Raport Badawczy

Research Report

RB/80/2011

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Control and Cybernetics

vol. **40** (2011) No. 4

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by

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Abstract: We propose a methodology to support decisions on how to construct rankings of objects which account for decision makers' preferences. As it is not always so that objects to be ranked are known upfront, the methodology is focused on constructing ranking algorithms rather than rankings themselves.

The methodology builds on Multiple Criteria Decision Making paradigms. To operationalize it we provide a consistent interactive framework which allows the decision maker to express his preferences with respect to objects directly, with respect to the criteria selection process (multiple criteria model building), and with respect to attributes resulting from the selected criteria.

The methodology is illustrated by a numerical example of municipality rankings.

Keywords: multiple criteria ranking, interactive multiple criteria decision making, holistic preferences, atomistic preferences, model building.

1. Introduction

In present-day life, object (variant, potential action) rankings are routinely used to make decisions. Because of their ubiquity, rankings are of extreme importance for quality of our lives. Depending on the context, consequences of a decision taken on the basis of rankings are observed directly, indirectly, instantly, or after some period of time, but in any case – only a posteriori.

In general, decision processes can be differentiated with respect to *participation rights*. In *autonomous processes* participation rights are held exclusively by the decision maker (DM). In consequence, the DM is sovereign as to the course

*Submitted: May 2011; Accepted: October 2011.

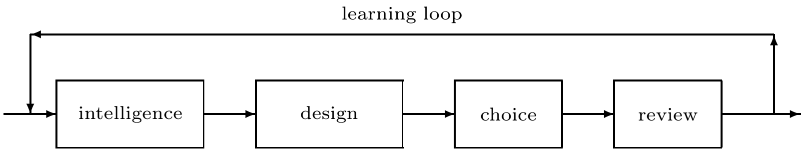

Figure 1 Four phases of decision making process

of the decision making process. In contrast to autonomous processes, in *nonautonomous processes* the course of the decision making process is conditioned by participation rights of other agents.

There is a widely accepted consensus that any rational decision making process consists of four phases closed by the learning loop, namely *intelligence*, *modeling*, *choice*, and *review* (Simon, 1977, see Fig. 1). Following the above definition, the decision process is autonomous if in every phase the DM is sovereign in his decisions.

A process ceases to be autonomous if DM's sovereignty with respect to the course of the decision process becomes constrained. Constraints (permanent or temporary) can be imposed on any stage of decision processes by external agents holding participation rights. A constraint of the sort can be constituted by, for example, the necessity to consult the manner of conducting the decision process, to adhere to predefined decision making scenarios, or to justify the selected scenario to a third party. This can go to the extreme, when participation rights are held by public parties, and the decision maker is bound to select an unequivocal decision making procedure and make it known to interested parties without possibility to make later changes in it. Such processes are called *frozen* (see Chmielewski, 2008). Phases of frozen processes are represented in Fig. 2.

Frozen processes are a rule in public tenders, where participation rights holders, next to the tender issuing party, are also bidders, supervisory agencies, and last but not least the whole society with its state and social (media!) monitoring institutions. Here consequences of selecting an erroneous decision making procedure (ranking algorithm) are borne in the first place by the issuing party, but also, directly or indirectly, by other participation rights holders.

Further instances of frozen processes are provided by procedures for person or institution evaluations of general public interest, for example — official rankings of academic institutions.

In a frozen ranking process the consequences of selecting an unsatisfactory ranking algorithm are irreversible. Moreover, if the selection of a ranking algorithm has to be made a priori, i.e. without knowing objects to be ranked, the chances to select a satisfactory algorithm are limited.

A satisfactory ranking algorithm is understood here as follows: the algorithm is satisfactory if the ranking derived by this algorithm is that which would be derived by the DM if the decision process were autonomous. An unsatisfactory

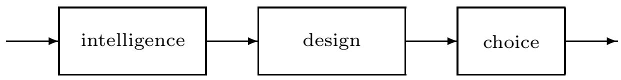

Figure 2 Phases of decision making when the process is frozen

algorithm (and therefore also an unsatisfactory ranking) is the source of DM's actual (posterior) *regret*.

Regret is a notion widely applied and investigated in decision theory, in the form minmax regret principle (see Luce, Raiffa, 1957; Savage, 1951), and also in a form being an extension of utility theory (see Bell, 1982; Lomes, Sugden, 1982). Both approaches rely on probabilities of events, this information being in the case of rankings in general hard to provide.

Therefore, in this work we seek inspiration in behavioral research on *expected regret* conducted by psychologists (Janis, Mann, 1977; see also Zeelenberg, 1999). Janis and Mann claim that anticipated occurrence of regret causes people to make more "rational" decisions – because of *anticipated regret* they think more and think more elaborately before the final decision is made. They state

"Before undertaking any enterprise of great pith and moment, we usually delay action and think about what might happen that could cause regret ... Such worries, which include anticipatory guilt and shame, provoke hesitation and doubt, making salient the realization that even the most attractive of the available choices might turn out badly."

In ranking processes the range of information the DM possess about objects can vary between two extremal cases: from the full extent of information (all objects are known to the DM before the ranking algorithm is selected, e.g. as in the case of ranking academic institutions), to the total lack of information (no object is known, e.g. as in the case of a public tender).

If objects to be ranked are not known before a ranking algorithm is selected, the DM still has possibilities to express his preferences. He can make use of *hypothetical objects*, which reflect his expectations, based on experience and possibly previous similar ranking problems, what real objects can be. Also criteria, which are an element of a DM preference structure, can be selected independently of objects to be ranked.

In this research we have been motivated by the fact that in nonautonomous processes, and specifically in ranking processes, actual regret can be "arbitrarily large". What we aim for is to provide the DM with means (methodologies and computer-based tools) similar to those he has in the case of autonomous ranking processes, namely possibilities to assess the consequences of decisions taken (rankings formed). But as when dealing with autonomous processes DM's goal is to construct the *most preferred ranking*[1], with nonautonomous processes his goal is to select a *ranking algorithm* which would preserve DM's preferences expressed during the ranking selection (decision) process. In other words, we aim to provide means to minimize actual regret understood as the result of comparing ex post DM's expectations with respect to *the most preferred ranking* (i.e. the ranking which the decision maker prefers the most) with the ranking provided by the ex-ante selected ranking algorithm. This, in turn, requires that appropriate (multiple criteria) model selection must be also a part of the ranking algorithm selection process.

The outline of the paper is as follows. In Section 2 we place our research in the context of other related works on the subject. In Section 3 we present the proposed methodology for supporting the DM in selecting a ranking algorithm. In Section 4 we give an illustrative numerical example based on a real problem of annual commune rankings to monitor their development dynamics. Section 5 concludes.

The notation and notions used in this paper are standard ones and therefore we abstain from presenting them at the beginning of the paper. However, to make the paper self-contained and to avoid any ambiguity, they all are collected in Appendix A.

2. Supporting the ranking process

In general, the ranking process consists of providing a *linear order* of objects, whereas in general DM's preferences provide for at most *an order* (also called *partial order*). Orders result from DM's preferences expressed with respect to objects directly, from DM's preferences expressed with respect to a set of criteria used to evaluate objects, or from DM's preferences expressed with respect to objects indirectly via *outcomes* (vectors of object attributes). Given DM preferences in the form of an order, the ranking process is enabled by asking the DM to supply additional information, which, when fed into by some formal constructs, yields eventually a ranking.

