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# An approach to finding trade-off solutions by a linear transformation of objective functions

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# An approach to finding trade-off solutions by a linear transformation of objective functions[1]

by

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**Abstract:** We address the problem of deriving Pareto optimal solutions of multiple objective optimization problems with predetermined upper bounds on trade-offs. As shown, this can be achieved by a linear transformation of objective functions. Each non-diagonal element of the transformation matrix is related to a bound on the trade-off between a pair of the objective functions.

**Keywords:** multiple objective optimization, trade-offs, linear transformation of objective functions.

## 1. Introduction

Consider the multiple objective optimization problem in the following general formulation:

$$\max_{x \in X} f(x), \tag{1}$$

where $X$ is the set of feasible solutions; $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$, $f_i : X \to \mathbf{R}$, $i \in N_k$, are objective functions; $N_k = \{1, 2, \ldots, k\}$. Solving this problem consists in deriving, in principle, all Pareto optimal elements of $X$ (see definition below).

Problem (1) is often used as an underlying model for Multiple Criteria Decision Making (MCDM) problems. Solving an MCDM problem means deriving an element of $X$, which is the most preferred for the Decision Maker (DM).

As a rule, it is impossible to obtain upfront complete information about DM's preferences, which is needed to derive the most preferred element of $X$. Therefore, methods of deriving and manipulating partial information about DM's preferences gain in importance. One approach to handling partial information

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about DM's preferences is to manipulate upper bounds on trade-offs between objective functions (see, for example, Kaliszewski, Michałowski, 1999; Kaliszewski, 2006). In particular, this approach is widely used in interactive methods of MCDM with relative preference expressing (see, for example, Roy and Wallenius, 1992; Kaliszewski, Michałowski, 1999; Kaliszewski, 2006; Kaliszewski and Zionts, 2004).

In this paper we address the problem of deriving Pareto optimal solutions of (1) with predetermined upper bounds on trade-offs (*trade-off solutions*). This problem was formulated and investigated earlier by Kaliszewski and Michałowski (1997); recent developments in this direction are presented in Kaliszewski (2006).

Computationally effective methods for deriving trade-off solutions are based on scalarizing the problem (1), i.e. replacing $f(x)$ by a surrogate parametric objective function. Surrogate functions used up to now have at most $k$ independent parameters, which can be used to enforce upper bounds on trade-offs for any solution of the scalarized problem. This means that when manipulating upper bounds on trade-offs one can have at most $k$ degrees of freedom, while the number of trade-offs can be as much as $k(k-1)$.

We propose a new form of scalarizing parametric function resulting from a linear transformation of objective functions. This form allows predetermining an upper bound on the trade-off for each pair of objective functions separately. This means that when manipulating upper bounds on trade-offs one can have up to $k(k-1)$ degrees of freedom.

## 2. Notation and definitions

Let us observe that in problem (1) each $x$, $x \in X$, is represented by its *evaluation* $y = f(x)$. Thus, for the purpose of MCDM, instead of problem (1) we can consider the problem

$$\max_{y \in Y} y, \tag{2}$$

where

$$Y = \{f(x) : x \in X\},$$

is the set of (vector) evaluations, $Y \subset \mathbf{R}^k$. Solving (2) consists in deriving all Pareto optimal evaluations (see definition below).

Given $Y$, the set of Pareto optimal evaluations $P(Y)$ and the set of weakly Pareto optimal evaluations $W(Y)$ are defined as

$$P(Y) = \{y \in Y : \nexists\, y' \in Y\ (y' \geqslant y \,\&\, y' \neq y)\},$$
$$W(Y) = \{y \in Y : \forall\, y' \in Y \backslash \{y\}\ \exists\, p \in N_k\ (y_p \geqslant y'_p)\}.$$

Feasible solution $x$ is called Pareto optimal (weakly Pareto optimal) solution of problem (1), if $f(x) \in P(Y)$ $(f(x) \in W(Y))$.

