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Minimax theorems for Φ−convex functions with applications*

by

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Abstract: We investigate minimax theorems for Φ−convex functions. As an application we provide a formula for the Φ−conjugation of the pointwise maximum of Φ−convex functions.

Keywords: abstract convexity, Φ−convexity, Φ−conjugation, convexlikeness, minimax theorems, joint Φ−convexlikeness, Φ−intersection property

1. Introduction

Let X, Y be nonempty sets and let $f : X \times Y \to \mathbb{R}$ be a function. Minimax theorems provide sufficient conditions for the equality

$$\inf_Y \sup_X f(x,y) = \sup_X \inf_Y f(x,y)$$

to hold. The first minimax theorem was given by Neumann (1928). Since then, generalizations of the original theorem have been proved under various conditions.

Following Simons (1994, 1995), the existing minimax theorems can be divided into three groups: topological, algebraic and mixed, according to the types of conditions which appear in their formulation.

In topological minimax theorems the crucial role is played by connectedness (e.g. Kindler and Trost, 1989; Ricceri, 1993, 2008; Simons, 1994; Tuy, 1974; Wu, 1959, and the references therein). Algebraic minimax theorems are based on some extensions or generalizations of convexlike properties (see, for example,

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Fan, 1953; Kindler, 1990; Stefanescu, 1985, 2007). Theorems with both algebraic and topological conditions can be found in Kindler (1990), Simons (1990), Terkelsen (1972), and the references therein.

In the present paper we prove minimax theorems for $\Phi-$convex functions.

$\Phi-$convex functions are abstract convex functions. Theory of $\Phi-$convex functions has been developed by Dolecki and Kurcyusz (1978), Pallaschke and Rolewicz (1997), Rubinov (2000), Singer (1997). $\Phi-$convex functions are defined as pointwise suprema of functions from a given class Φ. Such an approach to abstract convexity generalizes the classical fact that each lower semicontinuous convex function is the upper envelope of a certain set of affine functions.

Let Φ be a class of functions $\varphi : X \to \mathbb{R}$

which is closed under addition of constants, i.e. if $\varphi \in \Phi$, then $\varphi + c \in \Phi$ for any $c \in \mathbb{R}$. Classes Φ with this property were considered in Dolecki and Kurcyusz (1978); Pallaschke and Rolewicz (1997); Rubinov (2000).

Recall that a set $A \subset \Phi$ is called *conic* if for all $\varphi \in A$ and $k > 0$ we have $k\varphi \in A$. A set $K \subset \Phi$ is called *additive* if for all $\varphi_1, \varphi_2 \in K$ we have $\varphi_1 + \varphi_2 \in K$. A set $C \subset \Phi$ is called *convex* if for all $\varphi_1, \varphi_2 \in C$ and $t \in [0,1]$ we have $t\varphi_1 + (1-t)\varphi_2 \in C$.

For any $f, g : X \to \mathbb{R}$

$$f \leq g \iff f(x) \leq g(x) \quad \forall x \in X.$$

Definition 1 *Let $f : X \to \mathbb{R}$. The set*

$$supp(f, \Phi) := \{\varphi \in \Phi \ : \ \varphi \leq f\}$$

is called the support *of f with respect to Φ.*

In the sequel, we will use the notation $\text{supp}(f)$ if the class Φ is obvious from the context.

Definition 2 *(Pallaschke and Rolewicz, 1997; Rubinov, 2000) A function $f : X \to \mathbb{R}$ is called* $\Phi-$convex *if*

$$f(x) = \sup\{\varphi(x) \ : \ \varphi \in supp(f)\} \quad \forall \ x \in X.$$

By $H(\Phi)$ we denote the set of all $\Phi-$convex functions $f : X \to \mathbb{R}$ defined on X. In Section 2, we introduce two concepts of joint convexlikeness, namely, the joint convexlikeness for a given class of functions Φ (Definition 4), and joint $\Phi-$convexlikeness for $\Phi-$convex functions (Definition 5), and we discuss their properties. In Section 3 we prove that joint $\Phi-$convexlikeness implies the $\Phi-$intersection property (Definition 6). Although technically involved, the $\Phi-$intersection property is the main tool in proving our minimax theorems for $\Phi-$convex functions (Theorem 1, Section 4). Furthermore, Theorem 2 of Section 4, which is a corollary of Theorem 1, provides sufficient conditions for the minimax equality expressed in terms of jointly convexlike $\Phi-$convex functions. In Section 5 we give an example of a class Φ satisfying the assumptions of

Theorem 2 and such that the level sets of jointly convexlike Φ−convex functions are not necessarily connected. In Section 6, as an application of the results of Section 4, we provide a formula for Φ−conjugations of pointwise maxima of two Φ−convex functions.

2. Joint Φ−convexlikeness

Starting from the paper by Fan (1953), convexlike properties were used in those minimax theorems which do not refer to linear structures of the underlying spaces.

Let X be a set and Φ be a class of functions $\varphi : X \to \mathbb{R}$ defined on X. Following Fan (1953) we say that the class Φ is *convexlike on* X if for any $x_1, x_2 \in X$ and $t \in [0,1]$ there exists $x_0 \in X$ such that

$$\varphi(x_0) \le t\varphi(x_1) + (1-t)\varphi(x_2) \quad \text{for} \quad \varphi \in \Phi.$$

Numerous extensions and generalizations of convexlikeness have been proposed (see, for example, Fan, 1953; Kindler, 1990; Stefanescu, 1985, 2007). We introduce the concept of joint convexlikeness which generalizes the convexlikeness and is shaped for Φ-convex functions.

We start with two underlying concepts.