There are two basic approaches to ranking. The first ("the mechanics goes first") is to establish a formal construct (function or procedure) which yields a ranking of objects and then to make the DM supply all the necessary information (parameters) to make this work (see, e.g. Roy, 1990; Brans et al., 1986).

The second approach ("the DM goes first") is to let the DM express preferences in a holistic manner (reference ranking(s), reference sets, reference point(s)), and from them to derive a ranking yielding construct (see, e.g. Jacquet-Lagrèze, Siskos, 1982; Skulimowski, 1996; Wierzbicki, 1999; Greco et al., 2007).

Both approaches assume usually that the problem of criteria selection is exogenous to ranking. In this work we take the opposite stance. In accordance

[1]Even if the DM is guided by an algorithm, in the real world he is not constrained by it in arriving at what he regards as the most preferred ranking.

with Simon's aforementioned four-phased decision making model, we include criteria selection into the ranking process. We do that in a flexible manner, claiming only that

either the set of criteria selected (and hence the preference model with respect to criteria values) has to be consistent with preferences expressed prior to criteria selecting,

or preferences expressed prior to criteria selecting have to be modified to be consistent with the set of criteria selected.

3. The methodology

In what follows we propose to combine the two basic approaches to ranking mentioned in the previous section into one and to use the resulting methodology repetitively to account for DM's learning curve (see, e.g. Zangwill, Kantor, 1998). By this token we endow the DM with the chance to express his preferences in the form which he, at a given stage of a ranking process, considers appropriate and convenient. The resulting approach is applicable to decision processes with any spectrum of information about objects in DM's possession, including the case of frozen processes where information is extremely limited (compare the previous section).

3.1. Problem formulation

Consider a finite set X_0 of objects x, $X_0 \subseteq X$.[2] We assume that each object is characterized by the set L, $|L| = l,\ l \geq 2$, of attributes with values defined by criteria

$$f_i :\ X \to \mathcal{R},\ i = 1, \ldots, l,$$

i.e. $f_i(x)$ is the value of i-th attribute of x (see Appendix A (12)). We assume also that in X_0 as well as in $f(X_0)$, where $f = (f_1, \ldots, f_l)$, a partial information on DM's preferences is available and this information has the form of (partial) orders (see Appendix A).

What we aim at is to propose a rational methodology for ranking X_0 by selecting an appropriate ranking algorithm.

3.2. An interactive scheme for object ranking

The motivation of the proposed scheme is twofold.

[2] The approach we propose covers the case where set X_0 is given implicitly by a set of conditions as well as the case where sets X_0 and $f(X_0)$ are explicitly given and thus specifications of set X and functions f_i are irrelevant. The first case covers customarily the Multiple Criteria Decision Making framework, whereas the second the Multi Attribute Decision Making one.

First, it has been regarded desirable to allow the DM to express preferences with respect to objects and with respect to their attributes (outcomes) in a flexible way.

Second, it has been found that the existing methods of object ranking do not satisfactorily address the issue of the modeling phase of the decision making process, as framed by Simon, though the importance of the phase is widely acknowledged. Therefore, it is of utmost importance to provide guidelines for the DM for criteria selecting (multiple criteria model building) in a disciplined manner.

To ease the DM of the burden to provide a vast range of preference information at once, the proposed scheme follows the interactive decision making principle.

SCHEME

Preference elicitation with respect to objects.

Step 1. By pairwise comparisons of selected objects $x \in A$, $A \subseteq X_0$, the DM defines an order $\gg_A$ in A.

Remark. Order $\gg_A$ reflects DM's preferences expressed with respect to pairs of objects (x, x'), $x, x' \in A$, i.e. if x is preferred to x', then $x \gg_A x'$.

Step 2. The DM is presented for analysis with *feasible* rankings which at this step are rankings *consistent* with order $\gg_A$ in X_0 (for the definition of order consistency refer to Appendix A).

Remark. By order consistency we now have

$$\gg_A \subseteq \kappa ,$$

where κ is a feasible ranking of X_0 .

Depending on the results of the analysis, the DM can modify his preferences with respect to objects (modification of $\gg_A$, modification of A – moving to Step 1) or resort to preference expression with respect to criteria selection (Step 3).

Preference elicitation with respect to criteria selection.

Step 3. From the given set of criteria L, a subset $K \subseteq L$, $|K| = k$, $k \leq l$, is selected (without loss of generality we assume that criteria in K are numbered 1 to k) such that order $\gg_{X_0}$ induced in X_0 by *dominance relation* $\gg$ (for the definition of the dominance relation refer to Appendix A) defined on set of outcomes $Z = f(X_0)$, where $f = (f_1, \ldots, f_k)$, is consistent with order $\gg_A$ (i.e. $\gg_A \subseteq \gg_{X_0}$).

Step 4. The DM is presented with feasible rankings, which, at this step, are rankings of X_0 consistent with order $\gg_{X_0}$ induced in X_0 by dominance relation $\gg$, with $\gg$ and Z as in Step 3.

Remark. The presented rankings are in general quasi-rankings of X_0, i.e. $\gg_{X_0}$ is a linear quasi-order (see Lemma A.2).

By order consistency we now have

$$\gg_A \subseteq \gg_{X_0} \subseteq \kappa',$$

where κ' is a feasible ranking of X_0.

Depending on the results of the analysis, the DM can modify his preferences with respect to objects (modification of $\gg_A$, modification of A – moving to Step 1) or modify criteria selection (modification of K – moving to Step 3) or resort to preference expression with respect to outcomes (Step 5).

Preference elicitation with respect to outcomes.

Step 5. By pairwise comparisons of selected outcomes of T, $T \subseteq Z$, with Z as in Step 3, the DM defines order $\gg_T$ consistent with dominance relation $\gg$ (i.e. $\gg \subseteq \gg_T$). In this step, as explained in Subsection 3.4, order $\gg_T$ is associated to a family of functions, each function of this family represents a feasible ranking, and thus constitutes a ranking algorithm ("rank by the function value") which yields this ranking.

Remark. Order $\gg_T$ reflects DM's preferences revealed with respect to pairs of outcomes (y, y'), $y, y' \in T$, i.e. if y is preferred y', then $y \gg_T y'$.

Step 6. The DM is presented for analysis with feasible rankings which at this step are rankings consistent with order $\gg_{T_{X_0}}$ induced in X_0 by order $\gg_T$ of X_0.

Remark. The presented rankings are in general quasi-rankings of X_0.

By order consistency we now have

$$\gg_A \subseteq \gg_{X_0} \subseteq \gg_{T_{X_0}} \subseteq \kappa'',$$

where κ'' is a feasible ranking of X_0.

Depending on the results of the analysis the DM can modify his preferences with respect to objects (modification of $\gg_A$, modification of A – moving to Step 1) or modify criteria selection (modification of K – moving to Step 3) or modify his preferences with respect to outcomes (modification of $\gg_T$, modification of T – moving to Step 5).

END OF *SCHEME*

SCHEME terminates when the cardinality of feasible rankings equals 1 or the DM wishes to terminate.

