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Shape and topology optimization of distributed parameter systems*

by

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Abstract: The energy functional for an elliptic boundary value problem in two spatial dimensions is considered. The variations of shape functional resulting from the small shape-topological domain perturbations with the holes and inclusions in elastic body are determined.

The exact representation of solutions to the boundary value problem is exploited for the purposes of asymptotic analysis. To this end the perturbed solutions of the boundary value problem are expressed as the minimizers of perturbed energy functionals. The proposed method of asymptotic analysis results in the double asymptotic expansions, with respect to the size of a hole and to the contrast parameter of an inclusion with respect to the matrix, of solutions to the boundary value problems as well as of the associated energy functional.

The shape sensitivity analysis of the energy functional with respect of the boundary variations of an inclusion is performed. The further asymptotic analysis allows for the limit passage with the size of inclusion to zero. In this way the topological derivative of the energy functional is obtained.

The proposed analysis can be used in the shape and topology optimum design for elastic bodies governed by the stationary as well as by the time dependent elasticity boundary value problems in the framework of selfadjoint extensions of elliptic operators.

Keywords: shape optimization, topological derivative, asymptotic analysis, singularly perturbed geometrical domains, asymptotic expansion of energy functional

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1. Introduction

In the pioneering paper of the present authors (Sokolowski and Żochowski, 1999) the *topological derivative* of a shape functional in singularly perturbed geometrical domain was introduced and rigorously defined. The topological derivative of an energy functional for the elliptic boundary value problems can be recovered under regularity assumptions from the shape gradient of the same functional. From the point of view of applications, the useful formula for the topological derivative can be derived using the shape gradients of the shape functional under consideration. Actually, the topological derivative of a functional can be obtained as a singular limit for its shape gradients. On the other hand, the most interesting cases are those for which the topological derivative is determined by the singular limit of explicit solutions to the elliptic boundary value problems. This approach is used in the paper.

It turns out that the regular domain perturbations for elliptic boundary value problems are useful for applications. By regular domain perturbations we mean small inclusions with elastic properties different from the properties of the matrix material. In this way the notion of topological derivative can be extended to some evolution problems, by using the so-called selfadjoint extensions of the elliptic operators.

The problem of the influence of small, circular holes (empty or filled completely with different material) in the elastic body on the solutions of associated boundary value problems has been considered in many studies, see Garreau, Guillaume and Masmoudi (2001), Lewinski and Sokolowski (2003), Novotny et al. (2003), Sokolowski and Żochowski (2003). We consider a modification of such problems, with the partially filled hole, and with two different materials of the matrix and the inclusion, see Fig. 1 for an example. In this way the double asymptotic passage is considered, with two parameters, one parameter is small and governs the size of the hole, another parameter is the contrast between the matrix and the inclusion in elastic body.

From the point of view of mechanics, the proposed model describes, e.g., the hole with hardened walls.

The first order asymptotic expansions of energy type functionals for elliptic problems furnish also the so-called self-adjoint extension of the associated elliptic operator. In our context the self-adjoint extensions can be used for the defect modeling in elastic bodies. The self-adjoint extensions are introduced in mathematical physics for the purposes of point-interactions (Berezin and Fadeev, 1961), and are used as well in the evolution problems of the parabolic and hyperbolic types, hence for the wave equation (Kurasov and Posilicano, 2005; Kowalewski, Lasiecka and Sokolowski, 2012).

2. The model problem

We present a detailed analysis for the Laplacian in singularly perturbed domain. The singular perturbation contains a hole $B_{\lambda\rho}$ *reinforced* by a ring $C_{\lambda\rho,\rho}$, where the radius of the hole $\rho \to 0$ is a small parameter, and $0 < \lambda < 1$ is the contrast. This is the simplest setting, and the singular perturbation of the geometrical domain is governed by the small parameter $\rho > 0$. We introduce also the fictitious boundary Γ_R for the purpose of the domain decomposition technique. At the curve Γ_R the nonlocal Steklov-Poincaré boundary operator is defined which reflects the dependence of the energy functional in the truncated subdomain on the hole appearance inside the subdomain bounded by the curve. The simple geometric form of the fictitious domain allows for analytical evaluation of the asymptotics of the energy functional for $\lambda\rho \to 0$.