Definition 3 *Let* $\varphi_1, \varphi_2 : X \to \mathbb{R}$ *be two real-valued functions defined on* X*. We say that* φ_1 *and* φ_2 *are* jointly convexlike on X *if for every* $x_1, x_2 \in X$ *and* $t \in [0,1]$ *there exists* $x_0 \in X$ *such that*

$$\max\{\varphi_1(x_0),\ \varphi_2(x_0)\} \le \max\{t\varphi_1(x_1) + (1-t)\varphi_1(x_2),\ t\varphi_2(x_1) + (1-t)\varphi_2(x_2)\}. \tag{1}$$

Definition 4 *We say that the class* Φ *is* jointly convexlike on X *if any two* $\varphi_1, \varphi_2 \in \Phi$ *are jointly convexlike on* X*.*

If the class Φ consists of convex functions, then Φ is convexlike and jointly convexlike on X. When $\varphi_1 = \varphi_2 = \varphi$, the jointly convexlikeness reduces to the condition that for every $x_1, x_2 \in X$ and $t \in [0,1]$ there exists $x_0 \in X$ such that

$$\varphi(x_0) \le t\varphi(x_1) + (1-t)\varphi(x_2). \tag{2}$$

Hence, if Φ is convexlike on X, then Φ is jointly convexlike on X. In numerous important applications, the class Φ is jointly convexlike and not convexlike (see Example 1 below).

Note that functions φ_1, φ_2 are jointly convexlike on X if and only if the family $\Phi = \{\varphi_1, \varphi_2\}$ is weakly convexlike as defined in Stefanescu (2007).

The following definition is crucial for proving the minimax theorems of Section 4.

Definition 5 *Let* $f, g \in H(\Phi)$*. We say that* f *and* g *are* jointly Φ-convexlike on X *if every two* $\varphi_1,\ \varphi_2 \in \Phi$*,* $\varphi_1 \in supp(f)$*,* $\varphi_2 \in supp(g)$ *are jointly convexlike on* X*.*

If the class Φ is jointly convexlike on X then any $f, g \in H(\Phi)$ are jointly Φ-convexlike on X. An important feature of Definition 5 is that it is expressed in terms of functions $\varphi \in \Phi$ and not in terms of $\Phi-$convex functions f, g directly. In majority of applications, the functions $\varphi \in \Phi$ are of simple structure (e.g. quadratic functions, step functions) and are much easier to handle than generic $\Phi-$convex functions f, g (see Example 1 below).

Below, we give an example of a class Φ, which is jointly convexlike on $\mathbb{R}$ and not convexlike on $\mathbb{R}$.

Example 1 *Let $X = \mathbb{R}$ and $\Phi = \{\varphi_\theta\}$ be a class of functions indexed by a triplet $\theta = (u; c_1, c_2)$, where $u \in \mathbb{R}$, $c_1 \geq 0$, $c_2 \geq 0$. For a given $\theta = (u; c_1, c_2)$, function $\varphi_\theta : \mathbb{R} \to \mathbb{R}$ is given by the formula*

$$\varphi_\theta(x) := \begin{cases} c_1 & x < u \\ c_1 + c_2 & x = u \\ c_2 & x > u \end{cases} .$$

A function $p : \mathbb{R} \to [0, +\infty)$ is called a $P-$function if

$$p(\lambda x + (1-\lambda)y) \leq p(x) + p(y) \quad \text{for all} \quad \lambda \in (0,1) \quad \text{and} \quad x, y \in \mathbb{R}.$$

$P-$functions were investigated in Rubinov (2000), Chapter 6. By Proposition 6.16 in Rubinov (2000) $P-$functions are $\Phi-$convex with respect to our class $\Phi = \{\varphi_\theta\}$.

We show that any $f, g \in H(\Phi)$ are jointly $\Phi-$convexlike on $\mathbb{R}$. Let $\theta_1 = (u; c_1, c_2)$ and $\theta_2 = (w; d_1, d_2)$. Let $\varphi_1 := \varphi_{\theta_1}$ and $\varphi_2 := \varphi_{\theta_2}$. Without loss of generality we can assume that $u \leq w$. Then we need to show that for any $x_1, x_2 \in \mathbb{R}$ and $t \in [0,1]$ we have

$$\begin{aligned} &L := \inf_{x \in \mathbb{R}} \max\{\varphi_1(x), \varphi_2(x)\} \leq \\ &\max\{t\varphi_1(x_1) + (1-t)\varphi_1(x_2), t\varphi_2(x_1) + (1-t)\varphi_2(x_2)\} =: R. \end{aligned} \tag{3}$$

One can easily show that

$$\max\{\varphi_1(x), \varphi_2(x)\} = \begin{cases} \max\{c_1, d_1\} & x < u \\ \max\{c_1 + c_2, d_1\} & x = u \\ \max\{c_2, d_1\} & x \in (u, w) \\ \max\{c_2, d_1 + d_2\} & x = w \\ \max\{c_2, d_2\} & x > w \end{cases} . \tag{4}$$

Consider the following cases:

1. *$c_1 \leq c_2$ and $d_1 \leq d_2$. Then $L = \max\{c_1, d_1\}$. By elementary calculations, for every $x_1, x_2 \in \mathbb{R}$ and $t \in [0,1]$ we get $R \geq \max\{c_1, d_1\} = L$.*
2. *$c_1 > c_2$ and $d_1 \leq d_2$. Then $L = \max\{c_2, d_1\}$ and for every $x_1, x_2 \in \mathbb{R}$ and $t \in [0,1]$ we get $R \geq \max\{c_2, d_1\} = L$.*
3. *$c_1 \leq c_2$ and $d_1 > d_2$. Then $L = \max\{c_1, d_2\}$ and for every $x_1, x_2 \in \mathbb{R}$ and $t \in [0,1]$ we get $R \geq \max\{c_1, d_2\} = L$.*

4. $c_1 > c_2$ *and* $d_1 > d_2$*. Then* $L = \max\{c_2, d_2\}$ *and for every* $x_1, x_2 \in X$ *and* $t \in [0,1]$ *we get* $R \geq \max\{c_2, d_2\} = L$*.*

In this way we proved that inequality (1) *holds for every* $\varphi_1, \varphi_2 \in \Phi$*,* $x_1, x_2 \in \mathbb{R}$ *and* $t \in [0,1]$*. From here, taking into account* (4)*, one can easily deduce* (3)*, which means that every* $\varphi_1, \varphi_2 \in \Phi$ *are jointly convexlike on* $\mathbb{R}$*. Hence,* Φ *is jointly convexlike on* $\mathbb{R}$ *and, consequently, all* $f, g \in H(\Phi)$ *are jointly* $\Phi-$*convexlike on* $\mathbb{R}$*.*

3. The $\Phi-$intersection property

In the present section we show that any two jointly $\Phi-$convexlike functions $f, g : X \to \mathbb{R}$ satisfy the $\Phi-$intersection property defined below. The $\Phi-$intersection property is used in Section 4 in the proof of our minimax theorem.