SCHEME lets the DM reveal his preferences with respect to objects in three ways: directly and indirectly by criteria selection, and indirectly via outcomes. Only after the DM reveals a range of his preferences, *SCHEME* attempts to support him by providing the following information

— the actual number or at least an upper bound on the number of feasible rankings,
— all feasible rankings or at least a subset of them.

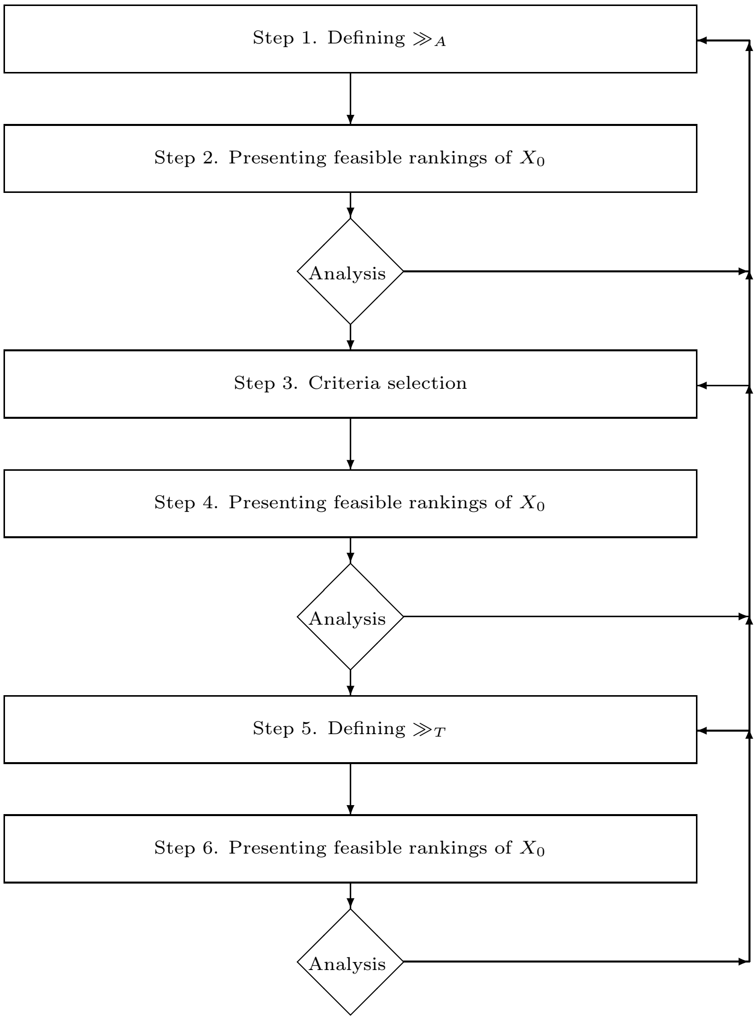

Figure 3 Flow diagram of *SCHEME*

A too large number of feasible rankings to enable their explicit inspection by the DM does not stop the decision process. On the opposite, such information can stimulate the DM to make efforts to reveal his preferences further.

Clearly, *SCHEME* ends up with a single ranking only if the DM reveals enough preference information to make the set of feasible rankings a singleton. Otherwise, upon termination, the *SCHEME* ends up with a multitude of feasible rankings, all equivalent with respect to preference information revealed by the DM.

The outline of the *SCHEME* presented here is purely conceptual and ignores such technicalities like which or how many feasible rankings are presented to the DM. We consider these questions later on and we show how they can be operationalized by an illustrative example in Section 4.

SCHEME can be justified step by step by the following arguments.

Step 1. In Step 1 the *SCHEME* offers the DM a possibility to reveal his preference with respect to objects by pairwise comparisons to the extent the DM regards appropriate.

We consciously depart from the assumption, very often adopted in the literature, on completness of preferences with respect to pairs of objects (see, e.g., works related to the AHP method, Saaty, 1980) because, whether in practice or laboratory experiments, even with a few objects this assumption is unrealistic. Similarly, we consciously abstain from adopting any, always to some extent arbitrary, method to fill lacking preferences, either with or without the controversial and contested assumption about the so called pairwise preference consistency (see, e.g. Fedrizzi, Giove, 2007). In consequence, after Step 1 the number of feasible rankings can be prohibitively large for explicit presentation and analysis. This is the price to be paid at an early stage of interacting with the *SCHEME* for the lack of any arbitrary assumption, which could result in fast, but often premature (from the point of view of DM's learning curve), reduction of the number of feasible rankings.

Step 2. The number of feasible rankings can be too large to be presented to the DM. In that case only a (reasonably limited) subset of rankings is presented, together with the number of feasible rankings or at least an upper bound on that number. Some results for calculating upper bounds on the number of rankings in a set with an order are presented in Appendix B.

The information that the number of feasible rankings is still large can be an incentive for the DM to engage further in preference revealing with respect to objects. The DM can do this by any number of moves from Step 2 back to Step 1 or to Step 3.

Step 3. Selecting criteria by the DM is always a subjective process. However, when selecting criteria, the preferences expressed by the DM with respect to objects before criteria selection should be taken into account to guarantee preference consistency.

To assist the DM in consistent criteria selection, which can necessitate modification of DM's preferences expressed with respect to objects, the concept of *discordance matrices* has been elaborated (see Appendix E).

Establishing consistency between preferences revealed by pairwise comparisons of objects and preferences implied by the selected criteria (the dominance relation) is an important step in building a consistent DM's preference structure. This process should be carried out till consistency is achieved (i.e. till inclusion $\gg_A \subseteq \gg_{X_0}$ is fulfilled) via modification of relation $\gg_A$, change of the set A, modification of criteria selection or any combination of those actions. The proposed *SCHEME* enables such actions in this very step or in Step 1.

The selected criteria enable the DM to further reveal his preferences (see Steps 3 and 5 of the *SCHEME*).

Step 4. Comments to Step 2 apply also here. This time, the calculations of upper bounds on the number of feasible rankings can be refined by the results presented in Appendix C, which exploit the dominance structure among outcomes derived in Step 3.

Step 5. In Step 5 the *SCHEME* offers the DM the possibility to reveal his preference with respect to objects by pairwise comparisons of outcomes to the extent the DM regards appropriate.

Establishing consistency between preferences expressed by pairwise comparisons of objects, preferences resulting from the selected criteria and the thus defined dominance relation, and preference expressed by pairwise comparisons of outcomes, is another important step to building a consistent DM's preference structure. This process should be carried out till full consistency is achieved (i.e. till inclusions $\gg_A \subseteq \gg_{X_0} \subseteq \gg_{T_{X_0}}$ are fulfilled) via modifications of relation $\gg_A$, change of the set A, modification of criteria selection, modification of relation $\gg_T$, or by any combination of these actions. The proposed *SCHEME* enables such actions in this very step or in Step 1 or Step 3.

An important part of this step is the association of order $\gg_T$ to a family of functions, elements of which constitute ranking algorithms. This is explained in detail below in Subsection 3.4 and illustrated in Section 4.

Step 6. Comments to Step 4 apply also here. The same results as in Step 4 can be used to calculate an upper bound on the number of feasible rankings.

The *SCHEME* could be presented without distinguishing the subsets A and T, and the DM would work then with the whole sets X_0 and Z, respectively. In practice, however, the DM tends to focus on a subset of objects or outcomes of limited cardinality, and preferences expressed with respect to elements of such sets are next applied to the whole set of objects.