Let us consider the domain Ω containing the hole with boundary made of modified material as depicted in Fig.1. For simplicity, the hole is located at the origin of the coordinate system. In order to write down the model problem, we introduce some notations:

$$
\begin{aligned}
B_s &= \{ x \in \mathbb{R}^2 \mid \|x\| < s \} \\
C_{s,t} &= \{ x \in \mathbb{R}^2 \mid s < \|x\| < t \} \\
\Gamma_s &= \{ x \in \mathbb{R}^2 \mid \|x\| = s \} \\
\Omega_s &= \Omega \setminus \overline{B}_s .
\end{aligned}
$$

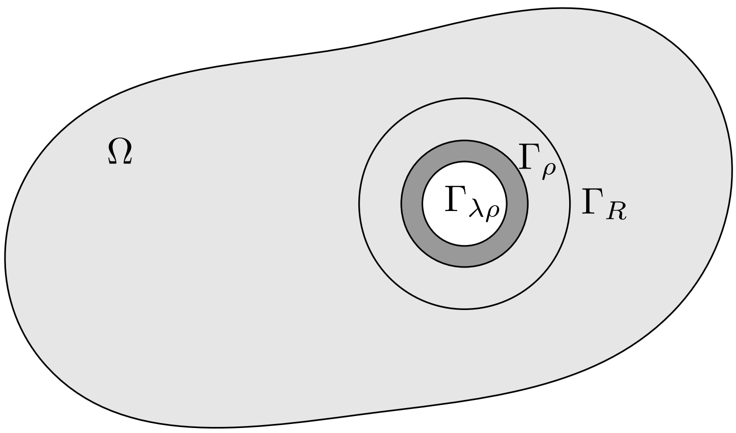

Figure 1. The domain with the hole and the surrounding circle

Then, the problem in the intact domain Ω has the form

$$
\begin{aligned}
k_1 \Delta w_0 &= 0 \quad \text{in } \Omega \\
w_0 &= g_0 \quad \text{on } \partial\Omega.
\end{aligned}
\tag{1}
$$

The model problem in the modified domain reads:

$$\begin{aligned} k_1\Delta w_\rho &= 0 \quad \text{in } \Omega_\rho \\ w_\rho &= g_0 \quad \text{on } \partial\Omega \\ w_\rho &= v_\rho \quad \text{on } \Gamma_\rho \\ k_2\Delta v_\rho &= 0 \quad \text{in } C_{\lambda\rho,\rho} \end{aligned} \tag{2}$$

$$\begin{aligned} k_2\frac{\partial v_\rho}{\partial n_2} &= 0 \quad \text{on } \Gamma_{\lambda\rho} \\ k_1\frac{\partial w_\rho}{\partial n_1} + k_2\frac{\partial v_\rho}{\partial n_2} &= 0 \quad \text{on } \Gamma_\rho \end{aligned}$$

where n_1 – exterior normal vector to Ω_ρ, n_2 – exterior normal vector to $C_{\lambda\rho,\rho}$, and $0 < \lambda < 1$.

We want to investigate the influence of the small inclusion in the form of a variable ring with respect to the small parameter ρ, the inclusion being made of a material with the properties different from the matrix material, on the asymptotic behaviour of the difference $w_\rho - w_0$ in Ω_R, where Γ_R surrounds $C_{\lambda\rho,\rho}$ and R is fixed. We assume that $\rho \to 0+$ and λ is considered at this stage of analysis as temporarily constant.

If we define

$$u_\rho = \begin{cases} w_\rho & \text{in } \Omega_\rho \\ v_\rho & \text{in } C_{\lambda\rho,\rho} \end{cases}$$

then the boundary value problem (2) is equivalent to minimization of the energy functional

$$\mathcal{E}_1(u_\rho) = \frac{1}{2}\int_{\Omega_\rho} k_1\nabla u_\rho \cdot \nabla u_\rho \, dx + \frac{1}{2}\int_{C_{\lambda\rho,\rho}} k_2\nabla u_\rho \cdot \nabla u_\rho \, dx \tag{3}$$

over the subset of $u_\rho \in H^1(\Omega_\rho)$ with the given trace $u_\rho = g_0$ on the exterior boundary $\partial\Omega$.