Let X be a set. For any function $f : X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ the strict lower level set of f is defined as

$$Z_\alpha(f) := \{x \in X,\ f(x) < \alpha\}.$$

Definition 6 *Let* $f, g \in H(\Phi)$*. We say that the* $\Phi-$intersection property holds for f and g at the level $\alpha \in \mathbb{R}$ *if there exist* $\varphi_f \in supp(f)$*,* $\varphi_g \in supp(g)$*, satisfying* $Z_\alpha(\varphi_f) \cap Z_\alpha(\varphi_g) = \emptyset$

such that for all

$$x_1 \in Z_\alpha(\varphi_g), \quad x_2 \in Z_\alpha(\varphi_f)$$

we have

$$(\alpha - \varphi_f(x_2))(\alpha - \varphi_g(x_1)) \leq (\alpha - \varphi_f(x_1))(\alpha - \varphi_g(x_2)). \tag{5}$$

The terminology is motivated by the fact that condition (6) ensures that the intersection

$$\{t \in [0,1]\ :\ t\varphi_f(x_1)+(1-t)\varphi_f(x_2) \geq \alpha\} \cap \{t \in [0,1]\ :\ t\varphi_g(x_1)+(1-t)\varphi_g(x_2) \geq \alpha\}$$

is nonempty.

In the proposition below we show that the joint $\Phi-$convexlikeness of f, g on X implies the $\Phi-$intersection property.

Proposition 1 *Let* $f, g \in H(\Phi)$*. If* f, g *are jointly* $\Phi-$*convexlike on* X*, and for every* $\alpha \in \mathbb{R}$ *there exist* $\varphi_f \in supp\ (f)$ *and* $\varphi_g \in supp\ (g)$ *such that* $Z_\alpha(\varphi_f) \cap Z_\alpha(\varphi_g) = \emptyset$*, then* f, g *satisfy the* $\Phi-$*intersection property at any level* $\alpha \in \mathbb{R}$*.*

Proof We proceed by contradiction. Suppose that the $\Phi-$intersection property does not hold for $f, g \in H(\Phi)$ at a certain level $\alpha \in \mathbb{R}$. Without loss of generality we can assume that $\alpha = 0$.

For every $\varphi_f \in \text{supp}(f)$, $\varphi_g \in \text{supp}(g)$, $Z_0(\varphi_f) \neq \emptyset$, $Z_0(\varphi_g) \neq \emptyset$, $Z_0(\varphi_f) \cap Z_0(\varphi_g) = \emptyset$, there exist $x_1 \in Z_0(\varphi_g)$, $x_2 \in Z_0(\varphi_f)$ such that

$$\varphi_f(x_2)\varphi_g(x_1) > \varphi_f(x_1)\varphi_g(x_2). \tag{6}$$

We show that the joint $\Phi-$convexlikeness of f, g on X together with inequality (6) leads to a contradiction with the fact that $Z_0(\varphi_f) \cap Z_0(\varphi_g) = \emptyset$.

Note that for $x_2 \in Z_0(\varphi_f)$ we have $\varphi_f(x_2) < 0$ and $\varphi_g(x_2) \geq 0$. Analogously, for $x_1 \in Z_0(\varphi_g)$ we have $\varphi_g(x_1) < 0$ and $\varphi_f(x_1) \geq 0$. Hence, inequality (6) is equivalent to

$$\frac{\varphi_f(x_1)}{\varphi_f(x_1) - \varphi_f(x_2)} < \frac{\varphi_g(x_1)}{\varphi_g(x_1) - \varphi_g(x_2)}. \tag{7}$$

So, there exists $t_0 \in (\frac{\varphi_f(x_1)}{\varphi_f(x_1)-\varphi_f(x_2)}, \frac{\varphi_g(x_1)}{\varphi_g(x_1)-\varphi_g(x_2)})$. Then, (7) implies that

$$t_0\varphi_f(x_2) + (1 - t_0)\varphi_f(x_1) < 0,$$

and

$$t_0\varphi_g(x_2) + (1 - t_0)\varphi_g(x_1) < 0.$$

From the assumption that f, g are jointly $\Phi-$convexlike on X, we infer that there exists $x_0 \in X$ such that

$$\begin{aligned}&\max\{\varphi_f(x_0),\ \varphi_g(x_0)\} \leq \\ &\max\{t_0\varphi_f(x_2) + (1 - t_0)\varphi_f(x_1),\ t_0\varphi_g(x_2) + (1 - t_0)\varphi_g(x_1)\} < 0.\end{aligned}$$

Hence,

$$\max\{\varphi_f(x_0),\ \varphi_g(x_0)\} < 0,$$

which means that $\varphi_f(x_0) < 0$ and $\varphi_g(x_0) < 0$. Hence, $x_0 \in Z_0(\varphi_f) \cap Z_0(\varphi_g)$ which is in contradiction to our assumption that $Z_0(\varphi_f) \cap Z_0(\varphi_g) = \emptyset$. □

Proposition 2 *If class Φ consists of convex functions, and for every $\alpha \in \mathbb{R}$ there exist $\varphi_f \in$ supp (f) and $\varphi_g \in$ supp (g) such that $Z_\alpha(\varphi_f) \cap Z_\alpha(\varphi_g) = \emptyset$ then any $f, g \in H(\Phi)$ satisfy the $\Phi-$intersection property at any level $\alpha \in \mathbb{R}$.*

Proof We noted above that if the class Φ consists of convex functions, then any $\Phi-$convex f, g are jointly $\Phi-$convexlike on X. Hence, by Proposition 1 we get the conclusion. □

4. Main results

Let X and Y be given sets and let $a : Y \times X \to \mathbb{R}$ be a function.

