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The market model of CDO spreads*

by

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Abstract: In this paper we present a new arbitrage-free bottom-up model of correlated defaults, based on a special approach to systematic and idiosyncratic risks for individual obligors. The model admits several attractive features, like consistency with currency and interest rate models, as well as numerical tractability and flexibility, making it capable to fit the market for practically all self-consistent CDO tranche prices. Its background is rather remote from other approaches, like copulas and point processes, so our presentation is detailed.

Keywords: CDO pricing, Marshall-Olkin copula, bottom-up approach

1. Introduction

There are two general classes of reduced-form models of collateralized debt obligation (CDO) spreads – those based on the copula functions and those based on point processes (Brémaud, 1980). The copula approach seems to be not flexible enough to model the term and capital structure of CDO spreads under various market conditions, while point process modeling is not sufficiently suitable to model bespoke products and individual obligors. We refer to Burtschell, Gregory and Laurent (2009) for an excellent account on copula models, and to Giesecke and Goldberg (2011), Lindskog and McNeil (2001), Longstaff and Rajan (2008), and Brigo, Pallavicini and Torresetti (2007a) for several examples of the point process approach to CDO modeling. Gaussian copula (Li, 2000) is the market standard and is used (or rather misused) in the worst possible way – every tranche for every maturity is priced with different model parameters without caring about model consistency and absence of arbitrage. Several attempts

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have been made to overcome these difficulties: Sidenius (2007) constructs a copula with term structure but for a finite number of tenors only, Brigo, Pallavicini and Torresetti (2007b) incorporate individual names into a top-down model by allowing many defaults of a single obligor.

Our purpose is to construct an arbitrage-free bottom-up market model approach to CDO tranche pricing, i.e. a model:

1. Admitting a natural interpretation of systematic and idiosyncratic risks in terms of individual spreads.
2. Having the ability to fit to all tranche spreads for all tenors.
3. Consistent with equity, currency and interest rate models, hence suitable for hybrid products.
4. Having a natural extension to dynamic modelling and credit options.

We will call the model 'market model' for these attractive features. We follow the general copula concept of systematic and idiosyncratic risks and interpolation between them, but our solution is different – more direct, flexible and intuitive. Because the research is new, all necessary formulas are given in extent. The above properties seem to answer almost all open questions in CDO pricing, although we are aware that it is the judgement of practitioners that decides. Our research is remotely based on an old idea by Giesecke (2003) and is similar in spirit to Balakrishna (2006) and Walker (2007).

2. The model

Let $F_i : R_+ \to [0,1]$ be a family of distribution functions, describing default times τ_i of a number of obligors, i.e. $F_i(t) = P(\tau_i \leqslant t)$ for $1 \leqslant i \leqslant N$. Default times τ_i, on the one hand, may be independent, on the other – co-monotonic. Correlations may vary between perfect and null, imposing changes in prices of CDO tranches. Finding correlations fitting market prices, being a kind of interpolation between independent and co-monotonic case, is one of most important tasks in credit derivatives modelling. The market standard interpolation method is the Gaussian single factor copula (Li, 2000), with the default times being constructed as follows:

$$\tau_i = F_i^{-1}\left(\Phi\left(\alpha_i \Phi^{-1}(X_0) + \Phi^{-1}(X_i)\sqrt{1-\alpha_i^2}\right)\right), \tag{1}$$

where $0 \leqslant \alpha_i \leqslant 1$ and $X_0, X_1, ..., X_d$ is a family of independent random variables uniformly distributed on [0,1], i.e. $P(X_i < t) = P(X_i \leqslant t) = t$. The random variables $X_1, ..., X_d$ are called idiosyncratic factors and the random variable X_0 is called systematic factor. There exist a number of other copulas called after their authors: Gumbel, Clayton, Frank, Student, and many others. The default times in a general single factor copula approach are constructed as follows:

$$\tau_i = F_i^{-1}\left(\Phi_{\alpha_i}\left(Q\left(\alpha_i, X_0, X_i\right)\right)\right),$$

where Q is a continuous function, such that the numbers $0 \leqslant \alpha \leqslant 1$ form an interpolation between independent and co-monotonic border cases, i.e. $Q(0, x, y) =$

y, and $Q(1,x,y) = x$, and Φ_α is a distribution of $Q(\alpha, X_0, X_i)$. The proper choice of the functions Φ_α and Q determines to which class the copula belongs. An interesting special case is the so called random factor loading, where the correlation parameter α is a function (see Andersen and Sidenius, 2005). We refer for details to Burtschell, Gregory and Laurent (2011).