3.3. Discussion

The assumption that the DM is at any time fully conscious of his preferences is widely questioned. Moreover, this assumption almost never holds in practice. It is expedient then to assume just the opposite, namely that at the beginning of the decision process DM's preferences are vague. The aim of the decision process is then to make those preferences gradually precise, along DM's learning curve, by confronting preferences revealed in the course of the process with the consequences represented by feasible rankings. In the case of frozen processes, proceeding in this fashion is of particular importance.

It is worth observing that the *SCHEME* applies to processes with any range of autonomy, from autonomous to frozen processes as the extremes.

It should be stressed here that because the assumed decisive role of the DM in ranking, in nonautonomous ranking processes there is no mechanism to *guarantee* confinement of actual regret. This is also not guaranteed by the use of the *SCHEME* but it provides for actual regret confinement. Indeed, by interacting with the *SCHEME* in what in fact is the Simon's learning loop (see Figs. 1 and 3), the DM expands his understanding of the ranking problem. This process is accompanied by more or less conscious accounting for expected regret. On the grounds of psychological findings mentioned in Introduction, it can be rationally expected that accounting for expected regret during the DM – *SCHEME* interactions can result in reduction of actual regret.

Adopting a framework in which the DM has a possibility (but no necessity) to act in two spaces to reveal his preferences – in the space of objects and in the space of outcomes – is in our opinion a novel approach. Besides evaluating objects via criteria (attributes), i.e. in *atomistic* manner, the DM can also evaluate objects in *holistic* (without paying regard to attributes) manner. In this approach criteria selection is the necessary step to resolve any inconsistencies between preferences expressed in these two spaces.

Up to now we have proposed how to support ranking processes in an interactive manner. Below we are concerned with how to select a ranking algorithm, which preserves DM's preferences expressed during the ranking process.

3.4. Establishing a ranking algorithm

To enable establishing ranking algorithms we extend Step 5 of the *SCHEME* and make use there of some interactive MCDM methodologies. We follow the approach to searching for the most preferred object as proposed in the Zionts-Wallenius method (Zionts, Wallenius, 1983) and its generalization, the Dell-Karwan method (Dell, Karwan, 1990).

We assume that the DM has implicit value function $v(\cdot)$, i.e. a function the form of which is unknown, or at least not revealed. The DM compares outcomes pairwise and consistently with his implicit value function decides whether he prefers one outcome over another. The mechanism employed to frame the preferences expressed by the DM is constituted by the parameters of some functions used as *proxy* to the implicit value function.

We assume the implicit value function to be *strongly monotone* (i.e. *strongly decreasing* or *strongly increasing*) on T, which is quite intuitive. Hence, the following relation holds

$$y \gg_T y',\ y \neq y',\ \Rightarrow\ v(y) > v(y'), \tag{1}$$

for strongly increasing functions, and

$$y \gg_T y',\ y \neq y',\ \Rightarrow\ v(y) < v(y'), \tag{2}$$

for strongly decreasing functions. Each of these relations establishes a condition for an implicit value function.

With the assumption of the implicit value function monotonicity and DM's preference consistency with that function there is no need to compare pairs of outcomes where one is dominated by another. In such a case DM's preference results directly from the property of monotonicity.

To represent implicit value functions we make use of explicit proxy value functions $g(\cdot)$. Consistently with (1) and (2), we assume that functions $g(\cdot)$ are strongly increasing or strongly decreasing, i.e.

$$y \gg_T y',\ y \neq y'\ \Rightarrow\ g(y) > g(y'),\quad y, y' \in T,\ T \subseteq Z, \tag{3}$$

or

$$y \gg_T y',\ y \neq y'\ \Rightarrow\ g(y) < g(y'),\quad y, y' \in T,\ T \subseteq Z, \tag{4}$$

respectively.

Values of functions $v(\cdot)$ and $g(\cdot)$ provide rankings of the outcome set Z, and therefore also quasi-rankings of the set X_0.

In particular, we make use of two families of strongly monotone proxy functions $g(\cdot)$ parameterized by the weight vector λ, $\lambda_i > 0,\ i = 1, \ldots, k$, widely used in MCDM, namely

— a family of linear functions (strongly increasing)

$$g(y) = \sum_{i=1}^{k} \lambda_i y_i\,, \tag{5}$$

— a family of regularized Tchebycheff functions (strongly decreasing for $y_i \leq y_i^*,\ i = 1, \ldots, k$)

$$g(y) = \max_i \lambda_i((y_i^* - y_i) + \rho e^k(y^* - y))\,, \tag{6}$$

where $\rho > 0$, e^k is k-dimensional vector $(1, ..., 1)$, and element y^* is a perturbation of *utopia point* (see Appendix A, (14)).

As no a priori specification of DM's implicit value function is made here, it is left to the DM to select the family he likes to work with or he can work simultaneously with both families (see the numerical example in Section 4).

Functions of both families are scalarizing functions (see Wierzbicki, 1999, also Miettinen, 1999; Ehrgott, 2005; Kaliszewski, 2006), i.e. $\bar{y}$ is an efficient outcome if

$$\bar{y} = \arg\max_{y \in Z} g(y), \tag{7}$$

for linear functions, and

$$\bar{y} = \arg\min_{y \in Z} g(y), \tag{8}$$

for regularized Tchebycheff functions.

The system of inequalities (3) or (4) imposes conditions on the proxy function $g(\cdot)$, which in the case of linear functions and regularized Tchebycheff functions reduces to conditions on their parameters λ_i .

With s pairwise comparisons of outcomes we get a system of at most s inequalities in the form

$$g(y^t) > g(y^{'t}),\ t = 1, ..., s, \tag{9}$$

for strongly increasing functions, and

$$g(y^t) < g(y^{'t}),\ t = 1, ..., s, \tag{10}$$

for strongly decreasing functions, where y^t and $y^{'t}$ are elements of pair t of compared outcomes.

For the family of linear proxy functions conditions (9) take the form of a system of constraints on vectors λ, namely

$$\sum_{i=1}^{k} \lambda_i y_i^t > \sum_{i=1}^{k} \lambda_i y_i^{'t},\ t = 1, ..., s. \tag{11}$$

Every vector $\lambda \in \Lambda$, where $\Lambda = \{\lambda \,|\, \lambda_i > 0,\ i = 1, ..., l\}$, satisfying (11), defines function $\sum_{i=1}^{k} \lambda_i y_i$, values of which yield a ranking X_0 consistent with preferences $y^t \gg_T y'^t,\ t = 1, ..., s$. The set of all such vectors, denoted $\bar{\Lambda}$, represents the extent of DM's flexibility (but also his arbitrariness) when selecting a ranking algorithm.

In an analogous way we specify the set $\bar{\Lambda}$ for the family of regularized Tchebycheff functions.

Without loss of generality we assume that vectors of Λ satisfy an additional condition $\sum_{i=1}^{k} \lambda_i = 1$.

The idea of selecting a ranking algorithm in Step 5 of the *SCHEME* reduces to selecting an element λ from the set $\bar{\Lambda}$ defined above. In this case a ranking algorithm has the form of a function from one of the considered families.