We use the following notation:

$$a_* := \sup_{y\in Y} \inf_{x\in X} a(y, x), \quad a^* := \inf_{x\in X} \sup_{y\in Y} a(y, x),$$

for every subset $C \subset Y$ and for every $x \in X$ and $y \in Y$ we write

$$X_\alpha(y) := \{x \in X : a(y,x) \le \alpha\}, \quad Y^C_\alpha(x) := \{y \in C : a(y,x) \ge \alpha\},$$

$$Y^C_\alpha(B) := \bigcap \{Y^C_\alpha(x) : \ x \in B\}, \quad \emptyset \ne B \subset X,$$

for any y_1, $y_2 \in Y$ we write

$$\operatorname{supp}_1 := \operatorname{supp}(a(y_1,\cdot)), \quad \operatorname{supp}_2 := \operatorname{supp}(a(y_2,\cdot)), \quad Z_0(\varphi) := Z(\varphi),$$

for any $x \in X$ we write $Y^C_0(x) := Y^C(x)$. When $C = Y$ we write $Y^C_\alpha(x) := Y_\alpha(x)$.

The proof of our minimax theorem (Theorem 1) is based on the immediate observation that $a_* = a^*$ if and only if for every $\alpha \in \mathbb{R}$ such that $\alpha < a^*$ the set $Y_\alpha(X)$ is nonempty.

We start with the following lemma.

Lemma 1 *Let X be a set and C be a convex subset of a vector space. Let Φ be a family of functions $\varphi : X \to \mathbb{R}$. Let $a : C \times X \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and $x_1, x_2 \in X$. If*
(i) *for any $y \in C$ the function $a(y,\cdot) : X \to \mathbb{R}$ is $\Phi-$convex on X,*
(ii) *for any $y_1, y_2 \in C$ the functions $a(y_1,\cdot)$ and $a(y_2,\cdot)$ have the $\Phi-$intersection property at level $\alpha \in \mathbb{R}$,*
(iii) *$Y^C_\alpha(x_1) \ne \emptyset$ and $Y^C_\alpha(x_2) \ne \emptyset$,*
(iv) *for any $x \in X$ the function $a(\cdot,x) : C \to \mathbb{R}$ is concave on C,*
then $Y^C_\alpha(\{x_1,x_2\})$ is nonempty.

Proof Let us recall that the class Φ is assumed to be closed under addition of constants, i.e. function $a(y,\cdot) - \alpha$ is $\Phi-$convex on X for every $y \in Y$. Hence, without loss of generality we can assume that $\alpha = 0$ (by replacing $a(y,x)$ with $a(y,x) - \alpha$ if necessary).

By contradiction, suppose that

$$Y^C(x_1) \cap Y^C(x_2) = \emptyset. \tag{8}$$

From (iii), there exist $y_1, y_2 \in C$, $y_1 \in Y^C(x_1)$ and $y_2 \in Y^C(x_2)$. By (8), $y_1 \notin Y^C(x_2)$ which means that $a(y_1,x_2) < 0$. Then

$$\forall \ \ \varphi_1 \in \operatorname{supp}_1 \qquad \varphi_1(x_2) < 0. \tag{9}$$

Again, by (8), $y_2 \notin Y^C(x_1)$ which means that $a(y_2,x_1) < 0$, and then

$$\forall \ \ \varphi_2 \in \operatorname{supp}_2 \qquad \varphi_2(x_1) < 0. \tag{10}$$

Consequently, by (9), $x_2 \in Z(\varphi_1)$ for any $\varphi_1 \in \operatorname{supp}_1$ and, by (10), $x_1 \in Z(\varphi_2)$ for any $\varphi_2 \in \operatorname{supp}_2$.

By (ii), the functions $a(y_1,\cdot)$ and $a(y_2,\cdot)$ have the $\Phi-$intersection property. This means that there exist $\varphi_1 \in \operatorname{supp}_1$, $\varphi_2 \in \operatorname{supp}_2$, $Z(\varphi_1) \ne \emptyset$, $Z(\varphi_2) \ne \emptyset$, satisfying

$$Z(\varphi_1) \cap Z(\varphi_2) = \emptyset \tag{11}$$

such that

$$x_1 \in Z(\varphi_2) \ \wedge \ x_2 \in Z(\varphi_1) \ \Rightarrow \ \varphi_1(x_2)\varphi_2(x_1) \le \varphi_1(x_1)\varphi_2(x_2). \tag{12}$$

From (11), $x_2 \notin Z(\varphi_2)$ and $x_1 \notin Z(\varphi_1)$. So, we have the following situation

$$\begin{array}{c} \varphi_1(x_1) \ge 0, \quad \varphi_1(x_2) < 0 \\ \text{and} \\ \varphi_2(x_1) < 0, \quad \varphi_2(x_2) \ge 0. \end{array} \tag{13}$$

Now, we show that there exists $\theta_0 \in [0,1]$ such that

$$(1-\theta_0)\varphi_1(x_1) + \theta_0\varphi_2(x_1) \ge 0 \ \text{ and } \ (1-\theta_0)\varphi_1(x_2) + \theta_0\varphi_2(x_2) \ge 0. \tag{14}$$

We start by noting that by (13), there exist $\theta_1, \theta_2 \in [0,1]$ such that

$$(1-\theta)\varphi_1(x_1) + \theta\varphi_2(x_1) \ge 0 \ \text{ for } \ \theta \in [0,\theta_1], \tag{15}$$

$$(1-\theta)\varphi_1(x_2) + \theta\varphi_2(x_2) \ge 0 \ \text{ for } \ \theta \in [\theta_2,1]. \tag{16}$$

Hence, $\varphi_1(x_1) + \theta(\varphi_2(x_1) - \varphi_1(x_1)) \ge 0$ and

$$0 \le \theta \le \frac{\varphi_1(x_1)}{\varphi_1(x_1) - \varphi_2(x_1)} = \theta_1 \le 1. \tag{17}$$

