + \mathcal{K}$.
Suppose on the contrary that (CP) does not hold for A. This means that there exist sequences $\{z_n\}$, $\{y_n\}$ such that

$$z_n \in A \setminus B(Min(A|\mathcal{K}), \varepsilon_0),$$

$y_n \in B(z_n, \frac{1}{n})$, and $y_n \notin \mathrm{cl}Min(A|\mathcal{K}) + \mathcal{K}$. By (DP),

$$z_n = \eta_n + k_n,$$

where $\eta_n \in Min(A|\mathcal{K})$, $k_n \in \mathcal{K}$, $\|k_n\| > \varepsilon_0$. Since $\mathcal{K}$ is based with a compact base Θ, $k_n = \lambda_n \theta_n$, with $\lambda_n > 0$, $\theta_n \in \Theta$. By the inequality

$$\varepsilon_0 < \|z_n - \eta_n\| = \lambda_n \cdot \|\theta_n\|,$$

we get $\varepsilon_0 < \lambda_n \cdot M$, and consequently, the sequence $\{\beta_n\}$, $\beta_n = \frac{1}{\lambda_n}$, is bounded. Without loss of generality, we can assume that $0 < \beta_n \leq 1$. By convexity of A,

$$\eta_n + \theta_n = \beta_n z_n + (1 - \beta_n)\eta_n \in A.$$

Since $\mathrm{cl}Min(A|\mathcal{K})$ is compact, $\{\eta_n\}$ contains a convergent subsequence with the limit point $\eta \in \mathrm{cl}Min(A|\mathcal{K})$. Without loss of generality we can assume that $\{\eta_n\}$ converges to η and $\{\theta_n\}$ converges to a $\theta \in \Theta$. The sequence $\{w_n\}$, $w_n = \eta_n + \theta_n$, for $n = 1, \ldots$, tends to $w = \eta + \theta \in A$. Clearly, $w \notin B(Min(A|\mathcal{K}), \varepsilon_2)$, $(0 < \varepsilon_2 < \varepsilon_1)$. We show that $w \in \partial(Min(A|\mathcal{K}) + \mathcal{K})$, where $\partial(\cdot)$ denotes the boundary of a set. On the contrary, if $w + B(0, \varepsilon_1) \subset Min(A|\mathcal{K}) + \mathcal{K}$, for some $\varepsilon_1 > 0$, then for some N

$$w_n \in w + B(0, \varepsilon_1/2) \quad \text{for} \quad n \geq N$$

and

$$z_n + B(0, \varepsilon_1/2) =$$
$$w_n + (\lambda_n - 1)\theta_n + B(0, \varepsilon_1/2) \in w + B(0, \varepsilon_1) + \mathcal{K} \subset Min(A|\mathcal{K}) + \mathcal{K},$$

contradictory to the fact that $\lim_n(z_n - \bar{y}_n) = 0$. Hence, $w \in \partial(Min(A|\mathcal{K}) + \mathcal{K})$, and $(w - \mathrm{int}\mathcal{K}) \cap A = \emptyset$, which proves that $w \in WMin(A|\mathcal{K})$. This contradicts the fact that $\mathrm{cl}Min(A|\mathcal{K}) = WMin(A|\mathcal{K})$.

□

It is easy to give examples showing that the equality $\mathrm{clMin}(A|\mathcal{K}) = \mathrm{WMin}(A|\mathcal{K})$ is important for (CP) property.

Example 2.1.1 *Let $A \subset R^2$, $\mathcal{K} = R^2_+$, and*

$$A = \{(x, y) \,|\, 0 \leq x \leq 1, \;\; 0 \leq y \leq 1\} \; .$$

Here $Min(A|R^2_+) = \{(0,0)\}$, $WMin(A|R^2_+) = \{(x, y) \in A \,|\, x = 0 or\; y = 0\}$, (DP) holds for A and (CP) does not.

The assumption that $\mathrm{Min}(A|\mathcal{K})$ is compact cannot be dropped. Also the convexity and the closedness of A cannot be weakened to $\mathcal{K}-$convexity and $\mathcal{K}-$closedness.

Corollary 2.3 *If A is convex and closed, $Min(A|\mathcal{K}) \neq \emptyset$, $clMin(A|\mathcal{K})$ is compact, $clMin(A|\mathcal{K}) = WMin(A|\mathcal{K})$, then (CP) holds for A.*

Proof . This follows from the result of Henig [40].

□

Theorem 2.1.2 *Let A be a convex and closed subset of R^m, and let $Min(A|\mathcal{K})$ be compact. The following conditions are equivalent:*

(i) $Min(A|\mathcal{K}) \neq \emptyset$, $Min(A|\mathcal{K}) = WMin(A|\mathcal{K})$,

(ii) (CP) holds for A.

Assume now that the set $A \subset R^m$ is polyhedral, ie., A is the solution set of a system of a finite number of linear inequalities,

$$A = \{y \in R^m \,|\, < b_i, y > \leq c_i \quad i \in I\} \; ,$$

and we prove an analogue of Theorem 2.1.1 without compactness of $Min(A|\mathcal{K})$. The recession cone of A, $Rec(A)$, is given by the system of homogeneous inequalities,

$$Rec(A) = \{y \in R^m \mid < b_i, y > \leq 0 \quad i \in I\} \ ,$$

and $Min(A|\mathcal{K}) \neq \emptyset$ if and only if $Rec(A) \cap [-\mathcal{K}] = \{0\}$ (Th.3.18 of [57]).

To make the presentation self-contained we prove closedness of $Min(A|\mathcal{K})$ and of $Min(A|\mathcal{K}) + \mathcal{K}$ whenever A is a polyhedral set. Usually, the closedness of $Min(A|\mathcal{K})$ is proved as a consequence of scalarization properties of linear multiobjective optimization problems with polyhedral cones. Here we prove the closedness of $Min(A|\mathcal{K})$ directly for any closed convex pointed cone $\mathcal{K}$.

Proposition 2.1.5 *If A is a polyhedral subset of R^m, and $\mathcal{K}$ is a closed convex pointed cone, then $Min(A|\mathcal{K})$ is closed.*

Proof. Suppose on the contrary that $Min(A|\mathcal{K})$ is not closed. There exists a sequence of minimal points $\{\eta_n\} \in Min(A|\mathcal{K})$ such that $\{\eta_n\}$ converges to $\eta \in A$ and $\eta \notin Min(A|\mathcal{K})$. Hence, there is an $\bar{\eta} \in A$ such that $\eta - \bar{\eta} \in \mathcal{K} \setminus \{0\}$.

Let us split the index set I into two subsets $I_1, I_2 \subset I$ such that

$$\langle b_i, \eta_n \rangle = c_i \quad i \in I_1 \quad \text{and} \quad \langle b_i, \eta_n \rangle < c_i \quad i \in I_2 \, .$$

Hence, $\langle b_i, \eta \rangle = c_i$, for $i \in I_1$ and $\langle b_i, \eta \rangle \geq \langle b_i, \bar{\eta} \rangle$ for $i \in I_1$, because $\bar{\eta} \in A$. Moreover, $\langle b_i, \bar{\eta} \rangle > \langle b_i, \eta \rangle$ for some $i \in I_2$, since otherwise $0 \neq -k = \bar{\eta} - \eta \in Rec(A)$. Thus, there are two index subsets $I_3, I_4 \subset I_2$, with $I_4 \neq \emptyset$, such that

$$\langle b_i, \bar{\eta} \rangle < \langle b_i, \eta \rangle \quad i \in I_3 \quad \text{and} \quad \langle b_i, \bar{\eta} \rangle > \langle b_i, \eta \rangle \quad i \in I_4 \, .$$

Consequently,

$$\begin{aligned} &\langle b_i, \bar{\eta} - \eta \rangle \leq 0 \quad i \in I_5 \supset I_1 \\ &\langle b_i, \bar{\eta} - \eta \rangle < 0 \quad i \in I_3 \\ &\langle b_i, \bar{\eta} - \eta \rangle > 0 \quad i \in I_4 \, , \end{aligned}$$

for some subset $I_5 \subset I$, $I_3 \cup I_4 \cup I_5 = I$.

For each n, put

$$\gamma_n = \min_{i \in I_4} \frac{c_i - \langle b_i, \eta_n \rangle}{\langle b_i, \bar{\eta} - \eta \rangle} > 0 \, ,$$

and consider $w_n = \eta_n + \gamma_n(\bar{\eta} - \eta)$. For $i \in I_3 \cup I_5$, we have $\langle b_i, w_n \rangle \leq c_i$, and for $i \in I_4$,

$$\langle b_i, \eta_n \rangle + \gamma_n \langle b_i, (\bar{\eta} - \eta) \rangle \leq \langle b_i, \eta_n \rangle + \frac{c_i - \langle b_i, \eta_n \rangle}{\langle b_i, \bar{\eta} - \eta \rangle} \cdot \langle b_i, (\bar{\eta} - \eta) \rangle = c_i .$$

Finally, $w_n \in A$, and $w_n \in \eta_n - \mathcal{K}$. This contradicts the minimality of η_n.