One option is to select λ as the middlemost element of $\bar{\Lambda}$. [3]

[3] With the Tchebycheff functions the set $\bar{\Lambda}$ can be a union of disjoint sets (see Dell, Karwan, 1990; Chmielewski, 2008; Chmielewski, Kaliszewski, 2009) and then the selection of λ gets more involved.

The case $\bar{\Lambda} = \emptyset$ signals inconsistency between DM's preferences specified and the monotonicity of the selected proxy functions. To restore consistency some of DM's evaluations are to be revoked.

4. An illustrative example

To illustrate the versatility of *SCHEME* we use data from the year 2004 for 177 municipalities of the Province of Łódź with 12 selected (out of several more) attributes used as criteria [4] (see Table 1). For the examplary ranking problem presented here we use data for the first 10 (alphabetic order) communes (municipalities). Original values of attributes were mapped into a numerical scale (points), as shown in Table 2 [5].

To proceed according to a credible decision making scenario, the ranking represented in Table 3 – Ranking 2004 – was used as a reference [6]. In the scenario assumed below for this example all preferences are consistent with Ranking 2004.

Below we present one pass of the *SCHEME* for the considered problem.

SCHEME

Step 1. According to the assumed scenario the DM defines the following preferences: Bełchatów-town is preferred to all remaining communes except Brzeziny-town, and Brzeziny-town is preferred to all remaining communes except Bełchatów-town.

Step 2. Feasible rankings are identified by enumeration. Enumeration is stopped whenever the number of identified feasible rankings reaches an adjustable threshold value, for a schematic description of the enumeration algorithm see Appendix D. Here the threshold value was set to 300 for all respective steps: Step 2, Step 4, and Step 6.

In this case the number of enumerated feasible rankings reached 300 and the enumeration algorithm stopped [7]

By Lemma B.2 the maximal number of feasible rankings is 907,200, hence the number of feasible rankings is between 300 and 907,200. To calculate this number some subsets of A (in this scenario $A = X_0$) have been selected such that in each subset the order $\gg_A$ is a ranking, and then the formula from Lemma B.2 has been used.

[4] All data used here come from a research conducted at the the Faculty of Administration and Social Sciences of the Technical University of Warsaw. The Authors are grateful to Professor Eugeniusz Sobczak from the Faculty for his kind assistance.

[5] "Złoty" is the Polish currency unit.

[6] This ranking was derived during the research mentioned in Footnote 4 before.

[7] In an implementation this algorithm can run in the background updating in real-time the lower bound on the number of feasible ranking.

Table 1 Criteria for municipality ranking

No	Criterion
1	Investments per capita
2	Transport and communication investments per capita
3	Share of investments in the commune budget
4	Employment per 1000 inhabitants
5	Unemployment per 1000 inhabitants
6	Migration inflow per year
7	Migration outflow per year
8	High school graduates per 1000 inhabitants per year
9	Percentage of population with access to water-pipe system
10	Percentage of population with access to sever-pipe system
11	Percentage of population with access to sever processing plant
12	The number of registered businesses per 1000 inhabitants

Table 2 Criteria mapping into a point scale

Criterion	Mapping
1	+1 point for each 10 złoty
2	+1 point for each 10 złoty
3	+1 point for each percent
4	+1 point for each person
5	-1 point for each person
6	+1 point for each person
7	-1 point for each person
8	+1 point for each person
9	+1 point for each percent
10	+1 point for each percent
11	+1 point for each percent
12	+1 point for each business

Table 3 Ranking 2004

No	Commune
1	Bełchatów-town
2	Brzeziny-town
3	Bolesławiec
4	Brójce
5	Andrespol
6	Bolimów
7	Bełchatów
8	Buczek
9	Brzeziny
10	Brąszewice

Step 3. The DM selects with the help of a discordance matrix (see Appendix E) presented in Table 5 a subset of six criteria: 4, 8, 9, 10, 11 and 12 (according to numbering in Table 1).

Step 4. The number of feasible rankings, as determined in Step 2, is over 300.

By Lemma B.2 the maximal number of feasible rankings is 907,200, hence the number of feasible rankings is between 300 and 907,200. [8]

Step 5. According to the assumed scenario, the preferences with respect to criteria values (outcomes) are defined for 12 pairs of communes. The *SCHEME*, accounting for transitivity of relation $\gg_T$, automatically identifies preferences for 7 other pairs. The list of pairs with preferences is presented in Table 4. Preferences resulting from transitivity of relation $\gg_T$ are indicated in the column *Comments*.

Step 6. The number of feasible rankings of X_0 is 102.

By Lemma B.2 the maximal number of feasible rankings is 25,200.

At the DM request the *SCHEME* can list out all 102 feasible rankings. At the DM request the *SCHEME* can also derive rankings for the middlemost elements (weights) of the set $\bar{\Lambda}$ for the selected proxy function. Rankings with middlemost weights clearly possess some extent of stability with respect to weight perturbations (Chmielewski, 2008). Such rankings (with $\rho = 0,0001$ for the class of regularized Tchebycheff functions) together with the corresponding middlemost weights λ are presented in Table 6.

END OF *SCHEME*

On the basis of this example it is worth observing that presenting rankings to the DM opens way to further supporting him in revealing his preferences. The DM either accepts a ranking or discards it. Discarding the ranking means that the DM does not accept precedence of objects as defined by the ranking. To eliminate this ranking from further considerations it is enough for the DM to reverse one pairwise preference consistent with this ranking and then proceed according to the *SCHEME*. This represents a rather radical change of preferences since such a move rends infeasible also all other currently feasible rankings.

[8] In this example we disregarded preferences introduced by the dominance relation. Had they been taken into account, they could have produced either the exact number of feasible rankings (if this number was less or equal 300) or a tighter upper bound.

Table 4 Pairs of outcomes with preferences

Lp	**Preferences**	**Comments**
1	Andrespol $\gg_T$ Bełchatów	
2	Bolesławiec $\gg_T$ Bolimów	
3	Bolesławiec $\gg_T$ Buczek	
4	Bolesławiec $\gg_T$ Bełchatów	
5	Bolesławiec $\gg_T$ Brąszewice	
6	Brójce $\gg_T$ Brzeziny	
7	Brójce $\gg_T$ Bełchatów	
8	Bolimów $\gg_T$ Bełchatów	
9	Bolimów $\gg_T$ Buczek	
10	Bolimów $\gg_T$ Brzeziny	
11	Buczek $\gg_T$ Brąszewice	
12	Bełchatów $\gg_T$ Buczek	
13	Bolesławiec $\gg_T$ Brzeziny	by transitivity of $\gg_T$
14	Bolimów $\gg_T$ Brąszewice	by transitivity of $\gg_T$
15	Andrespol $\gg_T$ Buczek	by transitivity of $\gg_T$
16	Brójce $\gg_T$ Buczek	by transitivity of $\gg_T$
17	Bełchatów $\gg_T$ Brąszewice	by transitivity of $\gg_T$
18	Andrespol $\gg_T$ Brąszewice	by transitivity of $\gg_T$
19	Brójce $\gg_T$ Brąszewice	by transitivity of $\gg_T$