Moreover, $\varphi_1(x_2) + \theta(\varphi_2(x_2) - \varphi_1(x_2)) \ge 0$ and

$$1 \ge \theta \ge \frac{-\varphi_1(x_2)}{\varphi_2(x_2) - \varphi_1(x_2)} = \theta_2 \ge 0. \tag{18}$$

By (12),

$$\varphi_1(x_2)\varphi_2(x_1) \le \varphi_1(x_1)\varphi_2(x_2). \tag{19}$$

Equivalently,

$$(-\varphi_1(x_2))(\varphi_1(x_1) - \varphi_2(x_1)) \le \varphi_1(x_1)(\varphi_2(x_2) - \varphi_1(x_2)),$$

and

$$\theta_2 = \frac{-\varphi_1(x_2)}{\varphi_2(x_2) - \varphi_1(x_2)} \le \frac{\varphi_1(x_1)}{\varphi_1(x_1) - \varphi_2(x_1)} = \theta_1.$$

Hence, there exists $0 \le \theta_0 \le 1$ such that (14) holds, which is the required conclusion.

In view of (14) we have

$$(1-\theta_0)a(y_1,x_1) + \theta_0 a(y_2,x_1) \ge 0 \ \text{ and } \ (1-\theta_0)a(y_1,x_2) + \theta_0 a(y_2,x_2) \ge 0.$$

By the concavity of $a(\cdot,x)$, for any $x \in X$ we have

$$a((1-\theta_0)y_1 + \theta_0 y_2, x_1) \ge 0 \ \text{ and } \ a((1-\theta_0)y_1 + \theta_0 y_2, x_2) \ge 0.$$

This shows that $(1-\theta_0)y_1+\theta_0 y_2 \in Y^C(x_1)\cap Y^C(x_2)$ contradictory to our assumption that $Y^C(x_1)\cap Y^C(x_2)=\emptyset$. Hence, for $x_1,x_2\in X$ we get $Y^C(x_1)\cap Y^C(x_2)\neq\emptyset$. □

Let us recall that a function $f: X\to\mathbb{R}$ is *upper semicontinuous (u.s.c.)* at $x_0\in X$ if for every $\varepsilon>0$ there exists a neighbourhood U_0 of x_0 such that

$$f(x) < f(x_0)+\varepsilon \text{ for } x\in U_0.$$

A function f is upper semicontinuous on X if f is upper semicontinuous at each $x_0\in X$. Let us recall that for any $\beta\in\mathbb{R}$ the upper level sets of an upper semicontinuous function f,

$$L_\beta := \{x\in X \mid f(x)\geq\beta\}$$

are closed in X (see Proposition 1.4, p. 12, Aubin, 1998).

Now, we can present our minimax theorem.

THEOREM 1 *Let X be a set and Y be a compact and convex subset of a topological vector space. Let Φ be a family of functions $\varphi: X\to\mathbb{R}$ and $a: Y\times X\to\mathbb{R}$. If*

(i) *for any $y\in Y$ the function $a(y,\cdot): X\to\mathbb{R}$ is $\Phi-$convex on X,*

(ii) *for any $y_1,y_2\in Y$ the functions $a(y_1,\cdot)$ and $a(y_2,\cdot)$ have the $\Phi-$intersection property at any level $\alpha<a^*$, $\alpha\in\mathbb{R}$,*

(iii) *for any $x\in X$ the function $a(\cdot,x): Y\to\mathbb{R}$ is concave and upper semicontinuous on Y,*

then $a_* = a^*$.

PROOF The proof consists of two steps.

Step 1. We show that for all $k\in\mathbb{N}$ we have

(k) $Y_\alpha(B)\neq\emptyset$ for every subset $B\subset X$, where the cardinality $|B|$ of B is k, i.e. $|B|=k$.

The proof proceeds by induction on k. Let $k=1$, $B=\{x\}$, $x\in X$, and $\alpha<a^*$. Then for every $x\in X$ there exists $y\in Y$ such that $a(y,x)>\alpha$ and $Y_\alpha(x)=Y_\alpha(B)\neq\emptyset$. Let $k=2$ and $B=\{x_1,x_2\}$, $x_1,x_2\in X$. From Lemma 1 we have $Y_\alpha(B)\neq\emptyset$.

Suppose that (k) holds for some $k\geq 2$. We show that (k) holds for $k+1$. Take any subset $D\subset X$ with $|D|=k+1$ and a subset $E\subset D$ such that $|D-E|=2$. From the inductive assumption we have $Y_\alpha(E)\cap Y_\alpha(x)=Y_\alpha^C(x)\neq\emptyset$ for $x\in X$, where $C:=Y_\alpha(E)$. Hence, $Y_\alpha(D)=Y_\alpha(E)\cap Y_\alpha(D-E)=Y_\alpha^C(D-E)$ and, by Lemma 1, we have $Y_\alpha(D)\neq\emptyset$.

Step 2. In Step 1 we have shown that the family $\{Y_\alpha(x),\ x\in X\}$ has the finite intersection property, i.e for every finite set B we have $Y_\alpha(B)\neq\emptyset$. By the upper semicontinuity of $a(\cdot,x)$, the sets $Y_\alpha(x)$ are closed for every $x\in X$ and $\alpha\in\mathbb{R}$. Since in a compact set every family of closed subsets with the finite intersection property has nonempty intersection (see Theorem III.5, p. 98, Nagata, 1985) we obtain $\bigcap_{x\in X} Y_\alpha(x)\neq\emptyset$. Then $a_*=a^*$. □

Theorem 2 *Let X, Y and Φ be as in Theorem 1. Let $a : Y \times X \to \mathbb{R}$. If*
(i) *for any $y \in Y$ the function $a(y, \cdot) : X \to \mathbb{R}$ is $\Phi-$convex on X,*
(ii) *for any $y_1, y_2 \in Y$ the functions $a(y_1, \cdot)$ and $a(y_2, \cdot)$ are jointly $\Phi-$convexlike on X, and for every $\alpha < \alpha*$ there exist $\varphi_1 \in supp_1$ and $\varphi_2 \in supp_2$ such that $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$,*
(iii) *for any $x \in X$ the function $a(\cdot, x) : Y \to \mathbb{R}$ is concave and upper semicontinuous on Y,*
then $a_* = a^*$.