This expression may be rewritten as

$$\begin{aligned} \mathcal{E}_1(u_\rho) = \frac{1}{2}&\int_{\Omega_R} k_1\nabla w_\rho \cdot \nabla w_\rho \, dx \\ &+ \frac{1}{2}\int_{C_{\rho,R}} k_1\nabla w_\rho \cdot \nabla w_\rho \, dx \\ &+ \frac{1}{2}\int_{C_{\lambda\rho,\rho}} k_2\nabla v_\rho \cdot \nabla v_\rho \, dx. \end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}\mathcal{E}_1(u_\rho) = \frac{1}{2}\int_{\Omega_R} k_1 \nabla w_\rho \cdot \nabla w_\rho \, dx \\ + \frac{1}{2}\int_{\Gamma\rho}\left(w_\rho k_1 \frac{\partial w_\rho}{\partial n_1} + v_\rho k_2 \frac{\partial v_\rho}{\partial n_2}\right) ds \\ + \frac{1}{2}\int_{\Gamma_R} k_1 w_\rho \frac{\partial w_\rho}{\partial n_3}\, ds,\end{aligned}$$

where n_3 – exterior normal to Ω_R.

Hence, due to boundary and transmission condition,

$$\mathcal{E}_1(u_\rho) = \frac{1}{2}\int_{\Omega_R} k_1 \nabla w_\rho \cdot \nabla w_\rho \, dx + \frac{1}{2}\int_{\Gamma_R} k_1 w_\rho \frac{\partial w_\rho}{\partial n_3}\, ds. \tag{4}$$

3. Steklov-Poincaré operator

Observe that $\mathcal{E}_1(w_0)$ corresponds to problem (1). Therefore, the main goal is to use the Steklov-Poincaré operator

$$\mathcal{A}_{\lambda,\rho} \;:\; w \in H^{1/2}(\Gamma_R) \;\longmapsto\; \frac{\partial w_\rho}{\partial n_3} \in H^{-1/2}(\Gamma_R), \tag{5}$$

where the normal derivative is evaluated from the auxiliary problem

$$\begin{aligned} k_1 \Delta w_\rho &= 0 \quad \text{in } \; C_{\rho,R} \\ w_\rho &= w \quad \text{on } \; \Gamma_R \\ w_\rho &= v_\rho \quad \text{on } \; \Gamma_\rho \\ k_2 \Delta v_\rho &= 0 \quad \text{in } \; C_{\lambda\rho,\rho} \\ k_2 \frac{\partial v_\rho}{\partial n_2} &= 0 \quad \text{on } \; \Gamma_{\lambda\rho} \\ k_1 \frac{\partial w_\rho}{\partial n_1} + k_2 \frac{\partial v_\rho}{\partial n_2} &= 0 \quad \text{on } \; \Gamma_\rho . \end{aligned} \tag{6}$$

The geometry of domains of definition for functions is shown in Fig. 2.

Now let us adopt the polar coordinate system around origin and assume the Fourier series form for w on Γ_R,

$$w = C_0 + \sum_{k=1}^{\infty} \left(A_k \cos k\varphi + B_k \sin k\varphi\right). \tag{7}$$

The general form of the solution w_ρ is

$$w_\rho = A^w + B^w \log r + \sum_{k=1}^{\infty} \left(w_k^c(r) \cos k\varphi + w_k^s(r) \sin k\varphi\right), \tag{8}$$

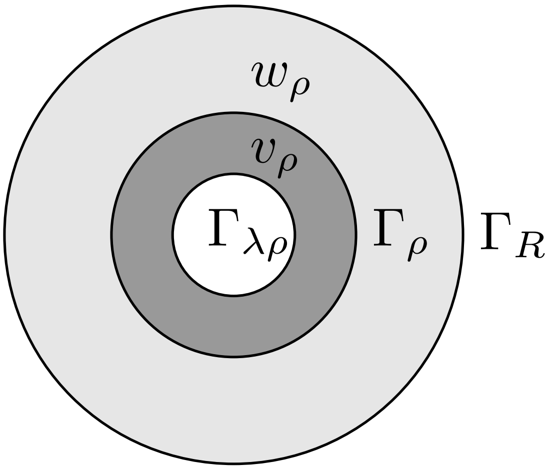

Figure 2. Domains of definition for w_ρ and v_ρ

where

$$w_k^c(r) = A_k^c r^k + B_k^c \frac{1}{r^k}, \quad w_k^s(r) = A_k^s r^k + B_k^s \frac{1}{r^k}.$$

Similarly for v_ρ:

$$v_\rho = A^v + B^v \log r + \sum_{k=1}^{\infty} \left(v_k^c(r) \cos k\varphi + v_k^s(r) \sin k\varphi \right), \tag{9}$$

where

$$v_k^c(r) = a_k^c r^k + b_k^c \frac{1}{r^k}, \quad v_k^s(r) = a_k^s r^k + b_k^s \frac{1}{r^k}.$$

Additionally, we denote the Fourier expansion of v_ρ on Γ_ρ by

$$v_\rho = c_0 + \sum_{k=1}^{\infty} \left(a_k \cos k\varphi + b_k \sin k\varphi \right). \tag{10}$$

From boundary conditions on $\Gamma_{\lambda\rho}$ it follows easily that $B^v = 0$, $A^v = c_0$, and then $B^w = 0$, $A^w = A^v = c_0 = C_0$.

