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Book review: Global Pseudo-Differential Calculus on Euclidean Spaces by Fabio Nicola and Luigi Rodino

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Global Pseudo-Differential Calculus on Euclidean Spaces

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Pseudo-differential operators are a generalization of differential operators. The study of these operators emerged in the 1960s, having its origins in the study of singular integro-differential operators. Since that time, pseudo-differential operators have proved useful in many areas of modern analysis and mathematical physics including Quantum Mechanics or Signal Analysis.

Pseudo-differential operator acting upon a function is defined as the inverse Fourier transform of a polynomial in the Fourier variable times the Fourier transform of the function. This integral representation leads to a generalization of differential operators, which correspond to functions other than polynomials in the Fourier variable, as far as the integral converges. In other words, this representation associates a given smooth complex valued function $p_\star(x, y)$ from $R^n \times R^n$, called the symbol, with a differential operator $p(x, \partial_x) : \mathcal{S} \to \mathcal{S}$ where $\mathcal{S}$ denotes the Schwartz space. The map $p_\star \to p$ is not unique. This gives rise to different theories of pseudo-differential calculus.

The book is concerned with the global pseudo-differential operator calculus on Euclidean spaces and its applications to non-commutative geometry and mathematical physics. It deals only with globally elliptic equations with emphasis on linear and non-linear operators of Quantum Physics or traveling wave equations. Especially, global regularity of pseudo-differential operators by means of construction of their parametrices in the pseudo-differential form is discussed.

The book is divided into two parts. The first part, addressed to non-experts, consisting of the first three chapters, presents the foundations of the simple pseudo-differential calculus. Having recalled standard notation and selected results from Real Analysis, in Chapter 1 essential definitions and properties concerning global pseudo-differential operator calculus have been provided. Definition of symbol classes used throughout the book as well as strong uncertainty principle and the results concerning the existence of symbols are given. Quantization of operators, adjoint and transposed operators, composition of operators are also discussed. The notions of global ellipticity and hypoellipticity of symbol classes are recalled. The results concerning the construction and the existence of the parametrix of pseudo-differential operator are proved. Let us recall that parametrix is an inverse of the elliptic operator modulo regularizing operator. It implies that the knowledge of the parametrix is essentially the same as having the inverse operator. The conditions of the boundedness of the pseudo-differential operators on $L^2(R^n)$ are formulated. Sobolev spaces associated to symbol classes satisfying strong uncertainty principle and used throughout the book are introduced. Fredholm properties of pseudo-differential operators are recalled. Anti-Wick quantization preserving the positivity of the operator and its properties are presented. Finally, the relationships between several types of quantizations, especially for polynomial symbols, are discussed. Chapter 2 deals with Γ-pseudo-differential operators as well as their generalizations, where the symbol of the operator is an H-polynomial in R^{2n} or multi-quasi-elliptic polynomial. Symbolic calculus satisfying the strong uncertainty principle is provided. Fredholm properties and L^p boundedness, $1 < p < \infty$, of the corresponding pseudo-differential operators are proved. Pseudo-differential operators with G-class of symbols are presented in Chapter 3. Classical G-symbols possessing three homogeneous principal parts and G-elliptic ordinary differential operators are studied. Results on global regularity of selected operators with polynomial coefficients are provided.

The second part of the book, consisting of Chapters 4 through 6 collects results obtained by the authors and their collaborators in the last ten years in the field of application of pseudo-differential operators calculus. Chapter 4 deals with Spectral Theory for pseudo-differential operators with hypoelliptic bounded symbols satisfying strong uncertainty principle. The spectrum of the closed extension of the operator is calculated. For generic weights appearing in the definition of symbol classes complex powers of hypoelliptic pseudo-differential operators are studied. The trace of the heat semigroup is computed in terms of its symbol by the provided integral formula. The asymptotic behaviour of this trace as parameter tends to zero is shown. Precise asymptotic expansions are deduced for the counting functions of self-adjoint operators in the case of Γ and G classes. In Chapter 5 the authors present results on non-commutative residue and Diximier's trace. The notion of the non-commutative residue of the operator from Γ or G classes is introduced. It is shown that this residue is the unique trace on the algebra of the classical Γ-operators, which vanishes on regularizing operators, up to a multiplicative constant. The coincidence of non-commutative residue map for Γ-operators in R^n with the Diximier trace is proved. Similar results are shown for the algebra of G-operators. The sufficient conditions for Diximier traceability in a case of operators and symbol classes with general weights are formulated. Chapter 6 deals with Γ-elliptic and G-elliptic equations in Gelfand-Shilov spaces rather than Schwartz spaces. Gelfand-Shilov space is a subspace of Schwartz space. The results concerning exponential decay and holomorphic extension of solutions to these equations are proved. Moreover, the existence and regularity of solutions to the semilinear equations having as linear part G-elliptic pseudo-differential operator are shown. The motivation of this type of study comes from the theory of traveling waves.

Each chapter is preceded by a short summary containing description of its contents. This summary illustrates and explains general ideas and lists models important in the applications. The notes at the end of each chapter provide detailed description of the bibliography associated with it and allow the reader to continue in depth studies on the presented topics. At the end of the book a bibliography consisting of 201 research papers and monographs is added. Moreover, the subject Index, as well as Index of Notations, are also provided.

The book describes in a clear way the basic theory as well as new trends and results in global pseudo-differential operators calculus for globally elliptic equations. The elementary knowledge of real analysis, functional spaces including Sobolev spaces, Fourier transforms and linear PDEs will allow the reader to follow easier the developed topics. The book is well written and organized at a difficulty level that precisely meets the target audience's needs.

Mathematics students as well as researchers in mathematical analysis will find this book an excellent resource to introduction into the field of pseudo-differential operator calculus. The book may also serve as a textbook for graduate–level courses in pseudo-differential operators. The advanced applications of pseudo-differential operators calculus in the Spectral Theory or in the holomorphic extension of the solutions of the semi-linear globally elliptic equations may also be useful and interesting for experienced PDEs researchers.

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