Proof Conclusion follows from Proposition 1 and Theorem 1. □

With the help of Theorem 2 we recover a classical minimax theorem.

Corollary 1 *Let X be a convex subset of a topological vector space and Y be a compact and convex subset of a topological vector space. Let $a : Y \times X \to \mathbb{R}$. If*
(i) *for any $y \in Y$ the function $a(y, \cdot) : X \to \mathbb{R}$ is convex and lower semicontinuous on X,*
(ii) *for any $x \in X$ the function $a(\cdot, x) : Y \to \mathbb{R}$ is concave and upper semicontinuous on Y,*
then $a_* = a^*$.

Proof By Proposition 3.1 of Ekeland and Temam (1976) under our assumption, functions $a(y, \cdot)$ are $\Phi-$convex with the class Φ of all affine functions defined on X. The conclusion follows from Proposition 2 and Theorem 1. □

5. Example

In the existing minimax theorems the connectedness of the level sets of the functions $a(\cdot, x)$, $x \in X$ is a crucial assumption (see Kindler and Trost, 1989; Ricceri, 1993, 2008).

Below, we give an example of a class Φ and a function $a(\cdot, \cdot)$, for which Theorem 2 holds and whose level sets are disconnected.

To present our example we introduce the following definition.

Definition 7 *A function $f : X \to \mathbb{R}$ is* minored *by the set Φ if there exists $\tilde{\varphi} \in \Phi$ such that*

$$f > \tilde{\varphi} \quad i.e. \quad f(x) > \tilde{\varphi}(x) \text{ for all } x \in X. \tag{20}$$

A set Φ is a *supremal generator* of the set Q of functions if every $f \in Q$ is Φ-convex.

Let X be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$. Consider

$$\Phi_q = \{\varphi : X \to \mathbb{R} \ : \ \varphi(x) = -a||x||^2 + \langle l, x \rangle - c \ : \ x \in X, \ l \in X, \ a \geq 0, \ c \in \mathbb{R}\}.$$

Proposition 3 *(Rubinov, 2000, Proposition 6.3) Let X be a Hilbert space. The set Φ_q is a supremal generator of the set of all lower semicontinuous functions, defined on X, minored by Φ_q.*

Proposition 4 *The class Φ_q is jointly convexlike on X.*

Proof Let $\varphi_1, \varphi_2 \in \Phi_q$, i.e.

$$\varphi_1(x) = -a_1||x||^2 + \langle l_1, x\rangle - c_1 \text{ and } \varphi_2(x) = -a_2||x||^2 + \langle l_2, x\rangle - c_2.$$

According to Definition 3 we show that for every $x_1, x_2 \in X$ and $t \in [0,1]$ there exists $x_0 \in X$ such that

$$\max\{\varphi_1(x_0),\ \varphi_2(x_0)\} \leq \max\{t\varphi_1(x_1) + (1-t)\varphi_1(x_2),\ t\varphi_2(x_1) + (1-t)\varphi_2(x_2)\}. \tag{21}$$

Let $x_1, x_2 \in X$ and $t \in [0,1]$. If $a_1 = 0$ and $a_2 = 0$, it is enough to take $x_0 = tx_1 + (1-t)x_2$.

Suppose now that $a_1 > 0$. Then, $\lim_{||x||\to+\infty} \varphi_1(x) = -\infty$. Consequently, there exists $\delta > 0$ such that for every $x \in X$ such that $||x|| > \delta$ we have

$$\varphi_1(x) \leq \max\{t\varphi_1(x_1) + (1-t)\varphi_1(x_2),\ t\varphi_2(x_1) + (1-t)\varphi_2(x_2)\}.$$

Take any $x \in X$ such that $||x|| > \delta$. The following situations may occur:
(i) $\varphi_2(x) \leq \max\{t\varphi_1(x_1) + (1-t)\varphi_1(x_2),\ t\varphi_2(x_1) + (1-t)\varphi_2(x_2)\}$,
(ii) $\varphi_2(x) > \max\{t\varphi_1(x_1) + (1-t)\varphi_1(x_2),\ t\varphi_2(x_1) + (1-t)\varphi_2(x_2)\}$.
If *(i)* holds, then $x_0 = x$, and the proof is completed. Suppose now that *(ii)* holds. If $a_2 > 0$, there exists $\delta' > 0$ such that for all $x \in X$ such that $||x|| > \delta'$ we have

$$\varphi_2(x) \leq \max\{t\varphi_1(x_1) + (1-t)\varphi_1(x_2),\ t\varphi_2(x_1) + (1-t)\varphi_2(x_2)\}.$$

Consequently, by taking $x' \in X$ such that

$$||x'|| > \max\{\delta', \delta\}$$

we get the conclusion.

Suppose now that $a_2 = 0$. Since the function φ_2 is affine, then in the set of all $x' \in X$ such that

$$\varphi_2(x') < t\varphi_2(x_1) + (1-t)\varphi_2(x_2)$$

we can find x' such that $||x'|| > ||x||$. Hence, (21) is satisfied with $x_0 = x'$. □

In view of Proposition 4, Theorem 2 for the class Φ_q takes the following form:

Theorem 3 *Let X be a Hilbert space and Y be a compact and convex subset of a topological vector space. Let $a : Y \times X \to \mathbb{R}$. If*

(i) *for any $y \in Y$ the function $a(y, \cdot) : X \to \mathbb{R}$ is lower semicontinuous on X and minored by Φ_q, and for any $y_1, y_2 \in Y$ and $\alpha < \alpha*$ there exist $\varphi_1 \in supp_1$ and $\varphi_2 \in supp_2$ such that $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$,*

(ii) *for any $x \in X$ the function $a(\cdot, x) : Y \to \mathbb{R}$ is concave and upper semicontinuous on Y,*

then $a_* = a^*$.