For any $y^* \in Y$ and any $j \in N_k$, let

$$Z_j^<(y^*, Y) = \{y \in Y : y_j < y_j^* \ \&\ \forall\, s \in N_k \backslash \{j\}\ (y_s \geqslant y_s^*)\}.$$

Definition 1 (Kaliszewski, Michałowski, 1997) *Let $i, j \in N_k$, $i \neq j$. If $Z_j^<(y^*, Y) \neq \emptyset$, then the number*

$$T_{ij}(y^*, Y) = \sup_{y \in Z_j^<(y^*, Y)} \frac{y_i - y_i^*}{y_j^* - y_j} \tag{3}$$

*is called trade-off between $i$-th and $j$-th objective functions for evaluation $y^*$. If $Z_j^<(y^*, Y) = \emptyset$, then by definition we assume $T_{ij}(y^*, Y) = -\infty$ .*

Trade-off $T_{ij}(y^*, Y)$ indicates how much at most the evaluation $y^*$ can be improved in the $i$-th component relative to its worsening in the $j$-th component during passage from $y^*$ to any other evaluation from $Y$, under the condition that the remaining components do not worsen.

## 3. The main result

Let us introduce a positive matrix $\mathrm{B} = (\beta_{ij})_{k \times k} \in \mathbf{R}^{k \times k}$ with the main diagonal elements equal to one, and the remaining elements satisfying the following conditions:

(a) $\beta_{pj}\beta_{ji} \leqslant \beta_{pi},\ i, j, p \in N_k,\ i \neq j,\ j \neq p,$
(b) $\beta_{ij} \leqslant 1/\beta_{ji},\ i, j, \in N_k,\ i \neq j.$

Let us consider the following linear transformation of $Y$ :

$$Y_{\mathrm{B}} = \{\mathrm{B}y : y \in Y\}.$$

Theorem 1 *Let $\mathrm{B}y^* \in W(Y_{\mathrm{B}})$. Then $y^* \in P(Y)$ and for any $i, j \in N_k$, $i \neq j$, we have*

$$T_{ij}(y^*, Y) \leqslant \frac{1}{\beta_{ji}}.$$

*Proof.* Let $\mathrm{B}y^* \in W(Y_{\mathrm{B}})$. From the definition of $W(Y_{\mathrm{B}})$ we have that for any $y \in Y$, $y \neq y^*$, there exists $p \in N_k$ such that $\mathrm{B}_p y^* \geqslant \mathrm{B}_p y$, where $\mathrm{B}_p$ is the $p$-th row of matrix B. Suppose $y^* \notin P(Y)$. Then there exists $y \in Y$ such that $y \geqslant y^*$ & $y \neq y^*$. From $\mathrm{B} > 0$ we get $\mathrm{B}_s y^* < \mathrm{B}_s y$ for any $s \in N_k$, which is a contradiction. Thus, $y^* \in P(Y)$.

Let $i, j \in N_k$. It remains to show that $T_{ij}(y^*, Y) \leqslant 1/\beta_{ji}$ under the condition $Z_j^<(y^*, Y) \neq \emptyset$, i.e. to show that any $y \in Z_j^<(y^*, Y) \neq \emptyset$ satisfies the inequality

$$\frac{y_i - y_i^*}{y_j^* - y_j} \leqslant \frac{1}{\beta_{ji}}. \tag{4}$$

Let $y \in Z_j^{<}(y^*, Y) \neq \emptyset$. Then, by definition, we have

$$y_s \geqslant y_s^* \text{ for any } s \in N_k \backslash \{j\}. \tag{5}$$

Moreover, $y^* \in W(Y_\mathrm{B})$ implies $\mathrm{B}_p y^* \geqslant \mathrm{B}_p y$ for some $p \in N_k$. By regrouping the elements in the last inequality we get

$$y_p^* - y_p \geqslant \sum_{s \in N_k,\, s \neq p} \beta_{ps}\,(y_s - y_s^*). \tag{6}$$

There are three possible cases.