We are strongly convinced that the copula approach is not very suitable for CDO modelling, and we are going to introduce a new approach related to reliability theory (Barlow and Proschan, 1965). First consider the border cases: if $\alpha_i \equiv 0$, then τ_i are independent, and if $\alpha_i \equiv 1$ for all i, then τ_i are co-monotonic. We know from Burtschell, Gregory and Laurent (2009) that co-monotonic defaults minimise the price of equity tranche. In other words, numbers $0 \leqslant \alpha_i \leqslant 1$ form interpolation between independent (high equity price) and co-monotonic (low equity price) border cases, preserving the marginal distribution $F_1(t), ..., F_d(t)$ of the random variables $\tau_1, ..., \tau_d$. We are going to construct a new interpolation method with more freedom in modelling correlation and simpler tranche pricing formulae. Let $G_i : R_+ \to [0,1]$ and $H_i : R_+ \to [0,1]$ be two families of distribution functions satisfying

$$(1 - H_i(t))(1 - G_i(t)) = 1 - F_i(t). \tag{2}$$

Define two families of random variables $Y_1, Y_2, ..., Y_d$ (idiosyncratic factors) and $Z_1, Z_2, ..., Z_d$ (systematic factors) by

$$Y_i = H_i^{-1}(X_i) \text{ and } Z_i = G_i^{-1}(X_0). \tag{3}$$

It should be kept in mind that $X_0, X_1, ..., X_d$ are independent random variables uniformly distributed on [0,1]. We denote by $t \wedge s = min\{t,s\}$. Calculate that F_i is the distribution function of $\tau_i = Y_i \wedge Z_i$:

$$\begin{aligned} &P(\tau_i > t) = P(Y_i \wedge Z_i > t) = P(Y_i > t, Z_i > t) = P(Y_i > t)P(Z_i > t) \\ &= P(X_i > H_i(t))P(X_0 > G_i(t)) = (1 - H_i(t))(1 - G_i(t)) = 1 - F_i(t). \end{aligned} \tag{4}$$

In particular: if $F_i(t) = H_i(t)$, then τ_i is independent of τ_j for any $i \neq j$, if $F_i(t) = G_i(t)$ for all i, then $\tau_i = G_i^{-1}(X_0)$, hence all random variables τ_i are co-monotonic. The economic interpretation of (2) is quite natural – the default may be caused either by external events (systematic) or by the company itself (idiosyncratic).

Define

$$\lambda_i(t) = -\frac{d \ln(1 - F_i(t))}{dt} = \frac{1}{1 - F_i(t)} \frac{dF_i(t)}{dt} \geqslant 0. \tag{5}$$

Obviously

$$F_i(t) = 1 - \exp\left(-\int_0^t \lambda_i(s) ds\right) \text{ and } \frac{dF_i(t)}{dt} = \lambda_i(t)(1 - F_i(t)).$$

The quantity $\lambda_i(t)$ admits natural interpretations as a hazard rate, namely

$$\lambda_i(t)dt = P(t \leqslant \tau_i < t + dt | t < \tau_i).$$

Let $\lambda_i^{sys}(t) \geqslant 0$ and $\lambda_i^{idio}(t) \geqslant 0$ be two hazard rates defined by

$$\lambda_i^{sys}(t) = -\frac{d \ln (1 - G_i(t))}{dt} \quad \text{and} \quad \lambda_i^{idio}(t) = -\frac{d \ln (1 - H_i(t))}{dt}.$$