Proof By Proposition 3, for each $y \in Y$ the function $a(y, \cdot)$ is Φ_q-convex. By Proposition 4, for every $y_1, y_2 \in Y$ the functions $a(y_1, \cdot)$ and $a(y_2, \cdot)$ are jointly Φ_q-convexlike on X. By Theorem 2, the conclusion follows. □

Based on the above discussion we give an example of the function a such that the Theorem 3 holds but there exists $\alpha \in \mathbb{R}$ and $y \in Y$ such that the set $X_\alpha(y)$ is not connected.

Example 2 *Let $X = \mathbb{R}$ and $Y = [0, 2]$, let*

$$\Phi_q = \{\varphi : X \to \mathbb{R} \ : \ \varphi(x) = -ax^2 + lx - c \ : \ x \in \mathbb{R}, \ l \in \mathbb{R}, \ a \geq 0, \ c \in \mathbb{R}\}.$$

Let $a : [0, 2] \times \mathbb{R} \to \mathbb{R}$ be defined as

$$a(y, x) := f(x) - y^2,$$

where

$$f(x) := \begin{cases} x^2 & x \leq 1 \\ (x-3)^2 & x > 1 \end{cases}.$$

In view of Proposition 4 all the assumptions of Theorem 3 are satisfied, since for all $y \in [0, 2]$ the function $a(y, \cdot)$ is lower semicontinuous on $\mathbb{R}$ and

$$a(y, x) > -x^2 - 5 =: \tilde{\varphi}(x) \quad \textit{for all} \quad x \in X,$$

where $\tilde{\varphi} \in \Phi_q$. Moreover, for all $x \in \mathbb{R}$ the function $a(\cdot, x) : [0, 2] \to \mathbb{R}$ is concave and continuous on $[0, 2]$. By Theorem 3, $a_* = a^*$. For $y \in [0, 2]$ take any $\alpha \in [-y^2, \ 1 - y^2]$. The sets

$$X_\alpha(y) = \{x \in X \ : \ a(y, x) \leq \alpha\} =$$
$$\left[-\sqrt{\alpha + y^2}; \ \sqrt{\alpha + y^2}\right] \cup \left[-\sqrt{\alpha + y^2} + 3; \ \sqrt{\alpha + y^2} + 3\right],$$

are disconnected.

6. Φ−conjugate of pointwise maximum of two functions

Many important facts from convex and nonsmooth analysis were investigated for Φ−convex functions, see e.g., Burachik and Jeyakumar (2005), Burachik and Rubinov (2008), Jeyakumar, Rubinov and Wu (2007).

In the present section we apply Theorem 2 to derive a formula for the Φ−conjugate of a pointwise maximum of two Φ−convex functions. Conjugates of pointwise maxima for proper convex lower semicontinuous functions in normed spaces were investigated e.g. in Boţ and Wanka (2008); Fitzpatrick and Simons (2000). We start by recalling the definition of Φ−conjugate function.

Let X be a set. Let $f \in H(\Phi)$.

Definition 8 *The function* $f^* : \Phi \to \mathbb{R}$, *defined as*

$$f^*(\varphi) := \sup_{x \in X} \{\varphi(x) - f(x)\},$$

is called the Fenchel-Moreau Φ−conjugate *of* f.

The Φ−conjugate has the following properties (see, for example, Proposition 1.2.2 in Rubinov, 2000)
(i) $f^*(\varphi) \geq g^*(\varphi)$ if and only if $f \leq g$,
(ii) $f^*(\varphi + c) = f^*(\varphi) + c$ for all $c \in \mathbb{R}$,
(iii) $(f^* + c)(\varphi) = f^*(\varphi) - c$ for all $c \in \mathbb{R}$,
(iv) $f(x) + f^*(\varphi) \geq \varphi(x)$ (Fenchel-Moreau inequality),
(v) if the class Φ is homogeneous, i.e $\alpha\varphi \in \Phi$ for all $\varphi \in \Phi$ and $\alpha \in \mathbb{R}$, then

$$(\alpha f)^*(\varphi) = \alpha f^*(\frac{\varphi}{\alpha}).$$

As stated in Proposition 2.2 of Jeyakumar, Rubinov and Wu (2007), if the set Φ is additive, then the set $H(\Phi)$ is additive, and if the set Φ is conic, then the set $H(\Phi)$ is also conic.

Consider the set-valued mapping $\mathrm{Supp} : H(\Phi) \rightrightarrows \Phi$, defined as

$$\mathrm{Supp}(f) := \mathrm{supp}(f, \Phi) \text{ for } \ f \in H(\Phi).$$

As observed in Proposition 2.3 of Jeyakumar, Rubinov and Wu (2007), if Φ is additive, then the mapping Supp is *superadditive*, i.e. for any $f, g \in H(\Phi)$ we have

$$\mathrm{Supp}(f + g) \supset \mathrm{Supp}(f) + \mathrm{Supp}(g), \tag{22}$$

where, for any sets A and B, $A+B$ is the Minkowski sum of A and B. Moreover, if Φ is conic, then $\mathrm{Supp}(\lambda f) = \lambda \mathrm{Supp}(f)$ for $\lambda > 0$.

We say that the mapping Supp is *additive in* $f, g \in H(\Phi)$ if

$$\mathrm{Supp}(f + g) = \mathrm{Supp}(f) + \mathrm{Supp}(g).$$

We say that the mapping Supp is *additive* if it is additive for every $f, g \in H(\Phi)$. Conditions ensuring that Supp is additive in $f, g \in H(\Phi)$ are discussed in Jeyakumar, Rubinov and Wu (2007).

Definition 9 *Let $h, j : \Phi \to \mathbb{R}$. The* infimal convolution $h \oplus j : \Phi \to \mathbb{R} \cup \{\pm\infty\}$ *of h and j is defined as*

$$h \oplus j(\varphi) := \inf_{\varphi=\varphi_1+\varphi_2, \varphi_1,\varphi_2 \in \Phi} \{h(\varphi_1) + j(\varphi_2)\}, \quad \varphi \in \Phi,$$

where the equality $\varphi = \varphi_1 + \varphi_2$ means that $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, $x \in X$. The infimal convolution is exact *provided the infimum is achieved for any $\varphi \in \Phi$.*

Theorem 4 *(Jeyakumar, Rubinov and Wu, 2007, Theorem 5.1) Let X be a set and let Φ be an additive set of functions $\varphi : X \to \mathbb{R}$ and let $f, g : X \to \mathbb{R}$ be $\Phi-$convex functions. Then the following are equivalent:*
(i) *the set-valued mapping Supp$(\cdot)$ is additive in f, g,*
(ii) *$(f + g)^*(\varphi) = f^* \oplus g^*(\varphi)$, where the infimal convolution is exact.*

We introduce the notation

$$f \vee g := \max\{f, g\},$$

where $\max\{f, g\}(x) := \max\{f(x), g(x)\}$.