□

Proposition 2.1.6 *For any polyhedral set $A \subset R^m$ and any closed convex pointed cone $\mathcal{K}$ in R^m, $Min(A|\mathcal{K}) + \mathcal{K}$ is closed.*

Proof. If $Min(A|\mathcal{K}) = \emptyset$, then $Min(A|\mathcal{K}) + \mathcal{K}$ is empty, hence closed. Assume that $Min(A|\mathcal{K}) \neq \emptyset$, and consider any convergent sequence $\{z_n\} \subset Min(A|\mathcal{K}) + \mathcal{K}$, $\lim_n z_n = z$. We have $z_n = x_n + \lambda_n \theta_n$, where $x_n \in Min(A|\mathcal{K})$, $\theta_n \in \Theta$, and $\lambda_n \geq 0$. By compactness of Θ, without loss of generality, we may assume that $\{\theta_n\}$ converges to $\theta \in \Theta$.
We start by showing that under our assumptions, $\{\lambda_n\}$ contains a bounded subsequence. Indeed, suppose on the contrary that $\{\lambda_n\}$ tends to $+\infty$. Then

$$\frac{1}{\lambda_n}[x_n + \lambda_n \theta_n] = \frac{1}{\lambda_n} x_n + \theta_n \rightarrow 0 ,$$

and, since $\theta_n \rightarrow \theta \neq 0$ $\lim_n \frac{1}{\lambda_n} x_n = -\theta$. On the other hand,

$$\langle b_i, \frac{1}{\lambda_n} x_n \rangle \leq \frac{1}{\lambda_n} c_i , \quad i \in I ,$$

and, by passing to the limit $\langle b_i, -\theta \rangle \leq 0$, ie., $-\theta \in Rec(A) \cap [-\mathcal{K}]$, contradictory to the assumption that $Min(A|\mathcal{K}) \neq \emptyset$ (see the remark above).
Consequently, $\{\lambda_n\}$ contains a convergent subsequence $\{\lambda_{n_\ell}\}$, $\lambda_{n_\ell} \rightarrow \lambda \geq 0$, and $\lambda_{n_\ell} \theta_{n_\ell} \rightarrow k \in \mathcal{K}$, and $x_{n_\ell} \rightarrow x \in Min(A|\mathcal{K})$, since, by Proposition 2.1.5, $Min(A|\mathcal{K})$ is closed, and finally $z = x + k \in Min(A|\mathcal{K}) + \mathcal{K}$.

□

In general, if $Min(A|\mathcal{K}) = WMin(A|\mathcal{K})$, and (DP) holds for A, then

$$A \subset WMin(A|\mathcal{K}) + \text{int}\mathcal{K} \cup \{0\}. \tag{4}$$

Theorem 2.1.3 *If $A \subset R^m$ is a polyhedral set, and $\mathcal{K} \subset R^m$ is a closed convex and pointed cone in R^m, then the following conditions are equivalent:*

(i) (DP) holds for A, and $Min(A|\mathcal{K}) = WMin(A|\mathcal{K})$,

(ii) (CP) holds for A.

Proof. The implication $(ii) \rightarrow (i)$ is obvious. To prove that $(i) \rightarrow (ii)$ suppose on the contrary that (CP) does not hold for A. Thus, there exists a sequence $\{y_n\} \subset A$ such that $y_n \notin B(Min(A|\mathcal{K}), \varepsilon_0)$ and $B(y_n, \frac{1}{n}) \cap [A + \mathcal{K}]^c \neq \emptyset$. By this, there exist $\{\bar{y}_n\} \subset [A + \mathcal{K}]^c$, $\lim_n(y_n - \bar{y}_n) = 0$, and $\{z_n\} \subset \partial(Min(A|\mathcal{K}) + \mathcal{K})$, $\lim_n(y_n - z_n) = 0$. If, for at least one n, $z_n \in A$, then $z_n \in WMin(A|\mathcal{K}) \setminus Min(A|\mathcal{K})$, a contradiction. Hence, for all n, $z_n \notin A$, and moreover,

$$(z_n - \mathcal{K}) \cap (Min(A|\mathcal{K}) + \mathcal{K}) \subset \partial(Min(A|\mathcal{K}) + \mathcal{K}), \tag{5}$$

since otherwise $z_n - \bar{k} + B(0,r) \subset Min(A|\mathcal{K}) + \mathcal{K}$ for some $\bar{k} \in \mathcal{K}$, and $r > 0$, and consequently, $z_n + B(0,r) \subset Min(A|\mathcal{K}) + \mathcal{K}$, which is impossible because $z_n \in \partial(Min(A|\mathcal{K}) + \mathcal{K})$.
By Proposition 2.1.6, $Min(A|\mathcal{K}) + \mathcal{K}$ is closed, and hence, $z_n = \eta_n + \lambda_n\theta_n$, where $\eta_n \in Min(A|\mathcal{K})$, $\theta_n \in \Theta$, and $\lambda_n \geq 0$. Moreover, $\lambda_n > \varepsilon_0/M$, and without loss of generality we can assume that $\lambda_n > 1$.
Since $z_n \notin A$, there is a subset κ of the index set I such that

$$\langle b_i, z_n\rangle > c_i \quad i \in \kappa \quad \text{and} \quad \langle b_i, z_n\rangle \leq c_i \quad i \in I \setminus \kappa.$$

Let us note that, if necessary, one can always shift slightly y_n, so as to have $\langle b_i, y_n\rangle = c_i$, for $i \in \kappa$. By compactness of Θ, we can assume that $\{\theta_n\}$ tends to $\theta \in \Theta$.
Suppose first that $\langle b_k, \theta\rangle = 0$, for $k \in \kappa$. For $i \notin \kappa$, $\langle b_i, \eta_n\rangle + \lambda_n\langle b_i, \theta_n\rangle \leq c_i$, and either $\langle b_i, \theta\rangle \leq 0$, or $\langle b_i, \theta\rangle > 0$. In the latter

case it must be $\langle b_i, \theta\rangle \leq c_i - \langle b_i, \eta_n\rangle$, because otherwise $\langle b_i, \theta\rangle > c_i - \langle b_i, \eta_n\rangle \geq \lambda_n \langle b_i, \theta_n\rangle$, and

$$\langle b_i, \eta_n\rangle - c_i + \lambda_n\langle b_i, \theta_n\rangle = [\langle b_i, \eta_n + \theta\rangle - c_i] + \langle b_i, \theta_n - \theta\rangle + (\lambda_n - 1)\langle b_i, \theta_n\rangle > 0 .$$

Hence, $\eta_n + \theta \in A$, and consequently, $\eta_n + \theta \in WMin(A|\mathcal{K})$, since $\eta_n + \theta \in \partial(Min(A|\mathcal{K}) + \mathcal{K})$, which is a contradiction.
Hence, it must be $\langle b_\ell, \theta\rangle > 0$, for some $\ell \in \kappa$, because $\langle b_k, \theta_n\rangle > 0$ for $k \in \kappa$. We show that each z_n can be represented in the form $z_n = \eta_n + \lambda_n\theta_n$, where $\eta_n \in Min(A|\mathcal{K})$, and $\langle b_\ell, \eta_n\rangle = c_\ell$. Indeed, we have

$$\langle b_i, \theta_n\rangle > 0 \;\; i \in \kappa_1 \;\; \text{and} \;\; \langle b_i, \theta_n\rangle = 0 \;\; i \in \kappa_2 \;\; \langle b_i, \theta_n\rangle < 0 \;\; i \in \kappa_3 ,$$

where $\kappa \subset \kappa_1$, $\kappa_2 = \{i \in I \mid \langle b_i, z_n\rangle = c_i\}$, $\kappa_3 \subset I \setminus \kappa$. Suppose that for all $k \in \kappa$ such that $(b_k, \theta) > 0$ we have $\langle b_k, \eta_n\rangle < c_k$, and put

$$\beta_n = 1/2 \min_{\substack{k \in \kappa \\ \langle b_k, \theta\rangle > 0}} \frac{c_k - \langle b_k, \eta_n\rangle}{\langle b_k, \theta_n\rangle} > 0 .$$

Observe first that for each $k \in \kappa$, $\lambda_n > \frac{c_k - \langle b_k, \eta_n\rangle}{\langle b_k, \theta_n\rangle} > \beta_n$. Now

$$\begin{aligned}
&\langle b_i, z_n - (\lambda_n - \beta_n)\theta_n\rangle = \langle b_i, \eta_n\rangle + \beta_n\langle b_i, \theta\rangle \leq \langle b_i, \eta_n\rangle + \tfrac{c_i - \langle b_i, \eta_n\rangle}{\langle b_i, \theta_n\rangle}\langle b_i, \theta_n\rangle = c_i \;\; i \in \kappa \\
&\langle b_i, z_n - (\lambda_n - \beta_n)\theta_n\rangle = \langle b_i, \eta_n\rangle + \beta_n\langle b_i, \theta_n\rangle \leq \langle b_i, \eta_n\rangle + \lambda_n\langle b_i, \theta_n\rangle \leq c_i \;\; i \in \kappa_1 \setminus \kappa \\
&\langle b_i, z_n - (\lambda_n - \beta_n)\theta_n\rangle = \langle b_i, \eta_n\rangle \leq c_i \;\; i \in \kappa_2 \\
&\langle b_i, z_n - (\lambda_n - \beta_n)\theta_n\rangle \leq \langle b_i, \eta_n\rangle \leq c_i \;\; i \in \kappa_3 ,
\end{aligned}$$

which means that $w_n = z_n - (\lambda_n - \beta_n)\theta_n \in A \cap (z_n - \mathcal{K})$, and, by (4),

$$w_n \in Min(A|\mathcal{K}) + \text{int}\mathcal{K} \subset \text{int}(Min(A|\mathcal{K}) + \mathcal{K}) ,$$

contrary to (5).This proves that for some $\ell \in \kappa$ such that $\langle b_\ell, \theta\rangle > 0$ it must be $\langle b_\ell, \eta_n\rangle = c_l$.
By letting $H_\ell = \{y \in R^m \mid \langle b_\ell, y\rangle = c_\ell\}$, we get

$$\|y_n - z_n\| \geq \text{dist}(z_n, H_\ell) = \frac{\langle b_\ell, z_n\rangle - c_\ell}{\sqrt{(b_\ell)^2}} = \frac{\lambda_n\langle b_\ell, \theta_n\rangle}{\sqrt{(b_\ell)^2}} ,$$

which implies that $\lambda_n \to 0$. This is a contradiction.