In other words

$$G_i(t) = 1 - \exp\left(-\int_0^t \lambda_i^{sys}(s) ds\right)$$

and

$$H_i(t) = 1 - \exp\left(-\int_0^t \lambda_i^{idio}(s) ds\right).$$

Obviously,

$$\lambda_i^{sys}(t) + \lambda_i^{idio}(t) = \lambda_i(t).$$

The family of functions

$$0 \leqslant \lambda_i^{sys}(t)/\lambda_i(t) \leqslant 1$$

form an interpolation between the independent and the co-monotonic case with term structure, hence richer than Gaussian copula interpolation, consisting of numbers $0 \leqslant \alpha_i \leqslant 1$. Setting the market model in terms of ratios $\lambda_i^{sys}(t)/\lambda_i(t)$, analogously as for the Gaussian copula, is not practical, because giving the complete capital and term structure would require a massive matrix, therefore other synthetic input data are needed. This topic will be briefly discussed in the last section. Similar construction for Gaussian copula is called 'random factor loading' (Andersen and Sidenius, 2005) and is much more technically complicated.

Obviously, $\tau_i | Z_i$ are conditionally independent and their distribution is

$$P(\tau_i > t | Z_i = z) = P(Y_i \wedge z > t) = (1 - H_i(t)) I(z > t).$$

Define point processes:

$$N(t) = \sum_{i=1}^{d} \mathrm{I}(\tau_i < t) - \text{total number of defaults}, \tag{6}$$

$$M(t) = \sum_{i=1}^{d} \mathrm{I}(Z_i < t) - \text{number of systematic defaults}, \tag{7}$$

and the number of idiosyncratic defaults in the set of names numbered from $j+1$ as

$$N_j(t) = \sum_{i=j+1}^{d} \mathrm{I}\left(Y_i < t\right), \tag{8}$$

where I is the indicator function. If we assume that Z_i are co-monotonic i.e. $Z_i \leqslant Z_{i+1}$, what is equivalent to $G_{i+1} \leqslant G_i$ for all i, then

$$P\left(M(t) = j\right) = P\left(Z_j < t < Z_{j+1}\right) = P\left(G_{j+1}(t) < X_0 < G_j(t)\right) = G_j(t) - G_{j+1}(t), \tag{9}$$

setting $G_0 \equiv 1$. In consequence

$$P\left(M(t) \geqslant i\right) = G_i(t) \tag{10}$$

and

$$P\left(N(t) = m\right) = \sum_{j=0}^{m} P\left(M(t) = j\right) P\left(N_j(t) = m - j\right) = \sum_{j=0}^{m} \left(G_j(t) - G_{j+1}(t)\right) P\left(N_j(t) = m - j\right). \tag{11}$$

Construction of the systematic factor as $Z_i = G_i^{-1}(X_0)$ is counter-intuitive, since the first default $Z_1 = G_1^{-1}(X_0)$ determines all next default times, i.e. $Z_i = G_i^{-1}(G_1(Z_1))$ – and this is not the way markets work. It is not disturbing for static models, but may give rise to a problem while modelling dynamics. There are several ways to construct a family of co-monotonic random variables, i.e. $Z_i \leqslant Z_{i+1}$, with given cumulative distributions G_i without counter-intuitive 'future prediction' property, via structural models, see, for instance Hull, Predescu and White (2005). Our construction is developed in terms of point processes with multiple defaults to keep consistent the point processes approach. Let $V_1, V_2, ..., V_d$ be a family of independent random variables uniformly distributed on [0,1], obviously independent of $X_0, X_1, ..., X_d$. Define

$$Z_d = G_d^{-1}(V_d) = \tilde{G}_d^{-1}(V_d), \tag{12}$$

$$Z_i = Z_{i+1} \wedge \tilde{G}_i^{-1}(V_i) \text{ for } i<d, \tag{13}$$

where $\tilde{G}_i(t)$ is a distribution function defined by

$$\tilde{G}_i(t) = 1 - \exp\left(-\int_0^t \left(\lambda_i^{sys}(s) - \lambda_{i+1}^{sys}(s)\right) ds\right) = 1 - \frac{1 - G_i(t)}{1 - G_{i+1}(t)} = \frac{G_i(t) - G_{i+1}(t)}{1 - G_{i+1}(t)}. \tag{14}$$