Table 5 The discordance matrix

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	$\mathbf{DN}_h$
$x^1 \gg_A x^3$	0	0	0	1	0	1	0	1	1	1	1	1	5
$x^1 \gg_A x^4$	0	0	0	1	0	0	1	1	1	1	1	1	5
$x^1 \gg_A x^5$	0	0	0	1	0	0	1	1	1	1	1	1	5
$x^1 \gg_A x^6$	0	1	0	1	0	1	0	1	1	1	1	1	4
$x^1 \gg_A x^7$	0	0	0	1	0	0	1	1	1	1	1	1	5
$x^1 \gg_A x^8$	0	0	0	1	0	0	1	1	1	1	1	1	5
$x^1 \gg_A x^9$	0	0	0	1	0	0	1	1	1	1	1	1	5
$x^1 \gg_A x^{10}$	0	0	0	1	0	1	1	1	1	1	1	1	4
$x^2 \gg_A x^3$	0	0	0	1	0	0	0	1	1	1	1	1	6
$x^2 \gg_A x^4$	0	0	0	1	0	0	0	1	1	1	1	1	6
$x^2 \gg_A x^5$	0	1	0	1	0	0	0	1	1	1	1	1	5
$x^2 \gg_A x^6$	0	1	0	1	0	0	0	1	1	1	1	1	5
$x^2 \gg_A x^7$	0	0	0	1	0	0	0	1	1	1	1	1	6
$x^2 \gg_A x^8$	0	0	0	1	1	0	0	1	1	1	1	1	5
$x^2 \gg_A x^9$	1	0	0	1	0	0	0	1	1	1	1	1	5
$x^2 \gg_A x^{10}$	0	0	0	1	0	0	0	1	1	1	1	1	6
$\mathbf{DN}_v$	15	13	16	0	15	13	10	0	0	0	0	0	

Table 6 The most stable feasible rankings

Linear function	**regularized Tchebycheff**
	$\rho = 0,0001$
$\lambda_1 = 0,085526$	$\lambda_1 = 0,143722$
$\lambda_2 = 0,179831$	$\lambda_2 = 0,169977$
$\lambda_3 = 0,193864$	$\lambda_3 = 0,169977$
$\lambda_4 = 0,177339$	$\lambda_4 = 0,169977$
$\lambda_5 = 0,179818$	$\lambda_5 = 0,176360$
$\lambda_6 = 0,183622$	$\lambda_6 = 0,169977$
Rankings	
1) Brzeziny-town	1) Brzeziny-town
2) Bełchatów-town	2) Bełchatów-town
3) Andrespol	3) Andrespol
4) Bolesławiec	4) Bolesławiec
5) Bolimów	5) Brójce
6) Brójce	6) Bolimów
7) Bełchatów	7) Buczek
8) Brzeziny	8) Bełchatów
9) Buczek	9) Brzeziny
10) Brąszewice	10) Brąszewice

5. Concluding remarks

The proposed approach gathers three major components pertaining to rankings, namely

- holistic assessment and elicitation of DM's partial preferences by pairwise comparisons of objects,
- building multicriteria (multiattribute) models of the ranking problems which preserve DM's partial preferences to objects,
- assessment and elicitation of DM's partial preferences by pairwise comparisons of objects via criteria values (attributes),

and provides a framework for maintaining consistency between these components. In the approach the distinction between Multiple Criteria Decision Making and Multiple Attribute Decision Making is not material.

The only underlying assumptions adopted in the approach are monotonicity of the implicit value function and of the proxy value functions used to represent DM's preferences, and consistency of DM's pairwise evaluations with the monotonicity assumption. As there is no built-in mechanism in the approach which, artificially or subjectively, limits the scope of DM's possible choices from among feasible rankings or biases him towards some specific feasible rankings, so we regard this approach as "preference fair".

As the result, the DM is confronted with a family of feasible rankings. Any ranking from that family consistently corresponds to DM preferences expressed in the course of the decision making (ranking building) process. At least an upper bound on the number of feasible rankings is provided.

In consequence, the DM has a clear alternative: either to select a ranking from the family of feasible rankings (such a selection can be either made at random or guided by some ”external hint”) or provide some more preferences to size down the set of feasible rankings to the point he regards appropriate.

As rankings in the approach have the form of functions, in each case such a function is a basis for a ranking algorithm.

It should be reiterated here that the proposed approach is uniform regarding problems with any extent of information about objects to be ranked, from the full extent of information to the total lack of information, and anything in between.

Acknowledgement

We would like to extend our sincere gratitude to Dmitry Podkopayev from the Systems Research Institute for his careful reading of the draft of this paper and his valuable comments and suggestions.

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A. Appendix – notions, definitions and notation

In the paper we refer to *partial orders* (*orders* for short).

Definition A.1 *Relation θ defined on X is an* order *if it satisfies the following conditions*

$$\begin{array}{rll} \forall x \in X: & x\theta x, & \textit{(reflexivity)} \\ \forall x,y,z \in X: & (x\theta y \wedge y\theta z) \Rightarrow x\theta z, & \textit{(transitivity)} \\ \forall x,y \in X: & (x\theta y \wedge y\theta x) \Rightarrow x = y. & \textit{(antisymmetry)} \end{array}$$

A set with an order is called *ordered set*.

Definition A.2 *Relation θ defined on X is a* quasi-order *if it satisfies conditions of reflexivity and transitivity.*

A set with a quasi-order is called *quasi-ordered set*.

Definition A.3 *Order θ defined on X is a* linear order *if it satisfies the following condition*

$$\forall x, y \in X : \quad x\theta y \ \vee \ y\theta x. \quad \textit{(completeness)}$$

Definition A.4 *Quasi-order θ defined on X is a* linear quasi-order *if it satisfies the completeness condition.*

A set with a linear order is called *linearly ordered set* and a set with linear quasi-order is called *linearly quasi-ordered set*.

A linear order is also referred to as *ranking* and a linear quasi-order as *quasi-ranking*.

In ordered sets *maximal elements* are distinguished.

Definition A.5 *Given set X with a linear order (ranking) θ and elements $x, y \in X,\ x \neq y$, element x is said to* precede *element y if $x\theta y$.*

Definition A.6 *Let θ be an order defined on X. Element $x^{max} \in X$ is called* maximal *if*

$$\sim \exists x \in X,\ x \neq x^{max} : \quad x\theta x^{max}.$$

In linearly ordered set the definition of maximal element is equivalent to the following definition.

Definition A.7 *Let θ be a linear order defined on X. Element $x^{max} \in X$ is called* maximal *if*

$$\forall x \in X : \quad x^{max}\theta x.$$

An element which is maximal in a linearly ordered set (ranking) is called *the first element*.

In a linearly ordered set there exists exactly one first element.

Definition A.8 *Let θ be a relation defined on X. We say that θ is* consistent *with relation $\succ$ defined on subset $A,\ A \subseteq X$, if for each $x, x' \in A$, the following holds*

$$x \succ x' \Rightarrow x\theta x',$$

in other words, $\succ \subseteq \ \theta$.

The Multiple Criteria Decision Making (MCDM) consists in selecting *the most preferred object (element of a set)*, i.e. an object, which the DM prefers most, in the presence of multiple criteria.