□

2.2 Upper Hausdorff continuity of minimal points for cones with nonempty interior.

We start with the main result of this section.

Theorem 2.2.1 (Bednarczuk, [16]) *Let U be a topological space and let Y be a Hausdorff topological vector space. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $int\mathcal{K} \neq \emptyset$. Suppose that*

(i) Γ is upper Hausdorff continuous at u_0,

(ii) Γ is $\mathcal{K}-$lower continuous at u_0, uniformly on $Min(\Gamma(u_0)|\mathcal{K})$,

(iii) (CP) holds for $\Gamma(u_0)$.

The minimal point multifunction M is upper Hausdorff continuous at u_0.

Proof. Let W_1, W be $0-$neighbourhoods, $W_1 + W_1 \subset W$. By Proposition 2.1.4, there exists a $0-$neighbourhood O such that any $z \in \Gamma(u_0)(W_1)$ can be represented in the form

$$z = \eta_z + k_z, \quad \eta_z \in Min(\Gamma(u_0)|\mathcal{K}), \quad k_z + O \subset \mathcal{K}. \quad (*)$$

Let O_1 be a $0-$neighbourhood such that $O_1 + O_1 \subset O$. By (i), there exists a neighbourhood U_0 of u_0 such that

$$\begin{aligned}\Gamma(u) &\subset \Gamma(u_0) + W_1 \cap O_1 \subset [\Gamma(u_0)(W_1) + W_1 \cap O_1] \\ &\cup [Min(\Gamma(u_0)|\mathcal{K}) + W_1 + W_1 \cap O_1],\end{aligned}$$

for $u \in U_0$. By (ii), there exists a neighbourhood U_1 of u_0 such that for $u \in U_1$

$$(\eta + O_1 - \mathcal{K}) \cap \Gamma(u) \neq \emptyset, \text{ for } \eta \in Min(\Gamma(u_0)|\mathcal{K}). \quad (**)$$

We show that

$$M(u) \cap [\Gamma(u_0)(W_1) + W_1 \cap O_1] = \emptyset \quad (***)$$

for $u \in U_0 \cap U_1$.

Take any $y \in \Gamma(u) \cap [\Gamma(u_0)(W_1) + W_1 \cap O_1]$. We have $y = \gamma + w$, $\gamma \in [\Gamma(u_0)(W_1)]$, $w \in W_1 \cap O_1$.

By $(*)$, γ can be represented in the form $\gamma = \eta_\gamma + k_\gamma$, $\eta_\gamma \in Min(\Gamma(u_0)|\mathcal{K})$, $k_\gamma + O \subset \mathcal{K}$. By $(**)$, there exists $\gamma_1 \in \Gamma(u)$ such that $\gamma_1 = \eta_\gamma - w_1 - k^1$, $w_1 \in O_1$, $k^1 \in \mathcal{K}$. Finally

$$y-\gamma_1 = \gamma+w-\gamma_1 = \eta_\gamma+k_\gamma+w-\eta_\gamma+w_1+k^1 \subset k_\gamma+k^1+O \subset \mathcal{K},$$

which proves $(***)$. Hence, for $u \in U_0 \cap U_1$,

$$M(u) \subset M(u_0) + W.$$

□

Below we give an example showing that the uniform $\mathcal{K}-$lower continuity assumption is essential in Theorem 2.2.1.

Example 2.2.1 *Let* $U = cl\{1/n \mid n = 1,...\}$ *with natural topology and let* $\Gamma : U \rightrightarrows R^2$, *be defined as follows*

$$\Gamma(0) = \{(y_1,y_2) \mid y_2 = -y_1\} \cup \bigcup_{k=1}^{\infty}(k,-k+1)$$
$$\Gamma(\tfrac{1}{n}) = \{(y_1,y_2) \mid y_2 = -y_1 + \tfrac{1}{n} \quad -n \le y_1 \le n\} \cup \bigcup_{k=1}^{\infty}(k,-k+1)$$

Now $Min(\Gamma(0)|R^2_+) = \{(y_1,y_2) \mid y_2 = -y_1\}$, *and*

$$Min(\Gamma(\frac{1}{n})|R^2_+) = \{(y_1,y_2) \mid y_2 = -y_1+\frac{1}{n}, \quad -n \le y_1 \le n\} \cup \bigcup_{k=n+1}^{\infty}(k,-k+1).$$

By Proposition 2.1.3, we obtain the following corollary.

Corollary 2.4 *Let* U *be a topological space and let* Y *be a Hausdorff topological vector space. Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in* Y, $int\mathcal{K} \neq \emptyset$. *Let* $\Gamma(u_0)$ *be a compact subset of* Y. *If*

(i) Γ *is upper Hausdorff continuous at* u_0,

(ii) Γ *is* $\mathcal{K}-$*lower continuous at* u_0, *uniformly on* $Min(\Gamma(u_0)|\mathcal{K})$,

(iii) (DP) holds for $\Gamma(u_0)$, *and* $Min(\Gamma(u_0)|\mathcal{K}) = WMin(\Gamma(u_0)|\mathcal{K})$,

then M *is upper Hausdorff continuous at* u_0.

In the proof of Theorem 2.2.1 we make use of Proposition 2.1.4 which holds true when int$\mathcal{K} \neq \emptyset$. There are numerous examples of cones satisfying this condition. For instance, cone R^n_+ of nonnegative elements in R^n, as well the cones of nonnegative elements in the spaces below have nonempty interiors.

Example 2.2.2 **1.** *In the space ℓ^∞ of sequences $s = \{s_i\}$, with real terms*

$$\ell^\infty = \{s = \{s_i\} \mid \sup_{i \in N} |s_i| < +\infty\}$$

the cone

$$\ell^\infty_+ = \{s = \{s_i\} \in \ell^\infty \mid s_i \geq 0\}$$

has nonempty interior.

2. *In the space $L^\infty(\Omega)$ of essentially bounded functions $f : \Omega \subset R^n \to R$, with $ess\sup_{x\in\Omega} |f(x)| < +\infty$, the natural ordering cone*

$$L^\infty(\Omega) = \{f \in L^\infty(\Omega) \mid f(x) \geq 0 \ \ almost\ everywhere\ on \Omega\}$$

has nonempty interior.

2.3 Weak containment property

As we have seen in the previous section in the proof of Theorem 2.2.1 we use Proposition 2.1.4 which holds true when int$\mathcal{K} \neq \emptyset$. However, in some important spaces, cones of nonnegative elements may have empty interiors. For example, in the space of integrable functions $L^p(\Omega)$, $1 \leq p < +\infty$, the cone $\mathcal{K}_{L^p(\Omega)}$ of nonnegative elements

$$\mathcal{K}_{L^p(\Omega)} = \{f \in L^p(\Omega) \mid f \geq 0 \text{ almost everywhere in } \Omega\},$$

as well as in the space ℓ^p, $1 \leq p < +\infty$, of summable sequences $s = \{s_i\}$ the cone

$$\mathcal{K}_{\ell^p(\Omega)} = \{s \in \ell^p \mid s_i \geq 0\}$$

have empty interiors (see [44]).

In this section we define weak containment property (WCP) which is a conterpart of the containment property (CP) for cones with possibly empty interiors.

Let Y be a Hausdorff topological vector space with topological dual Y^*. As defined in Section 1.1 the quasi-interior of $\mathcal{K}^*$ is given as

$$\mathcal{K}^{*i} = \{f \in Y^* \mid f(y) > 0 \ \text{ for all } \ y \in \mathcal{K} \setminus \{0\}\}.$$

As shown in Proposition 1.1.3 of Section 1.1, in locally convex spaces, $\mathcal{K}$ is based if and only if $\mathcal{K}^{*i}$ is nonempty. If int$\mathcal{K}$ is nonempty and $e \in \text{int}\mathcal{K}$, then $\Theta = \{f \in \mathcal{K}^* \mid f(e) = 1\}$ (see Theorem 1.1.2 of Section 1.1) is a base of $\mathcal{K}^*$.
Below we prove that $\mathcal{K}^{*i}$ is always based.