There remains to find $a_k, b_k, a_k^c, b_k^c, a_k^s, b_k^s, A_k^c, B_k^c, A_k^s, B_k^s$, assuming that A_k, B_k are given.

4. Asymptotic expansion

In order to eliminate the above mentioned coefficients we consider first the terms at $\cos k\varphi$. From boundary and transmission conditions we have for $k = 1, 2, \ldots$

$$
\begin{aligned}
A_k^c R^k + B_k^c \frac{1}{R^k} &= A_k \\
A_k^c \rho^k + B_k^c \frac{1}{\rho^k} - a_k &= 0 \\
a_k^c \rho^k + b_k^c \frac{1}{\rho^k} - a_k &= 0 \\
a_k^c (\lambda\rho)^{k-1} - b_k^c \frac{1}{(\lambda\rho)^{k+1}} &= 0 \\
k_1 A_k^c \rho^{k-1} - k_1 B_k^c \frac{1}{\rho^{k+1}} - k_2 a_k^c \rho^{k-1} + k_2 b_k^c \frac{1}{\rho^{k+1}} &= 0.
\end{aligned}
\tag{11}
$$

This may be rewritten in the matrix form: by grouping unknown parameters into a vector $p_k = [A_k^c, B_k^c, a_k^c, b_k^c, a_k]^\top$ we obtain

$$T(k_1, k_2, R, \lambda, \rho) p_k = R^k A_k e_1,$$

where

$$
T = \begin{bmatrix}
R^{2k} & 1 & 0 & 0 & 0 \\
\rho^{2k} & 1 & 0 & 0 & -\rho^k \\
0 & 0 & (\lambda\rho)^{2k} & 1 & -\rho^k \\
0 & 0 & (\lambda\rho)^{2k} & -1 & 0 \\
k_1\rho^{2k} & -k_1 & -k_2\rho^{2k} & k_2 & 0
\end{bmatrix}
\tag{12}
$$

with $e_1 = [1, 0, 0, 0, 0]^\top$.

It is easy to see that

$$p_k = p_k^0 A_k + \rho^{2k} p_k^1 A_k + o(\rho^{2k}), \tag{13}$$

where

$$p_k^0 = \lim_{\rho \to 0+} \lim_{\lambda \to 0+} \frac{p_k(k_1, k_2, R, \lambda, \rho)}{A_k}$$

and $p_k^0 = [1/R^k, 0, 0, 0, 0]^\top$, which corresponds to the ball B_R filled completely with material k_1.

Similar reasoning may be carried out for terms containing $\sin k\varphi$.

Ultimately,

$$\mathcal{A}_{\lambda,\rho} = \mathcal{A}_{0,0} + \rho^2 \mathcal{A}^1_{\lambda,\rho}(k_1, k_2, R, \lambda, \rho, A_1, B_1) + o(\rho^2). \tag{14}$$

The exact form of $\mathcal{A}^1_{\lambda,\rho}(k_1, k_2, R, \lambda, \rho, A_1, B_1)$ is obtained from the inversion of matrix T, but, what is crucial, it is linear in both A_1 and B_1. They, in turn, are computed as line integrals

$$A_1(w) = \frac{1}{\pi R^2} \int_{\Gamma_R} w x_1 \, ds, \quad B_1(w) = \frac{1}{\pi R^2} \int_{\Gamma_R} w x_2 \, ds.$$

As a result, for computing u_ρ we may use the following energy form

$$\begin{aligned}\mathcal{E}(u_\rho) = \frac{1}{2}\int_\Omega k_1 \nabla u_\rho \cdot \nabla u_\rho \, dx + \\ + \rho^2 Q(k_1, k_2, R, \lambda, \rho, A_1, B_1) + o(\rho^2)\end{aligned} \tag{15}$$

where $A_1 = A_1(u_\rho)$, $B_1 = B_1(u_\rho)$, and Q is a quadratic function of A_1, B_1. This constitutes a regular perturbation of the energy functional which allows for computing of perturbations for any functional depending on this solution and caused by small inclusion of the form described above.