Case 1. $p = i$. From (6) and (5) we have

$$y_i^* - y_i \geqslant \sum_{s \in N_k,\, s \neq i} \beta_{is}\,(y_s - y_s^*) \geqslant \beta_{ij}\left(y_j - y_j^*\right).$$

From (b) we obtain

$$\frac{y_i - y_i^*}{y_j^* - y_j} \leqslant \beta_{ij} \leqslant \frac{1}{\beta_{ji}}.$$

Case 2. $p = j$. From (6) and (5) we have

$$y_j^* - y_j \geqslant \sum_{s \in N_k,\, s \neq j} \beta_{js}\,(y_s - y_s^*) \geqslant \beta_{ji}\,(y_i - y_i^*) \text{ and } \frac{y_i - y_i^*}{y_j^* - y_j} \leqslant \frac{1}{\beta_{ji}}.$$

Case 3. $p \neq i$, $p \neq j$. Then $y_p^* - y_p \leqslant 0$. From (5) and (6) we have

$$\begin{aligned} 0 \geqslant y_p^* - y_p &\geqslant \sum_{\substack{s \in N_k, \\ s \notin \{p,i,j\}}} \beta_{ps}\,(y_s - y_s^*) + \beta_{pi}\,(y_i - y_i^*) + \beta_{pj}\left(y_j - y_j^*\right) \\ &\geqslant \beta_{pi}\,(y_i - y_i^*) + \beta_{pj}\left(y_j - y_j^*\right). \end{aligned}$$

Hence $\beta_{pi}(y_i - y_i^*) \leqslant \beta_{pj}(y_j^* - y_j)$. From (a) we have

$$\frac{y_i - y_i^*}{y_j^* - y_j} \leqslant \frac{\beta_{pj}}{\beta_{pi}} \leqslant \frac{1}{\beta_{ji}}.$$

Thus, in each case we obtained inequality (4). ■

## 4. A corollary

We shall now show that Theorem 1 is related to a result known from literature, formulated below as Corollary 1. To this aim we shall exploit the following fundamental result attributed usually to Bowman (1976) (see also some other references in Kaliszewski, 2006, p. 43).

PROPOSITION 1 *Let $Y \subset \boldsymbol{R}^k$, $y^* \in Y$, $y^0 \in \boldsymbol{R}^k$, $y_i^0 > y_i$, for all $y \in Y$, $i \in N_k$. Then $y^* \in W(Y)$ if and only if there exist $\lambda_i > 0$, $i \in N_k$, such that $y^*$ is a solution of the optimization problem*

$$\min_{y \in Y} \max_{i \in N_k} \lambda_i (y_i^o - y_i)$$

Let us rewrite Proposition 1 for $Y$ replaced by $Y_{\mathrm{B}}$, where as before, $Y_{\mathrm{B}} = \{\mathrm{B}y : y \in Y\}$.

PROPOSITION 2 *Let $y_B^* = By^*$, $y_B^0 = By^0$, where $y^*$ and $y^0$ are as in Proposition 1. Then $y_B^* \in W(Y_{\mathrm{B}})$ if and only if there exist $\lambda_i > 0$, $i \in N_k$, such that $y_B^*$is a solution of the optimization problem*

$$\min_{y \in Y_B} \max_{i \in N_k} \lambda_i ((y_B^o)_i - y_i) = \min_{y \in Y} \max_{i \in N_k} \lambda_i \left( \left(y_i^0 - y_i\right) + \sum_{\substack{j \in N_k \\ j \neq i}} \beta_{ij} \left(y_j^0 - y_j\right) \right). \quad (7)$$

We consider now a special instance of matrix B. Let $k$ positive numbers $\rho_i$, $i \in N_k$, be given. We define matrix B with the elements

$$\beta_{ii} = 1, \ \beta_{ij} = \frac{\rho_j}{1 + \rho_i}, \ i, j \in N_k, \ i \neq j. \quad (8)$$

These elements satisfy conditions (a) and (b). Indeed, for any $i, j, p \in N_k$, $i \neq j$, $j \neq p$, we have

$$\beta_{pi} - \beta_{pj}\beta_{ji} = \frac{\rho_i}{1 + \rho_p} - \frac{\rho_j}{1 + \rho_p} \frac{\rho_i}{1 + \rho_j} = \frac{\rho_i + \rho_i\rho_j - \rho_j\rho_i}{(1 + \rho_p)(1 + \rho_j)} > 0 \quad (9)$$

and

$$\beta_{ij}\beta_{ji} = \frac{\rho_j}{1 + \rho_i} \frac{\rho_i}{1 + \rho_j} = \frac{\rho_i\rho_j}{(1 + \rho_i)(1 + \rho_j)} < 1. \quad (10)$$