Proposition 2.3.1 *Let $\mathcal{K}$ be a closed convex cone in Y and let $\mathcal{K}^* \subset Y^*$ be its dual with $\mathcal{K}^{*i}$ nonempty. Then, for any $\theta_0 \in \mathcal{K} \setminus \{0\}$, the set*

$$\Theta^{*i} = \{f \in \mathcal{K}^{*i} \mid f(\theta_0) = 1\}$$

is a base of $\mathcal{K}^{*i}$.

Proof. Θ^{*i} is clearly convex and $0 \notin w{-}*{-}cl(\Theta^{*i})$. To see the latter note that if a net θ^*_α, $\theta^*_\alpha \subset \Theta^{*i}$, tends in the weak$-*-$topology to θ^*, then $\theta^* \neq 0$ since $\theta^*(\theta_0) = 1$. Moreover, each element $f \in \mathcal{K}^{*i}$ can be represented as a positive multiple of an element from Θ^{*i}. Indeed, to find $\lambda_f > 0$ and $\theta^* \in \Theta^i$ such that $f = \lambda_f \cdot \theta^*$, it is enough to note that

$$f(\theta_0) = \lambda_f \neq 0 \ \text{ and } \ \theta^* = \frac{f}{\lambda_f} \in \Theta^{*i}.$$

□

The bidual cone $\mathcal{K}^{**}$,

$$\mathcal{K}^{**} = \{y \in Y \mid f(y) \geq 0 \text{ for } f \in \mathcal{K}^*\},$$

is convex and weakly closed and in locally convex spaces $\mathcal{K} = \mathcal{K}^{**}$ if and only if $\mathcal{K}$ is convex and weakly closed (see Theorem 1.1.1 of Section 1.1, in normed spaces see Kurcyusz [54], Lemma 8.6).

Let $A \subset Y$ be a subset of Y and let $\mathcal{K}^*$ has a base Θ^*.

Definition 2.3.1 *The* **weak containment property** (WCP) *holds for* A *with respect to* Θ^* *if for every* $0-$*neighbourhood* W *there exists* $\delta > 0$ *such that for each* $y \in A(W)$ *there exists* $\eta_y \in Min(A|\mathcal{K})$ *satisfying*

$$\theta^*(y - \eta_y) > \delta \quad \text{for each } \theta^* \in \Theta^* .$$

Note that if $y - \eta_y$ satisfies $\theta^*(y - \eta_y) > \delta$ for some positive $\delta > 0$, and all $\theta^* \in \Theta^*$, then

$$y - \eta_y \in \{y \in Y \mid f(y) > 0 \text{ for all } f \in \mathcal{K}^* \setminus \{0\}\} \stackrel{def}{=} \mathcal{K}^i .$$

In general (WCP) depends upon base. However, we have the following proposition.

Proposition 2.3.2 *If* $\mathcal{K}^*$ *has a bounded base* Θ_0^*, *and* (WCP) *holds for* A *with respect to* Θ_0^*, *then* (WCP) *holds for* A *with respect to any base* Θ^* *of the form*

$$\Theta^* = \{\theta^* \in \mathcal{K}^* \mid \theta^*(\bar{y}) = 1 \quad \bar{y} \in \mathcal{K}^i\} .$$

Proof. For each $\theta^* \in \Theta^*$ there exists a $\theta_0^* \in \Theta_0^*$ such that

$$\theta^*(k) = \frac{1}{\theta_0^*(\bar{y})}\theta_0^*(k), \quad \text{for all } k \in \mathcal{K},$$

and, since Θ_0^* is bounded,

$$\inf_{\theta_0^* \in \Theta_0^*} \frac{1}{\theta_0^*(\bar{y})} \geq \frac{1}{\sup_{\theta_0^* \in \Theta_0^*} \theta_0^*(\bar{y})} \geq \kappa, \quad \text{for some } \kappa > 0 .$$

□

In the case of Θ_0^* unbounded, (WCP) holds for A with respect to any base Θ^* such that

$$\inf_{\theta^* \in \Theta^*} \frac{1}{\theta^*(\bar{y})} \geq \kappa .$$

Proposition 2.3.3 *Let* Y *be a locally convex space and let* $\mathcal{K} \subset Y$ *be a closed convex cone,* $int\mathcal{K} \neq \emptyset$. *For any subset* $A \subset Y$ *of* Y, (CP) *is equivalent to* (WCP).

Proof. Let W be a 0–neighbourhood. By (CP), there exists a 0–neighbourhood O such that for each $y \in A(W)$

$$y - \eta_y + O \subset \mathcal{K} \quad \text{for some} \quad \eta_y \in Min(A|\mathcal{K}) .$$

Take any $y_0 \in \mathcal{K}^i = \text{int}\mathcal{K}$. Since O can be assumed to be radial, $-\delta y_0 \in O$, for some $\delta > 0$, and

$$y - \eta_y - \delta y_0 \in \mathcal{K} ,$$

which means that (WCP) holds for A.
To see the converse implication, note that by Theorem 1.1.2, $\mathcal{K}^*$ has a weak$-*-$ compact, hence bounded, base Θ^*. By Proposition 2.3.2, (WCP) holds for Θ^*.

□

We have the following Proposition.

Proposition 2.3.4 *Let Y be a locally convex topological vector space and let $\mathcal{K} \subset Y$ be a closed convex cone with $\mathcal{K}^{*i} \neq \emptyset$. Then*

(i) $\mathcal{K}^i = \{y \in Y \mid f(y) > 0 \;\; \text{for all}\; f \in \mathcal{K}^* \setminus \{0\}\} \subset \mathcal{K} \setminus \{0\}$,

(ii) $w - * - cl\mathcal{K}^{*i} \subset \mathcal{K}^*$.

(iii) $\mathcal{K} = \{y \in Y \mid f(y) \geq 0 \; \text{for all}\; f \in \mathcal{K}^{*i}\}$,

(iv) $w - cl\{y \in Y \mid f(y) > 0 \;\; \text{for all}\; f \in \mathcal{K}^* \setminus \{0\}\} \subset \mathcal{K}$.

Proof. (i) follows from the fact that in a locally convex space $\mathcal{K} = \{y \in Y \mid f(y) \geq 0 \;\text{ for all } f \in \mathcal{K}^*\}$.
(ii) Since $\mathcal{K}^{*i} \subset \mathcal{K}^*$ and $\mathcal{K}^*$ is weakly $- * -$ closed, we get $w - * - cl\mathcal{K}^{*i} \subset \mathcal{K}^*$.
(iii) If $k \in \mathcal{K} \setminus \{0\}$, then $f(k) > 0$ for any $f \in \mathcal{K}^{*i}$, which proves that $\mathcal{K} \subset \{y \in Y \mid f(y) \geq 0 \text{ for all } f \in \mathcal{K}^{*i}\}$.
Suppose now that $y \notin \mathcal{K}$. Since Y is locally convex, there exists $f_0 \in \mathcal{K}^*$ such that $f_0(y) < 0$. Let $g \in \mathcal{K}^{*i}$. By choosing $\alpha > 0$ such that $f(y) + \alpha g(y) < 0$ we get $h = f + \alpha \cdot g \in \mathcal{K}^{*i}$ and $h(y) < 0$.
(iv) Since $\mathcal{K}$ is weakly closed, $w - cl\mathcal{K}^i \subset \mathcal{K}$.

□

The inclusion $\{y \in Y \mid f(y) \geq 0 \text{ for all } f \in \mathcal{K}^{*i}\} \subset \mathcal{K}$ is proved in [34], Lemma 5.5.

Let us note that if $\mathcal{K}^*$ has a base, then $\mathcal{K}^i \neq \emptyset$. Indeed, if Θ^* is a base of $\mathcal{K}^*$, then, since $0 \notin w-*-cl(\Theta^*)$, there exists $y_0 \in Y$ such that $f(y_0) \geq 1$ for all $f \in \Theta^*$. This means that $y_0 \in \mathcal{K}^i$.

It was shown in [44], Lemma 1.27, that if the algebraic interior $cor\mathcal{K}$ of a convex cone $\mathcal{K} \subset Y$ in a real linear space Y is nonempty, then the dual cone $\mathcal{K}^* \subset Y^*$ is pointed. Moreover, if $cor\mathcal{K}^* \neq \emptyset$, then $\mathcal{K}$ is based (see [42], Theorem I.5C).

By Theorem 1.1.3, $cor\mathcal{K} \subset \mathcal{K}^i$, and by Proposition 1.1.4, if $cor\mathcal{K} \neq \emptyset$, and $\mathcal{K}^* \neq \{0\}$, then $\mathcal{K}^*$ is based.