5. Asymptotics for plane elasticity boundary value problem

We extend the results obtained for the scalar elliptic boundary value problem to the plane elasticity. The general idea is explained, but we avoid to present the long formulae obtained as a result of the asymptotic expansions. The technique is exactly the same as in the case of Laplacian, with more applications in the optimum design in structural mechanics. The method presented seems to be new and well adapted to applications, since the closed form of the first order term of expansion can be obtained.

Let us consider the plane elasticity problem in the ring $C_{R,\rho}$. We use polar coordinates (r, θ) with $\mathbf{e}_r$ pointing outwards and $\mathbf{e}_\theta$ perpendicularly in the counter-clockwise direction. Then, there exists an exact representation of both solutions, obtained using the complex variable series. It has the form (Kachanov, Shafiro and Tsukrov, 2003; Lurie, 2005; Muskhelishvili, 1952)

$$\begin{aligned}\sigma_{rr} - i\sigma_{r\theta} &= 2\Re\phi' - e^{2i\theta}(\bar{z}\phi'' + \psi') \\ \sigma_{rr} + i\sigma_{\theta\theta} &= 4\Re\phi' \\ 2\mu(u_r + iu_\theta) &= e^{-i\theta}(\kappa\phi - z\bar{\phi}' - \bar{\psi}).\end{aligned} \tag{16}$$

The functions ϕ, ψ are given by complex series

$$\begin{aligned}\phi &= A\log(z) + \sum_{k=-\infty}^{k=+\infty} a_k z^k \\ \psi &= -\kappa\bar{A}\log(z) + \sum_{k=-\infty}^{k=+\infty} b_k z^k.\end{aligned} \tag{17}$$

Here, μ – Lamé constant, ν – Poisson ratio, $\kappa = 3 - 4\nu$ in the plain strain case, and $\kappa = (3-\nu)/(1+\nu)$ for the plane stress.

Similarly as in the simple case described in the former sections, the displacement data may be given in the form of Fourier series,

$$2\mu(u_r + iu_\theta) = \sum_{k=-\infty}^{k=+\infty} A_k e^{ik\theta}. \tag{18}$$

The traction-free condition on some circle means $\sigma_{rr} = \sigma_{r\theta} = 0$. From (16),(17) we get for displacements the formula

$$
\begin{aligned}
2\mu(u_r + iu_\theta) = 2\kappa A r \log(r)\frac{1}{z} - \bar{A}\frac{1}{r}z +& \\
+ \sum_{p=-\infty}^{p=+\infty} [\kappa r a_{p+1} - (1-p)\bar{a}_{1-p} r^{-2p+1}& \\
- \bar{b}_{-(p+1)} r^{-2p-1}] z^p.&
\end{aligned} \tag{19}
$$

Similarly, we obtain representation of tractions on some circle

$$
\begin{aligned}
\sigma_{rr} - i\sigma_{r\theta} = 2A\frac{1}{z} + (\kappa+1)\frac{1}{r^2}\bar{A}z& \\
+ \sum_{p=-\infty}^{p=+\infty} (1-p)[(1+p)a_{p+1} + \bar{a}_{1-p} r^{-2p}& \\
+ \frac{1}{r^2} b_{p-1}] z^p.&
\end{aligned} \tag{20}
$$

As we see, in principle it is possible to repeat the same procedure again, glueing solutions in two rings together and eliminating the intermediary Dirichlet data on the interface. The only difference lies in the considerably more complicated calculations, see, e.g., Gross (1957). This could be applied for making double asymptotic expansion, in term of both ρ and λ. However, in our case λ does not need to be small in comparison to ρ.

6. Conclusions

The explicit form of solutions in B_R allows us to conclude that for

$$\|w_\rho\|_{H^{1/2}(\Gamma_R)} \le \Lambda_0$$

the correction to the energy functional contains the part proportional to ρ^2 and the remainder of the order $\Lambda_0\rho^3$. This in turn, see Sokolowski and Żochowski (2005, 2008), implies the possibility of representation

$$w_\rho = w_0 + \rho^2 q + o(\rho^2) \quad \text{in} \quad H^1(\Omega_R)$$

for both linear and contact boundary value problems, justifying computations of topological derivatives.

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