Let x denote an object, X_0 a *set of (admissible) objects*, and X a *space of objects*. The underlying formal model of MCDM is as follows

$$\begin{aligned} &\text{"}vmax\text{"} f(x), \\ &x \in X_0 \subseteq X, \\ &f : X \to \mathcal{R}^k, \\ &f = (f_1, ..., f_k), \\ &f_i : X \to \mathcal{R}, \quad i = 1, ..., k, \quad k \geq 2, \end{aligned} \tag{12}$$

where $f_i,\ i = 1, ..., k$, are *criteria*, "*vmax*" denotes the operator of derivation of all *efficient objects* (see definition below) of X_0, $\mathcal{R}^k$ is *outcome space*. Without loss of generality we assume that all criteria are of the type "the more the better".

We introduce the following notation

$$y = f(x), \quad Z = f(X_0),$$

and elements of Z we call *outcomes*.

DEFINITION A.9 *Outcome $\bar{y}$ is called* efficient *if*

$$y_i \geq \bar{y}_j,\ i = 1, ..., k,\ y \in Z, \quad \textit{implies} \quad y = \bar{y}.$$

In Z the *dominance relation* is defined.

DEFINITION A.10 *Relation $\gg$ defined in $\mathcal{R}^k$ as*

$$y \gg y' \textit{ if } y_i \geq y'_i \textit{ for all } i,\ i = 1, ..., k,$$

$$\textit{and there exists } i,\ 1 \leq i \leq k,\ \textit{such that } y_i > y'_i$$

is called dominance relation.

Relation $\gg$ is not an order (for it is not reflexive, see Definition A.1).

Outcome y' for which there exists y such that $y \gg y'$ is called *dominated* and y is called *dominating*.

By Definition A.10 an outcome y is efficient if there is no outcome y' such that $y' \gg y$.

Let $\gg_T$ be an order defined on set T, $T \subseteq Z$.

DEFINITION A.11 *Relation $\gg_{T_{X_0}}$ defined as*

$$x \gg_{T_{X_0}} x' \Leftrightarrow y \gg_T y',$$

where $y = f(x), y' = f(x')$, is called induced in X_0 *by* $\gg_T$.

The relation induced by dominance relation $\gg$ is denoted by $\gg_{X_0}$.

The next two lemmas are direct consequences of Definition A.11.

Lemma A.1 *The relation induced in X_0 by an order defined in T, $T \subseteq Z$, is a quasi-order in X_0.*

Lemma A.2 *The relation induced in X_0 by a linear order defined in T, $T \subseteq Z$, is a linear quasi-order in X_0.*

Lemma A.3 *The relation induced in X_0 by dominance relation $\gg$ is a quasi-order.*

Proof. The proof follows from the fact that the dominance relation is a subset of the order defined as

$$y \widetilde{\gg} y' \quad \Leftrightarrow \quad y_i \geq y'_i, \quad i = 1, ..., k. \qquad \blacksquare$$

Definition A.12 *Function $u(y) : R^k \to R$ is called* strongly increasing *on T if*

$$y \gg_T y' \quad \textit{and} \quad y \neq y' \quad \Rightarrow \quad u(y) > u(y').$$

Definition A.13 *Function $u(y) : R^k \to R$ is called* strongly decreasing *on T if*

$$y \gg_T y' \quad \textit{and} \quad y \neq y' \quad \Rightarrow \quad u(y) < u(y').$$

Element $\hat{y}$ of R^k defined as

$$\hat{y}_i = \max_{y \in Z} y_i\,, \ i = 1, ..., k, \tag{13}$$

is called *utopia point*. We assume that $\hat{y}$ exists. We make use of element $y^* \in R^k$ defined as

$$y^* = \hat{y} + \bar{\epsilon}, \quad \bar{\epsilon} = \{\epsilon, \epsilon, ..., \epsilon\}, \ \epsilon > 0. \tag{14}$$

Definition A.14 *Object x is called efficient if outcome $y = f(x)$ is efficient.*

B. Appendix – Ordered sets and rankings – cardinality considerations

Let us observe that the number of rankings of G consistent with an order defined on its subset H (i.e. feasible rankings of G) depends on the number of rankings of H consistent with the order (feasible rankings of H). In particular, if a ranking of H is given, then the number of feasible rankings of G depends only on the number of rankings of $G \setminus H$.

Let $|G| = g$ and $|H| = h$, where $H \subseteq G$. Assume that a ranking of H is given.

LEMMA B.1 *The number of rankings of G when a ranking defined on H is given is equal to $\frac{g!}{h!}$.*

Proof. In $g!$ rankings (permutations) of g elements of G there are $h!$ rankings (permutations) of h elements of H.

If in $g!$ rankings of G each ranking of H is replaced by a given ranking, then the number of rankings of G is reduced $h!$ times. Hence, the number of rankings of G when a ranking of H is given is equal to $\frac{g!}{h!}$. ∎

Let $H^1, ..., H^t$ be disjoint subsets of G, i.e. $H^i \cap H^j = \emptyset,\ i,j = 1,...,t,\ i \neq j$, and $|H^i| = h^i,\ i = 1,2,...,t$. Assume that rankings of $H^i,\ i = 1,...,t$, are given.

LEMMA B.2 *The number of rankings of G when rankings of $H^1, ..., H^t$ are given is equal to $\frac{g!}{h^1! \cdot ... \cdot h^t!}$.*

Proof. For $t = 1$ the prof follows from Lemma B.1.

Assume $t = 2$. Because ranking of H^1 is given, by Lemma B.1 the number of rankings of G is equal to $\frac{g!}{h^1!}$.

If in $\frac{g!}{h^1!}$ rankings of G each ranking of H^2 is replaced by a given ranking, then the number of rankings of G is reduced $h^2!$ times. Hence, the number of rankings of G when rankings of H^1 and H^2 are given is equal to $\frac{g!}{h^1! \cdot h^2!}$.

The argument can be continued for any t. ∎

C. Appendix – Ordered sets and rankings – further cardinality considerations

Results of Appendix B do not account for the dominance relation and the fact that proxy functions $g(\cdot)$ we make use of are strongly increasing or decreasing. Yet since each such function "preserves order" (see Definitions A.12 and A.13), a dominated outcome does not precede in a ranking a dominating one, as shown by the next lemma.

LEMMA C.1 *In outcome rankings yielded by linear functions or regularized Tchebycheff functions, dominated outcomes do not precede their dominating outcomes.*

Proof. Assume that dominated outcome y precedes in a ranking its dominating outcome y'.

Consider linear functions first. If y precedes in a ranking y', then there exists $\lambda,\ \lambda_i > 0,\ i = 1,...,k$, such that $\sum_{i=1}^{k} \lambda_i y_i > \sum_{i=1}^{k} \lambda_i y'_i$ holds. But by the definition of the dominance relation we have $y'_i \geq y_i$ for $i = 1,...,k$ and $y'_i > y_i$ for some i, hence for any $\lambda_i > 0,\ i = 1,...,k$, we have $\sum_{i=1}^{k} \lambda_i y_i < \sum_{i=1}^{k} \lambda_i y'_i$, which is a contradiction.