Example 2.3.1 **1.** *Let $Y = R^m$, $\mathcal{K} \subset Y$ be a closed convex pointed cone. For any convex subset A, $cor(A)$ coincides with the topological interior of A. Hence, eg., for the cone $\mathcal{K} = \{(y_1, y_2) \mid y_1 \geq 0 \ \ y_1 = y_2\}$ we get $\mathcal{K}^* = \{(f_1, f_2) \mid f_2 \geq -f_1\}$ and $\mathcal{K}^i = \emptyset$.*

2. *For any $p \in [1, +\infty)$ consider the sequence space ℓ^p, of sequences $s = \{s_i\}$ with real terms,*

$$\ell^p = \{s = \{s_i\} \mid \sum_{i=1}^{\infty} |s_i|^p < +\infty\},$$

with the natural ordering cone

$$\ell^p_+ = \{s = \{s_i\} \in \ell^p \mid s_i \geq 0\}.$$

The ordering cone ℓ^p_+ has empty topological interior and empty algebraic interior, $cor(\ell^p_+) = \emptyset$. But

$$(\ell^p_+)^i = \{s = \{s_i\} \in \ell^p \mid s_i > 0\}.$$

3. *For any $p \in [1, +\infty)$, consider the space of all p-th Lebesque integrable functions $f : \Omega \to R$ with the natural ordering cone*

$$L^p_+ = \{f \in L^p \mid f(x) \geq 0 \text{ almost everywhere on } \Omega\}.$$

The topological interior $int(L^p_+)$ and $cor(L^p)$ are both empty but $\mathcal{K}^i \neq \emptyset$. To see this recall that

$$(L^p_+)^i = \{f \in L^p \mid \int_\Omega fg \, d\mu > 0 \ \ \textit{for all } g \in L^q_+ \setminus \{0\}\},$$

$\frac{1}{p} + \frac{1}{q} = 1$, *and*

$$(L^p_+)^i = \{f \in L^p \mid f(x) > 0 \ \ \textit{almost everywhere on } \Omega\}.$$

In locally convex spaces, if (WCP) holds for A, then

$$A \subset clMin(A|\mathcal{K}) + \mathcal{K}. \tag{6}$$

Indeed, if $a \in A \setminus clMin(A|\mathcal{K})$, there exists $\varepsilon > 0$ such that $a \notin B(Min(A|\mathcal{K}), \varepsilon)$. By (WCP), there exist $\eta_a \in Min(A|\mathcal{K})$ and $\delta > 0$ such that

$$\theta^*(a - \eta_a) > \delta$$

for each $\theta^* \in \Theta^*$, and hence $a - \eta_a \in \mathcal{K}^i \subset \mathcal{K}$.

Suppose now that Y is a Banach space. Let $\mathcal{K}^i \neq \emptyset$, $y_0 \in \mathcal{K}^i$, and

$$\Theta^* = \{\theta^* \in \mathcal{K}^* \mid \theta^*(y_0) = 1\}.$$

As we have shown before Θ^* is a base of $\mathcal{K}^*$. For any $k \in \mathcal{K}$,

$$\inf\{\theta^*(k) \mid \theta^*(y_0) = 1, \ \ \theta^* \in \mathcal{K}^*\}, \tag{7}$$

can be viewed as an *infinite-dimensional linear programming problem.* By applying the convex programming duality theory (see eg. Barbu, Precupanu [10], Ch.3, par.3, p.233) the dual takes the form

$$\sup\{r \mid k - r \cdot y_0 \in \mathcal{K}\}, \tag{8}$$

where r is a real number, $r \in R$ (compare also [10], Ch.3, Th.3.4.p.235). Since (8) is formulated with the help of $\mathcal{K}$ defined in the "primal" space Y, to preserve the consistency of terminology we refer to (8) and (7) as the primal and dual problems, respectively.
Since $r_0 = 0$ is feasible for (8), by Proposition 2.1, Ch.3, p.197 of [10], we have

$$0 \leq \sup\{r \mid k - r \cdot y_0 \in \mathcal{K}\} \leq \inf\{\theta^*(k) \mid \theta^*(y_0) = 1, \ \ \theta^* \in \mathcal{K}^*\}. \tag{9}$$

Suppose now that for a given $k \in \mathcal{K}^i$

$$\inf\{\theta^*(k) \mid \theta^*(y_0) = 1 \quad \theta^* \in \mathcal{K}^*\} = \bar{r} \geq 0\,.$$

Hence, for any $\theta^*(y_0) = 1\,, \quad \theta^* \in \mathcal{K}^*$, we have

$$\theta^*(k) \geq \bar{r}$$

which entails that $k - \bar{r}y_0 \in \mathcal{K}$ and

$$\bar{r} \leq \sup\{r \mid k - r \cdot y_0 \in \mathcal{K}\}$$

which proves that

$$\sup\{r \mid k - r \cdot y_0 \in \mathcal{K}\} = \inf\{\theta^*(k) \mid \theta^*(y_0) = 1\,, \quad \theta^* \in \mathcal{K}^*\}\,. \tag{10}$$

The function

$$q(k) = \sup\{r > 0 \mid r^{-1}k \in y_0 + \mathcal{K}\}\,,$$

has been also considered in other context (see Namioka[61]). It is superlinear, and the graph of q,

$$Graph(q) = \{(k, r) \mid q(k) \geq r\}$$

is a cone in $Y \times R$. Now the question arises when the optimal value $\bar{r}$ is nonzero. Clearly, if, for any $y_0 \in \mathcal{K}^i$ and any $k \in \mathcal{K}^i$, it would be $r > 0$ such that $k - ry_0 \in \mathcal{K}^i$, then $\mathcal{K}^i \subset \mathrm{cor}_{\mathcal{K}^i \cup (-\mathcal{K}^i)}(\mathcal{K}^i)$, ie., each $k \in \mathcal{K}^i$ belongs to the *core of* $\mathcal{K}^i$ *relative to* $\mathcal{K}^i \cup (-\mathcal{K}^i)$. It is easy to point out examples when $\bar{r} = 0$.

Example 2.3.2 *Let $p > 1\,, Y = \ell^p\,, \mathcal{K} = \ell^p_+$. As we observed before*

$$(\ell^p_+)^i = \{(s_i) \in \ell^p \mid s_i > 0 \quad \text{for each} \quad i \in N\}\,.$$

By taking $y_0 = (\frac{1}{i^2})$, and $k_0 = (\frac{1}{i^3})$, we see that for any $r > 0$ there exists an index I such that

$$\frac{1}{i^3} - r\frac{1}{i^2} < 0 \quad \text{for} \quad i > I\,,$$

and hence $\bar{r} = 0$.

Let

$$\mathcal{K}^{ii}_{y_0} = \{k \in \mathcal{K}^i \mid \inf\{\theta^*(k) \mid \theta^*(y_0) = 1, \ \ \theta^* \in \mathcal{K}^*\} > 0\}.$$

This is a nonempty subset of $\mathcal{K}^i$ since $y_0 \in \mathcal{K}^{ii}_{y_0}$. We have

$$\mathcal{K}^{ii}_{y_0} = \bigcup_{\delta>0}\{k \in \mathcal{K}^i \mid \inf\{\theta^*(k) \mid \theta^*(y_0) = 1, \ \ \theta^* \in \mathcal{K}^*\} \geq \delta\} = \bigcup_{\delta>0}\mathcal{K}^{ii}_{y_0}(\delta).$$

By (10), $k \in \mathcal{K}^{ii}_{y_0}(\delta)$ if and only if $k \in \delta \cdot y_0 + \mathcal{K}$. Now we can rewrite (WCP) property as follows: for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in A(\varepsilon)$ there exists $\eta_y \in Min(A|\mathcal{K})$ such that

$$y - \eta_y \in \mathcal{K}^{ii}_{y_0}(\delta),$$

or equivalently,

$$y - \eta_y \in \delta \cdot y_0 + \mathcal{K}. \tag{11}$$

When Y is an order complete vector lattice of minimal type (see [80], Ch.V, p.213), any point $k \in \mathcal{K}^i$ is proved to be a **quasi-interior** point of $\mathcal{K}$, where $k \in \mathcal{K}$ is said to be a quasi-interior point of $\mathcal{K}$ if the order interval $[0, k]$ is a total subset of Y in the sense that its linear hull is dense in Y (see Schaefer [80], Ch V. 8, Th.7.7, and Peressini [65],Ch.4.4). Moreover, each $k \in \mathcal{K}^i$ is a **weak order unit** (see [65]), ie., for each $y \in \mathcal{K}$ there exists $z \in \mathcal{K}$ with $z \leq y$ and $z \leq k$.

The role of $\mathcal{K}^i$ is similar to that of int$\mathcal{K}$ in the case when the latter is nonempty. To exploit this analogy we define **quasi-weakly minimal points** of a subset $A \subset Y$, $QWMin(A|\mathcal{K})$, as follows

$$QWMin(A|\mathcal{K}) = \{a \in A \mid (A - a) \cap (-\mathcal{K}^i) = \emptyset\}.$$

2.4 Upper Hausdorff continuity of minimal points for cones with possibly empty interiors via weak containment property

In the present section we use weak containment property (WCP) to give sufficient conditions for upper Hausdorff continuity of M (Theorem 2.4.1). Next, we modify (WCP) so as to avoid the necessity of assuming the weak$-*-$compactness of the dual cone base Θ^*, which is the case in Theorem 2.4.1, and we prove Theorem 2.4.2.