Obviously, $Z_i \leqslant Z_{i+1}$,$P\,(Z_d > t) = 1 - G_d(t)$, and, by induction,

$$\begin{aligned} P\,(Z_i > t) = P\left(\tilde{G}_i^{-1}(V_i) \wedge Z_{i+1} > t\right) = P\left(\tilde{G}_i^{-1}(V_i) > t, Z_{i+1} > t\right) \\ = P(V_i > \tilde{G}_i(t))P(Z_{i+1} > t) = (1 - \tilde{G}_i(t))(1 - G_{i+1}(t)) = 1 - G_i(t) \end{aligned} \tag{15}$$

and

$$\begin{aligned} P\,(M(t) = j) = P\,(Z_j < t \leqslant Z_{j+1}) = P\left(\tilde{G}_j^{-1}(V_j) < t \leqslant Z_{j+1}\right) \\ = \tilde{G}_j(t)\,(1 - G_{j+1}(t)) = G_j(t) - G_{j+1}(t). \end{aligned} \tag{16}$$

The so constructed jump process $M(t)$ is Markovian on $\{0, 1, \ldots, d\}$ with transition intensity

$$\begin{gathered} P\,(\,M(t + dt) = j|\,M(t) = i) = \\ \left(\lambda_j^{sys}(t)I(j > i) - \lambda_{j+1}^{sys}(t)\;\; I(j \geqslant i)\right)dt + I(i = j). \end{gathered}$$

We set $\lambda_{d+1}^{sys}(t) \equiv 0$. Probabilities given by the predicting approach coincide with those given by the Markovian one, so both models are indistinguishable from the pricing point of view. Notice that the market model admits multiple defaults, but their probability is very small when compared with single defaults.

Notice the dynamical structure of the model, since there is no transformation of time as in copula models, what makes it consistent with equity, currency and interest rate models, hence suitable for hybrid products. We may easily generalise the market model by making hazard rates $\lambda_i^{sys}(t)$ and $\lambda_i^{idio}(t)$ stochastic processes allowing for consistent modelling CDO options, tranches and tranche options. In such case the conditional default probability is given by the formula

$$P\,(\,\tau_i \geqslant t|\,F_s) = E\left\{ \exp\left(- \int_s^t \left(\lambda_i^{sys}(s) + \lambda_i^{idio}(s)\right) ds \right) \middle|\, F_s \right\} I(\tau_i \geqslant s),$$

where F_t is the natural filtration. Such a model will be investigated in the future, although its practical use is questionable these days – tranche options are practically not traded.

3. Calculation of probabilities

There exists a simple algorithm to calculate $P\,(N_j(t) = m)$ by Hull and White (2004), based on the Bernoulli triangle, namely

$$P\,(N_d(t) = 0) = 1, \quad P\,(N_d(t) = m) = 0 \;\text{ for }\; m > 0,$$

$$P\,(N_{j-1}(t) = 0) = \prod_{i=j}^{d}(1 - H_i(t)) = P\,(N_j(t) = 0)\,(1 - H_j(t))\,,$$

$$P\,(N_{j-1}(t) = m) = P\,(N_j(t) = m - 1)\,H_j(t) + P\,(N_j(t) = m)\,(1 - H_j(t))$$

for $m > 0$. In practice, $P(N_j(t) = m) = 0$ for large (exceeding 30) m. The point process $N_j(t)$ may be as well approximated by a Poisson process $\tilde{N}_j(t)$ with intensity $\sum\limits_{i=j+1}^{d} \lambda_i^{idio}(s)$. Hence,

$$P(N_j(t) = m) \approx P\left(\tilde{N}_j(t) = m\right) = \frac{1}{m!}\left(\int\limits_0^t \sum_{i=j+1}^{d} \lambda_i^{idio}(s)ds\right)^m \exp\left(-\int\limits_0^t \sum_{i=j+1}^{d} \lambda_i^{idio}(s)ds\right) = \frac{1}{m!}\left(-\sum_{i=j+1}^{d} \ln(1 - H_i(t))\right)^m \prod_{i=j+1}^{d} (1 - H_i(t)).$$