Now we are in a position to prove the formula for $\Phi-$conjugate function of a maximum of two $\Phi-$convex functions.

Theorem 5 *Let X be a set and let Φ be a convex set of functions $\varphi : X \to \mathbb{R}$ such that $-\varphi \in \Phi$ if $\varphi \in \Phi$. Let Φ be jointly convexlike (according to Definition 4) and $f, g \in H(\Phi)$. If*
(i) *the mapping Supp$(\cdot)$ is additive in f, g,*
then

$$(f \vee g)^*(\varphi) = \min_{0\le\lambda\le1} \{(\lambda f)^* \oplus ((1 - \lambda)g)^*(\varphi)\},$$

where the infimal convolution is exact.

Proof For any $x \in X$ we have

$$f \vee g(x) = \max_{0\le\lambda\le1} \{\lambda f(x) + (1 - \lambda)g(x)\}$$

and consequently

$$\begin{aligned} (f \vee g)^*_L(\varphi) &= \sup_{x\in X} \{\varphi(x) - (f \vee g)(x)\} \\ &= \sup_{x\in X} \{\varphi(x) - \max_{0\le\lambda\le1}\{\lambda f(x) + (1 - \lambda)g(x)\}\} \\ &= \sup_{x\in X} \min_{0\le\lambda\le1}\{\varphi(x) - \lambda f(x) - (1 - \lambda)g(x)\}. \end{aligned} \tag{23}$$

Let $a(\lambda, x) := \lambda\tilde{f}(x) + (1 - \lambda)\tilde{g}(x)$ where $\tilde{f} = f - \varphi$ and $\tilde{g} = g - \varphi$. For the function $a : [0, 1] \times X \to \mathbb{R}$ all the assumptions of Theorem 2 hold. It follows from the assumptions that the functions $a(\lambda,\cdot)$ are $\Phi-$convex for all $\lambda \in [0,1]$ and $a(\lambda_1,x)$ and $a(\lambda_2,x)$ are jointly $\Phi-$convexlike on X for every $\lambda_1,\lambda_2 \in [0,1]$. The functions $a(\cdot,x)$ are linear and continuous on Y. Therefore, by Theorem 2, the formula (23) takes the form

$$\begin{aligned}(f\vee g)^*(\varphi) &= \sup_{x\in X}\min_{0\le\lambda\le1}\{\varphi(x)-\lambda f(x)-(1-\lambda)g(x)\} \\ &= -\inf_{x\in X}\max_{0\le\lambda\le1} a(\lambda,x) \\ &= -\max_{0\le\lambda\le1}\inf_{x\in X} a(\lambda,x) \\ &= \min_{0\le\lambda\le1}\sup_{x\in X}\{\varphi(x)-\lambda f(x)-(1-\lambda)g(x)\} \\ &= \min_{0\le\lambda\le1}(\lambda f+(1-\lambda)g)^*(\varphi).\end{aligned} \tag{24}$$

By Theorem 4,

$$(f\vee g)^*(\varphi) = \min_{0\le\lambda\le1}\{(\lambda f)^*\oplus((1-\lambda)g)^*(\varphi)\},$$

were the infimal convolution is exact. □

7. Final remarks

Let us note that the proof of Theorem 5 is based on Theorem 2 applied to X being an arbitrary set, $Y := [0,1]$ and

$$\tilde{a}(\lambda,x) := \lambda\tilde{f}(x)+(1-\lambda)\tilde{g}(x),$$

where $\lambda\in[0,1]$, $x\in X$, $\tilde{f} := f-\varphi$, $\tilde{f} := f-\varphi$, $f,g : X\to\mathbb{R}$, $\varphi\in\Phi$, and $f,g : X\to\mathbb{R}$ are given $\Phi-$convex functions.

In this case Theorem 1 of Ricceri (1993) takes the following form:

Theorem 6 *(Ricceri, 1993) Let X be a topological space. Assume that*

(h0) *for each $\rho\in\mathbb{R}$, $\lambda\in[0,1]$ the sets*

$$\{x\in X:\ \tilde{a}(\lambda,x)\le\rho\}$$

are connected,

(h1) *$\tilde{a}(\lambda,\cdot)$ is lower semicontinuous in X for each $\lambda\in[0,1]$.*

Then

$$\sup_{\lambda\in[0,1]}\inf_{x\in X}\tilde{a}(\lambda,x) = \inf_{x\in X}\sup_{\lambda\in[0,1]}\tilde{a}(\lambda,x).$$

Consider the family of sets

$$\mathcal{O} = \{\{x\in X\ :\ \lambda\tilde{f}(x)+(1-\lambda)\tilde{g}(x)>\rho\}\ :\ \rho\in\mathbb{R},\ \lambda\in[0,1]\},$$

and the topology $\tau_{\mathcal{O}}$ generated by the family $\mathcal{O}$. Clearly, the topology $\tau_{\mathcal{O}}$ is the weakest topology in which all the sets of the form

$$\{x\in X:\ \tilde{a}(\lambda,x)\le\rho\},\quad \rho\in\mathbb{R},\ \lambda\in[0,1]$$

are closed. Hence, for each λ the function $\tilde{a}(\lambda,\cdot)$ is lower semicontinuous in the topology $\tau_{\mathcal{O}}$.

In this context the question arises whether the sets

$$\{x\in X:\ \tilde{a}(\lambda,x)\le\rho\},\quad \rho\in\mathbb{R},\ \lambda\in[0,1]$$

are connected in the topology $\tau_{\mathcal{O}}$. At the moment the question remains open.

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