A subset F of Y^* is **equicontinuous** ([42],12.D) if for any $\varepsilon > 0$ there exists a $0-$neighbourhood W such that

$$|f(W)| < \varepsilon$$

for any $f \in F$. Equivalently, there exists a balanced $0-$neighbourhood W such that

$$f(W) \leq 1$$

for each $f \in F$. According to the definition of the polar set A^o to a given set A, F is equicontinuous if and only if

$$F \subset W^o$$

for a balanced $0-$neighbourhood W. By Banach-Alaoglu theorem, W^o is relatively weakly$-*-$compact. When Y is a normed linear space, $F \subset Y^*$ is equicontinuous if and only if it is bounded in the norm topology of Y^*.

Proposition 2.4.1 *Let Y be a locally convex space. Let K be a closed convex pointed cone, int$\mathcal{K} \neq \emptyset$, and let $\mathcal{K}^*$ have an equicontinuous base. Then, for any subset $A \subset Y$, (CP) holds for A if and only if (WCP) holds for A.*

Proof. Suppose that (WCP) holds for A. Let W be a $0-$neighbourhood. There exists $\delta > 0$ such that for any $y \in A(W)$ there exists $\eta_y \in Min(A|\mathcal{K})$ satisfying

$$\theta^*(y - \eta_y) \geq \delta, \quad \text{for} \quad \theta^* \in \Theta^*.$$

Since Θ^* is equicontinuous, there exists a $0-$neighbourhood O such that $|\theta^*(q)| < \delta$ for $q \in O$, $\theta^* \in \Theta^*$. Hence,

$$\theta^*(y - \eta_y) \geq \delta > \theta^*(q),$$

and finally

$$\theta^*(y - \eta_y) + \theta^*(q) \geq \delta.$$

By Proposition **??**, the assertion follows.

□

Theorem 2.4.1 *Let U be a topological space and let Y be a Hausdorff locally convex topological vector space. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y and let $\mathcal{K}^*$ have an equicontinuous base Θ^*. If*

(i) Γ is upper Hausdorff continuous at u_0,

(ii) Γ is $\mathcal{K}-$lower continuous at u_0, uniformly on $Min(\Gamma(u_0)|\mathcal{K})$,

(iii) (WCP) holds for $\Gamma(u_0)$,

the minimal point multivalued mapping M is upper Hausdorff continuous at u_0.

Proof. Let W_1, W be $0-$neighbourhoods, $W_1 + W_1 \subset W$. By (iii), there exists $\delta > 0$ such that for $y \in \Gamma(u_0)(W_1)$ there exists $\eta_y \in Min(\Gamma(u_0)|\mathcal{K})$ satisfying

$$\theta^*(y - \eta_y) > \delta, \qquad (*)$$

for each $\theta^* \in \Theta^*$. By the equicontinuity of the base Θ^*, there exists a $0-$neighbourhood O such that

$$-\delta/4 < \theta^*(q) < \delta/4$$

for any $\theta^* \in \Theta^*$ and $q \in O$.
By (i), there exists a neighbourhood U_0 of u_0 such that

$$\begin{aligned} \Gamma(u) &\subset \Gamma(u_0) + W_1 \cap O \\ &\subset [\Gamma(u_0)(W_1) + W_1 \cap O] \\ &\cup [Min(\Gamma(u_0)|\mathcal{K}) + W_1 + W_1 \cap O], \end{aligned}$$

for $u \in U_0$.

By (ii), there exists a neighbourhood U_1 of u_0 such that

$$(\eta + O - \mathcal{K}) \cap \Gamma(u) \neq \emptyset, \quad \text{for } \eta \in Min(\Gamma(u_0)|\mathcal{K}), \quad u \in U_1 \ .(**)$$

We show that

$$M(u) \cap [\Gamma(u_0)(W_1) + W_1 \cap O] = \emptyset \quad (***)$$

for $u \in U_0 \cap U_1$. To this aim take any

$$y \in \Gamma(u) \cap [\Gamma(u_0)(W_1) + W_1 \cap O], \quad u \in U_0 \cap U_1 .$$

We have $y = \gamma + w$, $\gamma \in [\Gamma(u_0)(W_1)]$, $w \in W_1 \cap O$. By $(*)$, there exists $\eta_\gamma \in Min(\Gamma(u_0)|\mathcal{K})$ such that $\theta^*(\gamma - \eta_\gamma) > \delta$, for each $\theta^* \in \Theta^*$. By $(**)$, there exists $\gamma_1 \in \Gamma(u)$ such that $\gamma_1 = \eta_\gamma - w_1 - k^1$, $w_1 \in O$, $k^1 \in \mathcal{K}$. Thus,

$$\begin{aligned} \theta^*(y - \gamma_1) &= \theta^*(y - \gamma) + \theta^*(\gamma - \eta_\gamma) + \theta^*(\eta_\gamma - \gamma_1 - k^1) + \theta^*(k^1) \\ &> -\delta/4 + \delta - \delta/4 > \delta/2 . \end{aligned}$$

Consequently, $f(y - \gamma_1) > 0$ for any $f \in \mathcal{K}^* \setminus \{0\}$, and hence, $y - \gamma_1 \in \mathcal{K}^i \subset \mathcal{K}$. This implies $(* * *)$, which proves the assertion.

□

The following example shows that the above Theorem cannot be applied for some cones in finite-dimensional space.

Example 2.4.1 *Let $K \subset R^n$ be a convex closed cone in R^n with empty interior. Then $K^* \subset R^n$ has no base since the set $K^T = \{y \in K^* \mid y \cdot x = 0 \quad for\ each\ x \in K\}$ is a nontrivial linear subspace contained in K^* .*

In the definition of weak containment property and in the proof of Theorem 2.4.1 only two properties of base Θ^* are essential. Namely, in Definition 2.3.1 we use the fact that $0 \notin w - * - cl\Theta^*$, (since otherwise there would be no sets with (WCP) property), and in the proof of Theorem 2.4.1 we use the fact that Θ^* has the representation property given below. We do not exploit the convexity of Θ^* .

The assumption of equicontinuity of base Θ^* is restrictive. The cone of nonnegative elements in $L^p(\Omega)$, $1 < p < +\infty$, does not have an equicontinuous base since it does not have a bounded base (see [34]).

Below we propose to replace Θ^* with another set D which satisfies the relation

$$\{y \in Y \mid f(y) \geq 0 \text{ for all } f \in D\} \subset \mathcal{K} .$$

We start with the following definition.

Definition 2.4.1 *The set $D \subset Y^*$ has* **the representation property,** *whenever $y \in Y$ and $f(y) \geq 0$ for all $f \in D$, then $y \in \mathcal{K}$.*

In view of Proposition 2.3.4 one can choose sets D with the representation property basing on part (i) or (iii) of Proposition 2.3.4. In the case (i) any set D with the representation property is a subset of $\mathcal{K}^*$ and, in the case (iii), D is a subset of $\mathcal{K}^{*i}$.
Let $Y = (Y, \|\cdot\|)$ be a normed space and let $\mathcal{K}$ be a closed convex based cone in Y. By B^* we denote the unit ball in Y^*,

$$B^* = \{f \in Y^* \mid \|f\| \leq 1\},$$

and by $\mathcal{K}^{*i}$ the quasi-interior of $\mathcal{K}^*$, (see [44])

$$\mathcal{K}^{*i} = \{f \in \mathcal{K}^* \mid f(y) > 0 \text{ for all } y \in \mathcal{K} \setminus \{0\}\},$$

which is nonempty whenever $\mathcal{K}$ is based.

It was shown in [34] that for cones of nonnegative elements in R^n, ℓ^p, $1 \leq p \leq \infty$, L^p, $1 \leq p \leq \infty$, $C[0,1]$ the set $D = \mathcal{K}^{*i} \cap B^*$ has the representation property. This corresponds to case (iii) of Proposition 2.3.4.

In a locally convex space, if $\mathcal{K}$ is convex closed, and hence weakly closed, we have (see Lemma 8.6 of [54])

$$\mathcal{K} = \mathcal{K}^{**} = \{y \in Y \mid f(y) \geq 0 \text{ for each } f \in \mathcal{K}^*\}.$$

In consequence, if $y \notin \mathcal{K}$, there exists $f \in \mathcal{K}^*$ such that $f(y) < 0$, and, for

$$\bar{f} = f/\|f\| \in \bar{D} = \mathcal{K}^* \cap B^*,$$

we get $\bar{f}(y) < 0$ and

$$\begin{aligned} &\{y \in Y \mid f(y) \geq 0 \text{ for all } f \in \bar{D}\} = \\ &= \{y \in Y \mid f(y) \geq 0 \text{ for all } f \in \mathcal{K}^*\} = \mathcal{K}. \end{aligned}$$

Let $0 < \kappa < 1$. Define

$$D(\kappa) = \mathcal{K}^* \cap \{f \in Y^* \mid \kappa \leq \|f\| \leq 1\}.$$

Proposition 2.4.2 *Let Y be a normed space and let $\mathcal{K} \subset Y$ be a closed convex cone in Y with $\mathcal{K}^{*i} \neq \emptyset$. For any $0 < \kappa < 1$ the set*

$$D(\kappa) = \bar{D} \cap \{f \in Y^* \mid \kappa \leq \|f\|\}$$

has the representation property, ie, if $f(y) \geq 0$ for each $f \in D(\kappa)$, then $y \in \mathcal{K}$.