The processes $\tilde{N}_j(t)$ are Markovian in the Poisson approximation with transition intensity

$$P\left(\tilde{N}_j(t+dt) = k \middle| \tilde{N}_j(t) = i\right) = \begin{cases} \sum\limits_{i=j+1}^{d} \lambda_i^{idio}(t)dt & \text{if} \quad i = k-1, \\ 1 - \sum\limits_{i=j+1}^{d} \lambda_i^{idio}(t)dt & \text{if} \quad i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $Y_{first} = \min\{Y_i : 1 \leqslant i \leqslant d\}$ the first default of idiosyncratic type, whilst Z_1 is the first default of systematic type. Only systematic defaults may be multiple. Calculate that

$$P(Y_{first} \geqslant t) = \prod_{i=1}^{d} (1 - H_i(t)) = \exp\left(-\int\limits_0^t \sum_{i=1}^{d} \lambda_i^{idio}(s)ds\right)$$

and

$$P(Z_1 \geqslant t) = 1 - G_1(t) = \exp\left(-\int\limits_0^t \lambda_1^{sys}(s)ds\right).$$

Hence

$$\frac{P(t \leqslant Z_1 \leqslant t + dt \,|Z_1 > t)}{P(t \leqslant Y_{first} \leqslant t + dt \,|Y_{first} > t)} = \frac{\lambda_1^{sys}(t)}{\sum\limits_{i=1}^{d} \lambda_i^{idio}(t)},$$

what represents the common market view – multiple defaults may happen but their probability is much smaller than probability of single defaults. This inequality is also an informal proof that idiosyncratic hazard rates are responsible for equity prices, while the systematic hazard rates deal with the more senior tranches, demonstrated more rigorously in Burtschell, Gregory and Laurent (2009).

Let us collect the properties of the market model:

- The process $M(t)$ is Markovian on $\{0, 1, \dots, d\}$,
- The processes $N_j(t)$ are Markovian on all subsets of $\{j+1, j+2, \dots, d\}$,
- There exists a Markovian approximation on $\{j, j+1, j+2, \dots, d\}$ of the process $N_j(t)$,
- Transition probabilities for the process $M(t)$ admit closed formulae,
- Transition probabilities for the processes $N_j(t)$ admit accurate approximate formulae and fast calculation algorithms,
- The model admits single defaults with large probability and multiple defaults with very small probability.

4. Index and tranche pricing

The credit spread *Spread* for the premium index leg is calculated as

$$Spread(T_N) \sum_{i=1}^{N} \delta_i DF(0, T_i) \left(d - EN(T_i)\right) = LGD \cdot ELoss, \tag{17}$$

where T_j are fee payment days, $\delta_j = T_j - T_{j-1}$ is accrual period, $DF(0,t)$ is a discount factor and the expected loss *ELoss* is defined by

$$ELoss = \int_0^{T_N} DF(0,t) EN(dt) =$$

$$\sum_{i=1}^{d} \int_0^{T_N} DF(0,t) F_i(dt) = \sum_{i=1}^{d} \int_0^{T_N} DF(0,t) \lambda_i(t)(1 - F_i(t)) dt.$$

Since $\lambda_i^{sys}(t) + \lambda_i^{idio}(t) = \lambda_i(t)$, we have the following separated representation for the expected loss

$$ELoss = ELoss^{sys} + ELoss^{idio},$$

where

$$ELoss^{sys} = \sum_{i=1}^{d} \int_0^{T_N} DF(0,t) \lambda_i^{sys}(t)(1 - F_i(t)) dt$$

and

$$ELoss^{idio} = \sum_{i=1}^{d} \int_0^{T_N} DF(0,t) \lambda_i^{idio}(t)(1 - F_i(t)) dt.$$

By the formula (17) all spreads may be decomposed into idiosyncratic and systematic parts, as well. Therefore, $\lambda_i^{sys}(t)$ and $\lambda_i^{idio}(t)$ have one more natural economic interpretation – they are just infinitesimal increases of both expected losses $ELoss^{sys}$ and $ELoss^{idio}$, associated with the i-th obligor. The market model may be quoted in terms of both hazard rated and expected losses.