With this instance of B, Proposition 2 takes the following form:

PROPOSITION 3 *Let $y^* \in Y$, $y^0 \in \boldsymbol{R}^k$, $y_i^0 > y_i$ for all $y \in Y$, $i \in N_k$. Let $\rho_i > 0$, $i \in N_k$, and B be the matrix with elements defined by (8). Then $By^* \in W(Y_{\mathrm{B}})$ if and only if there exist $\lambda_i > 0$, $i \in N_k$, such that $y^*$ is a solution of the optimization problem*

$$\min_{y \in Y} \max_{i \in N_k} \lambda_i \left( \left(y_i^0 - y_i\right) + \sum_{\substack{j \in N_k \\ j \neq i}} \frac{\rho_j}{1 + \rho_i} \left(y_j^0 - y_j\right) \right). \quad (11)$$

To make one more modification of this proposition, we will use the following evident lemma.

LEMMA 1 *Let $y^* \in Y$, $y^0 \in \boldsymbol{R}^k$, $y_i^0 > y_i$ for all $y \in Y$, $i \in N_k$. Let $\rho_i > 0$, $i \in N_k$. Then, following two statements are equivalent:*

1. *there exist $\lambda_i > 0$, $i \in N_k$, such that $y^*$ is a solution of the optimization problem (11).*
2. *there exist $\lambda_i^{'} > 0$, $i \in N_k$, such that $y^*$ is a solution of the following optimization problem:*

$$\min_{y \in Y} \max_{i \in N_k} \lambda_i' (1 + \rho_i) \left( \left( y_i^0 - y_i \right) + \sum_{\substack{j \in N_k \\ j \neq i}} \frac{\rho_j}{1 + \rho_i} \left( y_j^0 - y_j \right) \right).$$

Taking into account the following evident equality

$$\lambda_i' (1 + \rho_i) \left( \left( y_i^0 - y_i \right) + \sum_{\substack{j \in N_k \\ j \neq i}} \frac{\rho_j}{1 + \rho_i} \left( y_j^0 - y_j \right) \right) =$$
$$= \lambda_i' \left( \left( y_i^0 - y_i \right) + \sum_{j \in N_k} \rho_j \left( y_j^0 - y_j \right) \right),$$

from Proposition 3 and Lemma 1 we obtain

PROPOSITION 4 *Let $y^* \in Y$, $y^0 \in \boldsymbol{R}^k$, $y_i^0 > y_i$ for all $y \in Y$, $i \in N_k$. Let $\rho_i > 0$, $i \in N_k$, and $\mathrm{B}$ be the matrix with elements defined by (8). Then $By^* \in W(Y_\mathrm{B})$ if and only if there exist $\lambda_i > 0$, $i \in N_k$, such that $y^*$ is a solution of the optimization problem*

$$\min_{y \in Y} \max_{i \in N_k} \lambda_i \left( \left( y_i^0 - y_i \right) + \sum_{j \in N_k} \rho_j \left( y_j^0 - y_j \right) \right). \tag{12}$$

Applying Theorem 1, we obtain, as a corollary, the following result of Kaliszewski and Michałowski (1997).

COROLLARY 1 *Let $y^* \in Y$, $y^0 \in \boldsymbol{R}^k$, $y_i^0 > y_i$ for all $y \in Y$, $i \in N_k$ and let $\rho_i > 0$, $i \in N_k$. If $y^*$ is a solution of problem (12) for some $\lambda_i > 0$, $i \in N_k$, then $y^* \in P(Y)$ and*

$$T_{ij}(y^*, Y) \leqslant \frac{1 + \rho_j}{\rho_i} \quad \textit{for all } i, j \in N_k,\ i \neq j.$$

## 5. Discussion

Proposition 4 states that problem (12) considered by Kaliszewski and Michałowski gives the same solutions and the same upper bounds on trade-offs as the problem (7) in the case of matrix B with elements defined by (8). But in contrast to the previous method, with the latter problem we can vary upper bounds on trade-offs separately with $k(k-1)$ degrees of freedom.