Proof. Take any $y \notin \mathcal{K}$. Since $\mathcal{K}$ is weakly closed, there exists $f \in \mathcal{K}^*$ such that $f(y) < 0$. Then $f_1 = \kappa \frac{f}{\|f\|} \in D(\kappa)$ and $f_1(y) < 0$.

□

Definition 2.4.2 *The* $\kappa-$**weak containment property** ($\kappa-WCP$) *holds for A if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $y \in A \setminus B(Min(A|\mathcal{K}), \varepsilon)$ there exists $\eta_y \in Min(A|\mathcal{K})$ satisfying*

$$\theta^*(y - \eta_y) > \delta$$

for each $\theta^* \in D(\kappa)$.

Let us note that the set $D(\kappa)$ is norm bounded and hence equicontinuous.

Theorem 2.4.2 *Let U be a topological space and let Y be a normed space. Let $\mathcal{K} \subset Y$ be a closed convex cone in Y with $\mathcal{K}^{*i} \neq \emptyset$. If*

(i) Γ is upper Hausdorff continuous at u_0,

(ii) Γ is $\mathcal{K}-$lower continuous at u_0, uniformly on $Min(\Gamma(u_0)|\mathcal{K})$,

(iii) $(\kappa - WCP)$ holds for $\Gamma(u_0)$,

then M is upper Hausdorff continuous at u_0.

Proof. The proof follows the same lines as the proof of Theorem 2.4.1 with Θ^* replaced by $D(\kappa)$.

□

2.5 Upper Hausdorff continuity of minimal points for cones with possibly empty interiors via approximation.

Let Y be a locally convex space and let $\mathcal{K}$ be a closed convex cone. By $N(0)$ we denote a family of all balanced convex $0-$neighbourhoods. Following [35] we introduce the **expansion**, or **Henig delating** cones of $\mathcal{K}$. Let Θ be a base of $\mathcal{K}$. Since $0 \notin cl\Theta$, there exists a balanced convex open neighbourhood $V_0 \in N(0)$ such that $V_0 \cap (\Theta + V_0) = \emptyset$. Let $N_1 = \{V \in N(0) \mid V \subset V_0\}$. For $V \in N_1(0)$, define

$$\mathcal{K}_V = cl\ cone(\Theta + V).$$

For any $V \in N_1(0)$, $\mathcal{K}_V$ is a closed convex cone. We have

$$\mathcal{K}_V \neq Y, \quad \mathcal{K} \setminus \{0\} \subset \text{int}\mathcal{K}_V, \quad \bigcap_{V \in N_1(0)} \mathcal{K}_V = \mathcal{K}. \tag{12}$$

For any $u \in U$, $V \in N_1(0)$, denote

$$M(u,V) = Min(\Gamma(u)|\mathcal{K}_V).$$

Let $A \subset Y$ be a subset of Y. For any $V \in N_1(0)$, $W \in N(0)$ denote

$$A(V,W) = A \setminus [Min(A|\mathcal{K}_V) + W].$$

Definition 2.5.1 *The* **approximate containment property,** (ACP) *holds for A if for every $V \in N_1(0)$ the containment property (CP) holds for A with respect to cone $\mathcal{K}_V$, ie., for any $W \in N(0)$ and any $V \in N_1(0)$ there exists $O_V \in N(0)$ such that*

$$A(W,V) + O_V \subset Min(A|\mathcal{K}_V) + \mathcal{K}_V.$$

Theorem 2.5.1 *Let Y be a locally convex topological vector space and let $\mathcal{K} \subset Y$ be a closed convex and based cone. Let U be a topological space. If*

(i) Γ is upper Hausdorff continuous at u_0,

(ii) Γ is $\mathcal{K}-$lower continuous at u_0, uniformly on $Min(\Gamma(u_0)|\mathcal{K})$,

(iii) (ACP) holds for $\Gamma(u_0)$,

then M is upper Hausdorff continuous at u_0.

Proof. Let W, W_1 be 0−neighbourhoods, and $W_1 + W_1 \subset W$. By Theorem 2.2.1, for any $V \in N_1(0)$ there exists a neighbourhood U_0 of u_0 such that

$$M(u,V) \subset M(u_0,V) + W_1$$

for $u \in U_0$. By examining the proof of Theorem 2.2.1, a neighbourhood U_0 is chosen independently of $\mathcal{K}$. Now, by the uniform lower Hausdorff continuity of $M(u,V)$ around u_0 with respect to V, there exists $V_0 \in N_1(0)$ such that

$$M(u) \subset M(u,V) + W_1 \quad \text{for} \quad V \subset V_0 \quad u \in U_0 .$$

Finally,

$$M(u) \subset M(u,V) + W_1 \subset M(u_0,V) + W_1 + W_1 \subset M(u_0) + W .$$

□

2.6 Multiobjective optimization problems

In this section we consider multiobjective optimization problems

$$\mathcal{K} - Min\ f(x)$$
$$s.t.\ \ x \in A_0,$$

where $f = (f_1, ..., f_m) : R^n \to R^m$, $A_0 \subset R^n$, $\mathcal{K} \subset R^m$ a closed convex and pointed cone.

Theorem 2.6.1 *If f_i, $i = 1, ..., m$ are linear functions,*

$$A_0 = \{x \in R^n \mid \langle b_i, x\rangle \le c_i\,,\ i \in I\},$$

and $Min(f(A_0)|\mathcal{K}) \neq \emptyset$, $Min(f(A_0)|\mathcal{K}) = WMin(f(A_0)|\mathcal{K})$, then $f(A_0)$ has the containment property (CP).

Proof. It is enough to observe that $f(A_0)$ is a polyhedral set and apply Theorem 2.1.3.

□

Theorem 2.6.2 *Suppose that f_i, $i = 1, ..., m$, are linear, $A_0 \subset R^n$ is convex, and $Min(f(A_0)|\mathcal{K}) \neq \emptyset$. If $Min(A|\mathcal{K})$ is compact, then $f(A_0)$ has the containment property (CP).*

Proof. It is enough to note that $f(A_0)$ is convex, since $f(\lambda a_1+(1-\lambda)a_2) = \lambda f(a_1) + (1-\lambda)f(a_2) \in f(A_0)$, and apply Theorem 5.3.3.

□

Consider now parametric multiobjective problems

$$\mathcal{K} - Min\ f(u,x)$$
$$s.t.\ \ x \in A(u),$$

where $f : U \times R^n \to R^m$, is a continuous function on $U \times R^n$, with U being a topological space, $A : U \to R^n$ is a multivalued mapping.

Now we apply Theorem 2.2.1 to the above parametric problem. We start with the following stability results.

Theorem 2.6.3 *Let $f = (f_1, ..., f_m) : U \times R^n \to R^m$ be linear with respect to $x \in R^m$ and let $A : U \to R^n$ be a feasible set multifunction given by a system of inequalities*

$$A(u) = \{x \in R^n \mid g_j(u,x) \leq 0 \ \ j \in J\},$$

where, for each $j \in J$, the function $g_j(u_0, \cdot) : R^n \to R$ is convex. If

$\Gamma : U \to R^m$, $\Gamma(u) = f(u, A(u))$, is u.H.c. and l.c. at u_0,

$Min(\Gamma(u_0)|\mathcal{K})$ is nonempty and compact, $Min(\Gamma(u_0)|\mathcal{K}) = WMin(\Gamma(u_0)|\mathcal{K})$,

then $M(u) = Min(\Gamma(u)|\mathcal{K})$ is u.H.c. at u_0.

Proof. Since f is linear and $g_j(u_0, \cdot)$, $j \in J$, are convex, the set $\Gamma(u_0) = f(u_0, A(u_0))$ is convex. By Theorem 2.1.1, and 2.6.2, (CP) holds for $\Gamma(u_0)$. By Theorem 2.2.1, the conclusion holds.

□

Now consider a feasible set multifunction $A : U \to R^n$ given by a system of inequalities of the form

$$A(u) = \{x \in R^n \mid g_j(u,x) \leq 0 \ \ j \in J\}, \qquad (13)$$

where, for each $j \in J$, $g_j : U \times R^n \to R$ is a linear function with respect to x, $g_j(u,x) = \langle b_j(u), x\rangle - c_j(u)$, $j \in J$, $b_j : U \to R^n$,

$c_j : U \to R$. Lower and upper semicontinuities of the multifunction A have been investigated by many authors (see [?]). According to Propositions 4.1. and 4.2 of [?], in order to apply Theorem 2.2.1 we need sufficient conditions for u.H.c. and uniform lower semicontinuity of A.