Taking into consideration formula (11) and Section 2 we may easily price the CDO tranches. Assume deterministic recovery rate R and loss-given-default $LGD = 1 - R$ as a fraction of the nominal for all obligors and the notional of every obligor to be equal LGD^{-1}. Define the tranche as $[k,K]$ and let the CDO consist of d obligors with equal notional. The survival amount associated to the tranche $[k,K]$ is defined as

$$S_{kK}(t) = \left(\frac{Kd}{LGD} - N(t)\right)^+ - \left(\frac{kd}{LGD} - N(t)\right)^+. \tag{18}$$

Hence, the expected survival amount is equal

$$ES_{kK}(t) = Q_K(t) - Q_k(t), \tag{19}$$

where

$$Q_k(t) = E\left(\frac{kd}{LGD} - N(t)\right)^+ = \sum_{j=0}^{n(k)} P(N(t) = j)\left(\frac{kd}{LGD} - j\right),$$

$$n(k) = \max\{n : LGD \cdot n < kd\}.$$

The upfront payment *Upfront* and credit spread *Spread* for the premium leg are calculated as

$$\begin{aligned} &Spread_{kK}(T_N)\sum_{i=1}^{N}\delta_i DF(0,T_i)ES_{kK}(T_i) + \\ &Upfront\frac{d(K-k)}{LGD} + \int_0^{T_N} DF(0,t)ES_{kK}(dt) = 0. \end{aligned} \tag{20}$$

This result reflects the picture of a CDO tranche as a portfolio of CDS swaps. The following trapezoidal approximation is commonly used

$$\begin{aligned} &-\int_{T_j}^{T_{j+1}} DF(0,t)EA_{kK}(dt) \approx \\ &\frac{DF(0,T_j) + DF(0,T_{j+1})}{2}\left(EA_{kK}(T_j) - EA_{kK}(T_{j+1})\right). \end{aligned}$$

5. Examples and Markov approximation

The number of potential calibration parameters is enormous – 6 tenors for 125 obligors are quoted by the market, what makes possible 750 partition parameters of hazard rates into systematic and idiosyncratic parts. We propose here two natural ways of aggregating these data in order to avoid overfitting:

Example 1 *Marshall-Olkin copula (shock model)*

An important special case, when $G_i = G_1$ for all i, is called Marshall-Olkin copula or "shock model" and represents the systematic factor of catastrophic character – forcing defaults of all obligors at the same time Z_1. For the shock model the number of defaults formula takes the form

$$P\left(N(t) \geqslant m\right) = \left(1 - G_1(t)\right) P\left(N_0(t) \geqslant m\right) + G_1(t).$$

Example 2 *Stepwise systematic hazard rate*

In this case $G_i = G_{i+1}$ for $1 \leqslant i < k_j$ and $1 \leqslant j \leqslant g$. In other words, there is only a small number of systematic distributions and obligors default in clusters. The Marshall-Olkin copula is a special case of this example.

There exists a Markov approximation of the process $N(t)$. Remember that the process of systematic defaults $M(t)$ is Markov with transition probability

$$P\left(M(T_{j+1}) = n \,|M(T_j) = i\,\right) \approx \begin{cases} 0 \qquad for\; n < i, \\ 1 - \sum\limits_{k=i+1}^{d} \left(\tilde{G}_k(T_{j+1}) - \tilde{G}_k(T_j)\right) for\; n = i, \\ \tilde{G}_n(T_{j+1}) - \tilde{G}_n(T_j)\; for\; n > i. \end{cases}$$

If all idiosyncratic spreads are identical i.e. $H_i = H_1$ for all i, then the process of all idiosyncratic defaults

$$N_0(t) = \sum_{i=1}^{d} \mathrm{I}\left(Y_i < t\right)$$

is Markov with transition probability

$$P\left(N_0(T_{j+1}) = n \,|N_0(T_j) = i\,\right) \approx \begin{cases} 0\; for\; n < i\; and\; n > i+1, \\ 1 - (d-i)\left(H_1(T_{j+1}) - H_1(T_j)\right)\; for\; n = i, \\ (d-i)\left(H_1(T_{j+1}) - H_1(T_j)\right)\; for\; n = i+1. \end{cases}$$

We may construct the process $N(t)$ as a composition of processes $M(t)$ and $N_0(t)$, giving its transition probability

$$\begin{aligned} &P\left(N(T_{j+1}) = n \,|N(T_j) = i\,\right) \\ &= P\left(M(T_{j+1}) = n \,|M(T_j) = i\,\right) + P\left(N_0(T_{j+1}) = n \,|N_0(T_j) = i\,\right) - I(i = n). \end{aligned}$$

This approximation slightly underestimates the expected loss.