Indeed, consider matrix B with the elements defined by (8) as a point in space $\mathbf{R}^{k(k-1)}$ with coordinates $\beta_{ij}$, $i, i \in N_k$, $i \neq j$. Since inequalities (9) and (10) are strict, they define in $\mathbf{R}^{k(k-1)}$ an open set containing B. Therefore, B has a neighborhood in $\mathbf{R}^{k(k-1)}$, where inequalities (9) and (10) hold. In this neighborhood we can change all $k(k-1)$ coordinates $\beta_{ij}$ separately.

Example 1 Consider problem (2) for $k$=3. Let $\rho_1$=0.25, $\rho_2$=0.5 and $\rho_3$=1. Then, according to Corollary 1, the solutions of problem (12) for any $\lambda_i > 0$, $i \in N_3$, have upper bounds on trade-offs as shown in Table 1.

Table 1.

| *i* \ *j* | **1** | **2** | **3** |
|---|---|---|---|
| **1** | | 6 | 8 |
| **2** | 2.5 | | 4 |
| **3** | 1.25 | 1.5 | |

According to Proposition 4, these solutions are weakly Pareto optimal solutions of the transformed problem

$$\mathrm{B}y \to \max$$
$$y \in Y$$

where

$$\mathrm{B} = \begin{pmatrix} 1 & 2/5 & 4/5 \\ 1/6 & 1 & 2/3 \\ 1/8 & 1/4 & 1 \end{pmatrix}.$$

These solutions can be found by solving scalarized problem (7).

Modifications of B allow deriving evaluations with upper bound patterns, which cannot be achieved by manipulating $\rho_i$'s. For example, it is possible to decrease upper bounds on trade-offs between the first and the second objective from 6 to 4 and between the first and the third objective from 8 to 3, and enforce upper bounds on trade-offs as shown in Table 2.

To enforce these bounds, we only need to change the value of $\beta_{21}$ to 1/4 and of $\beta_{31}$ to 1/3. The modified matrix takes the form

$$\mathrm{B} = \begin{pmatrix} 1 & 2/5 & 4/5 \\ 1/4 & 1 & 2/3 \\ 1/3 & 1/4 & 1 \end{pmatrix}.$$

Table 2.

| $i \backslash j$ | **1** | **2** | **3** |
|---|---|---|---|
| **1** | | 4 | 3 |
| **2** | 2.5 | | 4 |
| **3** | 1.25 | 1.5 | |

It is easy to verify that conditions (a) and (b) hold. By Proposition 2 solutions of (7) with modified B belong to $W(Y_{\mathrm{B}})$. Then by Theorem 1 they are Pareto optimal solutions of (2) with the required upper bounds on trade-offs as shown in Table 2.

Note that with problem (12) it is impossible to enforce upper bounds on trade-offs set as in Table 2. Indeed, it is easy to verify that there do not exist positive numbers $\rho_1$, $\rho_2$, $\rho_3$ such that

$$\frac{1+\rho_2}{\rho_1} = 4,\ \frac{1+\rho_3}{\rho_1} = 3,\ \frac{1+\rho_1}{\rho_2} = 2.5,\ \frac{1+\rho_3}{\rho_2} = 4,$$
$$\frac{1+\rho_1}{\rho_3} = 1.25,\ \frac{1+\rho_2}{\rho_3} = 1.5.$$

## 6. Concluding remarks

We have shown how to reduce the problem of deriving solutions of multiple objective optimization problems with predetermined upper bounds on trade-offs to the problem of deriving weakly Pareto optimal solutions. Such a reduction is obtained by a linear transformation of objective functions. Our approach provides more flexibility in setting upper bounds on trade-offs than other known results, since its offers $k(k-1)$ parameters for manipulation, in contrast to $k$ parameters in traditional approaches.

Another advantage of our result is that it allows formulating properties of multiple objective problems in algebraic terms.

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