Let $F, F_0, F_1 : U \to R^n$, be multifunctions,

$$F(u) = F_1(u) \cap F_0(u),$$

and

$$F_0(u) = \{x \in R^n \mid g_j(u,x) < 0 \quad j \in J_0\}$$
$$F_1(u) = \{x \in R^n \mid g_j(u,x) = 0 \quad j \in J_1\},$$

where, for $j \in J = J_0 \cup J_1$ the functions $g_j(u,x)$ are continuous on $U \times R^n$, and, for each $u \in U$, the functions $g_j(u,\cdot)$ are convex functions on R^n.

The following result has been proved in [?].

Theorem 2.6.4 ([?],Th 3.2.2.) *If*

(i) $F_1(u) \neq \emptyset$ *for* $u \in U$,

(ii) $F_1(u)$ *forms an affine set in* R^n,

(ii) *dim lin*$F_1(u) =$ *dim lin*$F_1(u_0)$ *for* $u \in U$.

then F_1 *and* F *are l.s.c. at* u_0.

We prove the following auxiliary lemma.

Lemma 2.1 *If a multifunction* $A : U \to R^n$, *of the form (13) is lower semicontinuous at* u_0, *the functions* $b_j : U \to R^n$, *and* $c_j : U \to R$ *are continuous for* $j \in J$, *then* A *is uniformly lower semicontinuous.*

Proof. Let $x_0 \in A(u_0)$, ie., $\langle b_j(u_0), x_0\rangle = c_j(u_0)$ $j \in J_1$, $\langle b_j(u_0), x_0\rangle < c_j(u_0)$ $j \in J_2$. By the lower semicontinuity

$$\forall_Q \exists_{W_0} \forall_{u \in W_0} \quad (Q + x_0) \cap A(u) \neq \emptyset,$$

ie., there is a $x_u \in (Q+x_0) \cap A(u)$, $u \in W_0$, and moreover x_u can be chosen in such way that $\langle b_j(u), x_u\rangle = c_j(u)$ $\quad j \in J_1$, $\langle b_j(u), x_u\rangle < c_j(u)$ $\quad j \in J_2$.

Let us take any $x_1 \in A(u_0)$ such that $x_1 \in \mathcal{A}(u_0) = \{x \in A(u_0)\ \exists U_1$ of u_0 $\langle b_j(u) - b_j(u_0), x_1 - x_0 \rangle \leq 0$.

$$\langle b_j(u_0), x_1 \rangle = c_j(u_0) \quad j \in J_1 \quad \langle b_j(u_0), x_1 \rangle < c_j(u_0) \quad j \in J_2 .$$

and consider $y_u = x_1 + [x_u - x_0]$, for $u \in W_0$.

To prove the inequality $\langle b_j(u), y_u \rangle \leq c_j(u)$ for $j \in J$ and $u \in W_0$ we need to show that

(I) for any $j \in J_1$, $\langle b_j(u), x_u \rangle = c_j(u) \Rightarrow \langle b_j(u), [x_1 - x_0] \rangle \leq 0$,

(II) for any $j \in J_2$, $\langle b_j(u), x_u \rangle < c_j(u) \Rightarrow \langle b_j(u), [x_1 - x_0] \rangle \leq c_j(u) - \langle b_j(u), x_u \rangle$,

Let us show (I). By assumption, $\langle b_j(u), x_u \rangle = c_j(u)$, and hence, by continuity of b_j and c_j it must be $j \in J_1$. Thus, $\langle b_j(u_0), x_0 \rangle = c_j(u_0)$, $\langle b_j(u_0), x_1 \rangle = c_j(u_0)$, and $\langle b_j(u_0), [x_1 - x_0] \rangle = 0$. We have

$$\begin{aligned}\langle b_j(u), y_u \rangle &= \langle b_j(u), x_u \rangle + \langle b_j(u), [x_1 - x_0] \rangle \\ &= c_j(u) + \langle b_j(u_0), [x_1 - x_0] \rangle + \langle [b_j(u) - b_j(u_0)], [x_1 - x_0] \rangle \\ &\leq c_j(u) .\end{aligned}$$

This proves I.

To show (II) suppose that $\langle b_j(u), x_u \rangle < c_j(u)$, and $\langle b_j(u_0), x_0 \rangle - c_j(u_0) = \alpha < 0$. Then by taking any $\alpha_1 > 0$ such that $\alpha + \alpha_1 < 0$ one can find a neighbourhood U_1 of u_0 such that

$$\langle b_j(u), x_u \rangle - c_j(u) < \alpha + \alpha_1 < 0 .$$

On the other hand, there exists a neighbourhood U_2 of u_0 such that

$$\begin{aligned}\langle b_j(u), [x_1 - x_0] \rangle &= \langle b_j(u), x_u \rangle - \langle b_j(u), x_u \rangle + \langle b_j(u), [x_1 - x_0] \rangle \\ &= \langle b_j(u), [x_u - x_0] \rangle - \langle b_j(u), x_u \rangle + \langle b_j(u_0), x_1 \rangle - \langle b_j(u_0), x_1 \rangle + \langle b_j(u), \\ &\langle b_j(u), [x_u - x_0] \rangle - c_j(u) + c_j(u_0) + \langle [b_j(u) - b_j(u_0)], x_1 \rangle .\end{aligned}$$

Let $j \in J_2$, ie., $\langle b_j(u_0), x_0 \rangle < c_j(u_0)$. Then for all u in some neighbourhood U_4 of u_0, $c_j(u) = \langle b_j(u), x_u \rangle < c_j(u_0)$. If $j \in J_3$, ie., $\langle b_j(u_0), x_1 \rangle = c_j(u_0)$, then

$$\begin{aligned}\langle b_j(u), x_u \rangle &= \langle b_j(u_0), x_0 \rangle + \langle b_j(u_0), [x_u - x_0] \rangle + \langle [b_j(u) - b_j(u_0)], x_u \rangle \\ &= c_i(u_0) + \langle b_j(u_0), [x_u - x_0] \rangle + \langle [b_j(u) - b_j(u_0)], x_u \rangle \leq c_j(u)\end{aligned}$$

□

Theorem 2.6.5 *Let $f = (f_1, ..., f_m) : U \times R^n \to R^m$ be a linear function of $x \in R^m$ and let $A : U \to R^n$ be a feasible set multifunction given by a system of inequalities*

$$A(u) = \{x \in R^n \mid g_j(u,x) \leq 0 \quad j \in J\},$$

where, for each $j \in J$, $g_j : U \times R^n \to R$ is a linear function with respect to x, $g_j(u,x) = \langle b_j(u), x\rangle - c_j(u)$, $j \in J$, $b_j : U \to R^n$, $c_j : U \to R$. If

$\Gamma : U \to R^m$, $\Gamma(u) = f(u, A(u))$, is u.H.c. and l.c. at u_0,

$Min(\Gamma(u_0)|\mathcal{K})$ is nonempty, and $Min(\Gamma(u_0)|\mathcal{K}) = WMin(\Gamma(u_0)|\mathcal{K})$,

then $M(u) = Min(\Gamma(u)|\mathcal{K})$ is u.H.c. at u_0.

Proof. According to Theorem 2.2.1 we need to show that Γ, $\Gamma(u) = f(u, A(u))$, is uniformly lower semicontinuous on $Min(\Gamma(u_0)|\mathcal{K})$.

Let $\varepsilon > 0$. According to the assumptions, for $i \in I$, $f_i(u,x) = \langle f_i(u), x\rangle$,

$$|\langle f_i(u), x\rangle - \langle f_i(u_0), x_0\rangle| \leq |\langle f_i(u), x\rangle - \langle f_i(u_0), x\rangle| + |\langle f_i(u_0), x\rangle - \langle f_i(u_0), x_0\rangle|,$$

and there exists a neighbourhood U_0 of u_0 and a $0-$ neighbourood $V = B(0, M)$, in R^n, $M = \min_{i \in I}\{M_i\}$, where $M_i < \frac{\varepsilon}{2\|f_i(u_0)\|}$, such that

$$|\langle f_i(u), x\rangle - \langle f_i(u_0), x\rangle| \leq \|f_i(u_0) - f_i(u)\| \, \|x\| < \varepsilon/2,$$

for $u \in U_0$, and

$$|\langle f_i(u_0), x\rangle - \langle f_i(u_0), x_0\rangle| \leq \|f_i(u_0)\| \, \|x - x_0\| < \varepsilon/2$$

for $x \in x_0 + V$. By Lemma 2.1, A is uniformly lower semicontinuous at u_0. Hence, there exists a neighbourhood U_1 of u_0 such that for each $x_0 \in A(u_0)$

$$(x_0 + B(0,M)\,) \cap A(u) \neq \emptyset \quad u \in U_1,$$

ie., there exists $x_u \in (x_0 + B(0,M)\,) \cap A(u)$. Now, for $u \in U_0 \cap U_1$

$$\|f(u,x_u) - f(u_0,x_0)\| \leq \|f(u,x_u) - f(u_0,x_u)\| + \|f(u_0,x_u) - f(u_0,x_0)\| < \varepsilon.$$

Since $f(u, x_u) \in \Gamma(u)$, and $y_0 = f(u_0, x_{u_0}) \in \Gamma(u_0)$, the above inequality proves that, for $u \in U_0 \cap U_1$,

$$\Gamma(u) \cap (y_0 + B(0, \varepsilon)\,) \neq \emptyset\,.$$

$\square$

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