6. Multifactor case

We need to introduce more factors to assure a better fit of the market, especially under stressed conditions. This construction is analogous to the one already introduced, although some extra notation is needed, what may make the presentation less clear.

Let $U_1, U_2, ..., U_k, \left(X_j^i\right)_{1\leqslant j\leqslant d_i, 1\leqslant i\leqslant k}$ be a family of independent random variables on [0,1], uniformly distributed. Let $G_i^j : R_+ \to [0,1]$, $H_i : R_+ \to [0,1]$ and $F_i : R_+ \to [0,1]$ be three families of distribution functions such that

$$(1 - G_i^j(t))(1 - H_i^j(t)) = 1 - F_i^j(t).$$

Define two families of random variables $\left(Y_j^i\right)_{1\leqslant j\leqslant d_i, 1\leqslant i\leqslant k}$ (idiosyncratic factors) and $\left(Z_j^i\right)_{1\leqslant j\leqslant d_i, 1\leqslant i\leqslant k}$ (k systematic factors) by

$$Y_j^i = \left(H_j^i\right)^{-1}(X_j^i) \tag{21}$$

and $Z_j^i = \left(G_j^i\right)^{-1}(U_i)$.

Calculate that F_j^i is the distribution function of $\tau_j^i = Y_j^i \wedge Z_j^i$:

$$\begin{aligned} P\left(\tau_j^i > t\right) &= P\left(Y_j^i > t, Z_j^i > t\right) = P(Y_j^i > t)P(Z_j^i > t) \\ &= (1 - H_j^i(t))(1 - G_j^i(t)) = 1 - F_j^i(t). \end{aligned} \tag{22}$$

Define point processes:

$$N(t) = \text{card}\left(i, j : \tau_j^i < t\right), \quad N^i(t) = \text{card}\left(j : \tau_j^i < t\right),$$

$$M^i(t) = \text{card}\left(j : Z_j^i < t\right)$$

and

$$N_j^i(t) = \text{card}\left(k > j : Y_k^i < t\right). \tag{23}$$

If we assume that Z_j^i are co-monotonic, i.e. $Z_j^i \leqslant Z_{j+1}^i$, then analogously as in the single-factor case

$$P\left(M^i(t) = m\right) = G_m^i(t) - G_{m+1}^i(t) \tag{24}$$

and therefore

$$P\left(N(t) = m\right) = \sum_{m_1+...+m_k=m} \prod_{i=1}^{k} P\left(N^i(t) = m_i\right), \tag{25}$$

where

$$\begin{aligned} P\left(N^i(t) = m\right) &= \sum_{j=0}^{m} P\left(M^i(t) = j\right) P\left(N_j^i(t) = m - j\right) = \\ &\sum_{j=0}^{m} \left(G_j^i(t) - G_{j+1}^i(t)\right) P\left(N_j^i(t) = m - j\right). \end{aligned}$$

The model may be approximated by a k-dimensional Markov process as it was introduced in the previous section.

7. Numerical results

Let us consider ITRX.9 quotations of the day 30/05/2008 (given in Table 1).

Table 1. ITRX.9 quotations of the 30/05/2008

Tranche Name	**5 Years**	**7 Years**	**10 Years**
ITRX.9_0%-3%	33.75%	41.75%	47.13%
ITRX.9_3%-6%	300.00	398.00	520.00
ITRX.9_6%-9%	188.00	234.00	300.00
ITRX.9_9%-12%	128.00	151.00	190.00
ITRX.9_12%-22%	63.00	73.00	88.00
ITRX.9_0%-100%	80.00	86.00	91.00

Calculations were performed according to a proprietary approach of functional dependence between idiosyncratic and systematic hazard rates, leading to very good fitting to all tranches with three consecutive tenors simultaneously – 5, 7 and 10 years. Its general concept is as follows:

1. Hazard rates are divided into systematic part and idiosyncratic part and systematic hazard rates are divided into fatal part and mezzanine part, i.e. $\lambda_i^{fatal}(t) + \lambda_i^{mezz}(t) = \lambda_i^{sys}(t)$.
2. There is one universal factor of the catastrophe character, applied to all names at the same level i.e. fatal hazard rate is equal for all names $\lambda_i^{fatal}(t) = \lambda^{fatal}(t)$.
3. All idiosyncratic hazard rates $\lambda_i^{idio}(t)$ have the same level provided the total spread is larger than the fatal hazard rate, i.e. $\lambda_i^{idio}(t) = \min\left\{\lambda^{idio}(t), \lambda_i(t) - \lambda^{fatal}(t)\right\}$.
4. The mezzanine hazard rate $\lambda_i^{mezz}(t)$ is distributed among three factors chosen to improve the fitting of mezzanine tranches, i.e. $\lambda_i^{mezz}(t) = \lambda_i(t) - \lambda_i^{fatal}(t) - \lambda_i^{idio}(t)$.

Table 2. Hazard rates

Hazard rate	**5 Y**	**7 Y**	**10 Y**
Idiosyncratic	41.43	30.72	11.95
Fatal	101	84	105
Mezzanine	$\lambda_i^{mezz}(t) = \lambda_i(t) - \lambda_i^{fatal}(t) - \lambda_i^{idio}(t)$		

We see that in Spring 2008 the market was afraid not of individual independent defaults but of defaults of the whole economy. And it was right. In our opinion this presentation is a much more transparent indicator than compound or base correlation, telling in practice nothing.

We see that setting the model in terms of ratios $\lambda_i^{sys}(t)/\lambda_i(t)$, analogously as for the Gaussian copula, would not be practical, since the ratio $\lambda_i^{sys}(t)/\lambda_i(t)$

strongly depends on the level of hazard rates, which vary from small numbers for investment grade obligors to large ones for nearly defaulted companies.

The errors are less than 1 basis point and they are well within bid-ask spreads (see Table 3 and Fig. 1).

Our choice of the market model setting was of course arbitrary, since this is the first research in the field, and may be improved by further study.

Table 3. Quality of fit

Tranche Name	**5 Y**	**7 Y**	**10 Y**
ITRX.9_0%-3%	-0.12%	0.05%	0.00%
ITRX.9_3%-6%	0.50	-0.48	0.00
ITRX.9_6%-9%	0.27	-0.20	-0.55
ITRX.9_9%-12%	0.44	-0.13	-0.09
ITRX.9_12%-22%	0.14	0.12	0.06
ITRX.9_0%-100%	-0.00	-0.00	-0.00

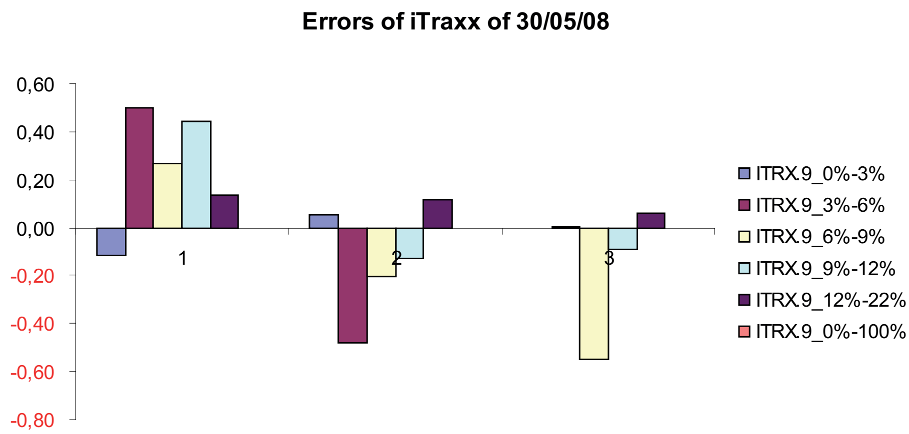

Figure 1. Quality of fit

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