Consider now regularized Tchebycheff functions. If y precedes y' in a ranking, then there exists $\lambda > 0,\ \lambda_i > 0,\ i = 1,...,k$, such that $\max_i \lambda_i((y^*_i - y'_i) +$

$\rho(y^* - y')) > \max_i \lambda_i((y_i^* - y_i) + \rho(y^* - y))$ holds. But by the definition of the dominance relation we have $y_i' \geq y_i$ for $i = 1, ..., k$, and $y_i' > y_i$ for some i, hence $(y_i^* - y_i') + \rho(y^* - y') < (y_i^* - y_i) + \rho(y^* - y)$, $i = 1, ..., k$, and for any λ, $\lambda_i > 0$, $i = 1, ...,$ we have $\max_i \lambda_i((y_i^* - y_i') + \rho(y^* - y')) < \max_i \lambda_i((y_i^* - y_i) + \rho(y^* - y))$, which is a contradiction. ■

From Lemma C.1, Lemma C.2 immediately follows.

Lemma C.2 *In outcome rankings yielded by linear functions or regularized Tchebycheff functions a dominated outcome cannot be the first outcome (i.e. the first element in the linearly ordered set of outcomes).*

Let G be a set of n outcomes, p the number of efficient outcomes in G, p^e the number of efficient outcomes located on the convex hull of G, and $\hat{y}^{min}$ an element defined as

$$\hat{y}_i^{min} = \min_{y^e \in E} y_i^e, \quad i = 1, \ldots, k,$$

where E is the set of efficient outcomes.

Let $G^1 = G \setminus G^2$, where $G^2 = \{y \in G \mid \hat{y}_i^{min} > y_i,\ i = 1, \ldots, k\}$, and let $r = |G^2|$.

All p efficient outcomes can be determined with regularized Tchebycheff functions with $\rho < \rho_{max}$, where ρ_{max} is a number specific for each problem considered. For a method to calculate or assess ρ_{max} see Kaliszewski (1987).

Lemma C.3 *The number of rankings of G resulting from regularized Tchebycheff functions with $\rho < \rho_{max}$ is bounded from above by*

$$p\frac{(n-r-1)!}{(n-r-p)!}(n-p)!\ .$$

Proof. There are $n! = n(n-1)(n-2)\ldots 1$ possible rankings of G. By Lemma C.2 the first elements in rankings can be only p efficient outcomes and they all can be determined with regularized Tchebycheff functions with $\rho < \rho_{max}$. By Lemma C.1 none of elements on positions $2, \ldots, p-1$ in a ranking can be an outcome dominated by $\hat{y}^{min}$. On these positions we have $\binom{n-1-r}{p-1}$ possible combinations of elements and each combination corresponds to $(p-1)!$ rankings. Hence

$$(p-1)!(\frac{n-1-r}{p-1}) = (p-1)!\frac{(n-1-r)!}{(p-1)!(n-1-r-p+1)!} = \frac{(n-r-1)!}{(n-r-p)!}$$

On the remaining $(n-p)$ positions (the positions from $p+1$ to n) at most $(n-p)!$ rankings can be formed. ■

LEMMA C.4 *The number of rankings of G resulting from linear functions is bounded from above by*

$$p^e \frac{(n-r-1)!}{(n-r-p^e)!}(n-p^e)! \ .$$

Proof. The proof is analogous to the proof of Lemma C.3. ∎

In general, determining p, p^e and r can be a problem in itself. Lemma C.3 and Lemma C.4 (as well as Lemma C.5 and Lemma C.6) are of practical value only if those numbers are easily accessible.

Let $H^{11}, \ldots, H^{1t}$ be disjoint subsets of G^1, i.e. $H^{1i} \cap H^{1j} = \emptyset, \ i,j = 1,\ldots,t, \ i \neq j$, and $|H^{1i}| = h^{1i}, \ i = 1,2,\ldots,t$. Similarly, let $H^{21},\ldots,H^{2u}$ be disjoint subsets of G^2, i.e. $H^{2i} \cap H^{2j} = \emptyset, \ i,j = 1,\ldots,u, \ i \neq j$, and $|H^{2i}| = h^{2i}, \ i = 1,2,\ldots,u$.

LEMMA C.5 *The number of rankings of G resulting from regularized Tchebycheff functions with $\rho < \rho_{max}$ is bounded from above by*

$$p \times \frac{(n-r-1)!}{(h^{11}! \cdot \ldots \cdot h^{1t}!)(n-r-p)!} \times \frac{(n-p)!}{(h^{21}! \cdot \ldots \cdot h^{2t}!)} \ .$$

Proof. The proof follows directly from Lemma C.3 and Lemma B.2. ∎

LEMMA C.6 *The number of rankings of G resulting from linear functions is bounded from above by*

$$p^e \times \frac{(n-r-1)!}{(h^{11}! \cdot \ldots \cdot h^{1t}!)(n-r-p^e)!} \times \frac{(n-p^e)!}{(h^{21}! \cdot \ldots \cdot h^{2t}!)} \ .$$

Proof. The proof follows directly from Lemma C.4 and Lemma B.2. ∎

D. Appendix – Enumeration of feasible rankings

To enumerate feasible rankings a simple algorithm is proposed.

This algorithm makes use of the notion of *forests with roots*. The idea of the algorithm is to build all paths from roots to leaves, where each path represents one feasible ranking of n objects, the first object is the object represented by the root and the last object is the object represented by a leaf. A path with $m < n$ nodes can be expanded by the node representing object x' and by the edge connecting the last node of the paths and the node representing x' only if on this path there is no node corresponding to object x such that $x' \succ x$, where $\succ$ is a preference relation.

The algorithm starts with the number of roots equal to the number of objects to be ranked. On successive iterations all different paths are build till leaves are reached.

In computations for the *SCHEME* the algorithm stops when a threshold value for the number of complete (i.e. composed of $n-1$ edges) paths representing feasible rankings is reached.

Example D.1 *Let us consider rankings of a set of four objects. There are* $4! = 24$ *possible rankings. Assume that the DM has defined the following preferences (order* $\succ$*) among objects:* $a \succ b$*,* $c \succ a$ *and* $d \succ a$*. Fig. 4 illustrates the working of the algorithm. The only feasible rankings, consistent with the preferences (order) are rankings:* c, d, a, b *and* d, c, a, b*.*

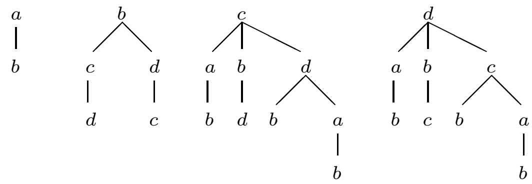

Figure 4 Rooted forest for Example D.1

E. Appendix – Discordance matrices

To assist the DM in criteria selection the concept of discordance matrices has been elaborated. In a discordance matrix each row corresponds to a pair of objects with defined preference and each column corresponds to a criterion f_i, $i = 1, \ldots, l$. Element i, j indicates if for i-th pair of objects x, x' such that $x \gg_A x'$, the relation $f_i(x) \geq f_i(x')$ holds (1 – holds, 0 – otherwise).

To satisfy the condition of consistency of induced relation $\gg_{X_0}$ with relation $\gg_A$, only criteria for which the corresponding column contains 1 in each row can be selected.

Whenever the DM is not able to ensure consistency between relations $\gg_{X_0}$ and $\gg_A$ (via modifications of relation $\gg_A$, set A, or selection of k criteria) the decision process is suspended or terminated.

To ease the process of criteria selection, for each column and each row the *discordance number*, i.e. the number of appearances of 0, is calculated.

The discordance matrix for the problem solved in Section 4 is presented in Table 5.