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# Radial basis function level set method for structural optimization

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# Radial basis function level set method for structural optimization*

by

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**Abstract:** This paper is concerned with simultaneous topology and shape optimization of elastic contact problems. The structural optimization problem for an elastic contact problem consists in finding such topology as well as such shape of the boundary of the domain occupied by the body that the normal contact stress is minimized. Shape and topological derivatives formulae of the cost functional obtained using material derivative and asymptotic expansion methods, respectively, are recalled. These derivatives are employed to formulate the necessary optimality condition and to calculate a descent direction in a numerical algorithm. Level set based method is employed in numerical algorithm for tracking the evolution of the domain boundary on a fixed mesh and finding an optimal domain. In order to increase the efficiency of the level set based numerical algorithm, the radial basis function approach is used to solve the equation governing domain boundary evolution. Numerical examples are provided and discussed.

**Keywords:** contact problem, structural optimization, level set method, radial basis functions.

## 1. Introduction

The paper is concerned with the numerical solution of a structural optimization problem for an elastic body in unilateral contact with a rigid foundation. The contact with a given friction, described by Coulomb law, is assumed to occur at a portion of the boundary of the body. The displacement field of the body in unilateral contact is governed by an elliptic variational inequality of the second order. The results concerning the existence, regularity and finite-dimensional approximation of solutions to contact problems are given in Haslinger and Mäkinen (2003), Hlavaček et al. (1986). Primal-dual algorithms

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for numerical solving of contact problems are developed in Hüeber et al. (2008) and Stadler (2004). The structural optimization problem for the elastic body in contact consists in finding such topology and such shape of the boundary of the domain occupied by the body that the normal stress along a contact boundary is minimized. The volume of the body is assumed to be bounded.

Shape optimization of contact problems is considered, in particular, in Haslinger and Mäkinen (2003), Sokołowski and Zolésio (1992), where necessary optimality conditions, results concerning convergence of finite-dimensional approximation and numerical results are provided. The material derivative method is employed in the monograph of Sokołowski and Zolésio (1992) to calculate the sensitivity of solutions to contact problems as well as the derivatives of domain depending functionals with respect to variations of the boundary of the domain occupied by the body. Shape optimization of a dynamic contact problem with Coulomb friction and heat flow is considered in Myśliński (2004). In this paper the level set based method is applied to find numerically the optimal solution.

Topology optimization deals with the optimal material distribution within the body resulting in its optimal shape, Sokołowski and Żochowski (2004). The topological derivative is employed to account for variations of solutions to state equations or shape functionals with respect to emergence of small holes in the interior of the domain occupied by the body. The notion of the topological derivative and results concerning its application in optimization of elastic structures are reported in a series of papers: Burger et al. (2004), Fulmański et al. (2007), Garreau et al. (2001), De Gourmay (2006), Myśliński (2005), Novotny et al. (2005), Sokołowski and Żochowski (2003, 2004, 2005). In particular, paper by Sokołowski and Żochowski (2005) deals with the calculation of topological derivatives of solutions to Signorini and elastic contact problems. Asymptotic expansion method, combined with transformation of energy functional, are employed to calculate these derivatives. Simultaneous shape and topology optimization of Signorini and elastic frictionless contact problems are analyzed by Fulmański et al. (2007) and Myśliński (2005), respectively. In these papers the level set method is incorporated in the numerical algorithm.

In structural optimization the level set method, Chopp and Dolbow (2002), Osher and Fedkiw (2003), Wang et al. (2003), is employed in numerical algorithms for tracking the evolution of the domain boundary on a fixed mesh and finding an optimal domain. This method is based on an implicit representation of the boundaries of the optimized structure. A level set model describes the boundary of the body as an isocountour of a scalar function of a higher dimensionality. The evolution of the boundary of the domain is governed by the Hamilton - Jacobi equation. While the shape of the structure may undergo major changes, the level set function remains simple in its topology. Level set methods are numerically efficient and robust procedures for the tracking of interfaces, which allow domain boundary shape changes in the course of iteration. Applications of the level set methods in structural optimization or optimal control can be found, for instance, in Allaire et al. (2004), Burger et al. (2004), Myśliński (2005), Wang et al. (2003) and Hintermüller and Ring (2004). The speed vector field in the Hamilton - Jacobi equation, driving the propagation of the level set function, is given by the Eulerian derivative of an appropriately defined cost functional with respect to the variations of the free boundary. Recently, in the series of papers by Gomes and Suleman (2006), De Gourmay (2006), He et al. (2007), Norato et al. (2007), Xia et al. (2006), Wang et al. (2007) different numerical improvements of the level set method employed for the numerical solution of the structural optimization problems have been proposed and numerically tested. These improvements include, in particular, approximation of the level set function using Fourier series expansion in Gomes and Suleman (2006), velocity extension by solving an auxiliary elliptic equation in the optimized domain in De Gourmay (2006), incorporating topological derivative into the Hamilton - Jacobi equation under additional regularity assumptions in He et al. (2007), solving Hamilton - Jacobi equation using semi-Lagrangian methods or radial basis functions as approximating the level set function in Xia et al. (2006), Wang et al. (2007).

Over the last several decades the radial basis functions (RBFs) have been found to be widely successful for the interpolation of scattered data. A radial basis function is a continuous univariate function that has been radialized by composition with the Euclidean norm. More recently, the RBF methods have emerged as an important type of method for the numerical solution of partial differential equations (see Douan, 2008; Larsson and Forenberg, 2003; Wang et al., 2007). RBF methods can be as accurate as spectral the methods without being tied to a structured computational grid. This leads to ease of application in complex geometries in any number of space dimensions.

This paper deals with topology and shape optimization of an elastic contact problem. The structural optimization problem for the elastic contact problem is formulated. Shape as well as topological derivatives formulae of the cost functional are recalled from Myśliński (2006, 2008). These derivatives are employed to formulate necessary optimality condition for simultaneous shape and topology optimization. Level set based numerical algorithm to solve the shape optimization problem is proposed. Radial basis function approach is used to approximate the level set function and to solve the equation governing boundary evolution. The finite element method (see Haslinger and Mäkinen, 2003) is used as the discretization method of the contact problem. Numerical examples are provided and discussed.

## 2. Problem formulation

Consider deformations of an elastic body occupying two–dimensional domain $\Omega$ with a smooth boundary $\Gamma$ (see Fig. 1). Assume $E \subset \Omega \subset D$, where $E \subset R^2$ is a given domain such that $E \subset \Omega$ and all perturbations $\delta\Omega$ of it satisfy $E \subset \delta\Omega$. $D$ is a bounded smooth hold–all subset of $R^2$, i.e., domain $\Omega$ and all its perturbations $\delta\Omega$ are contained in $D$. The body is subject to body forces

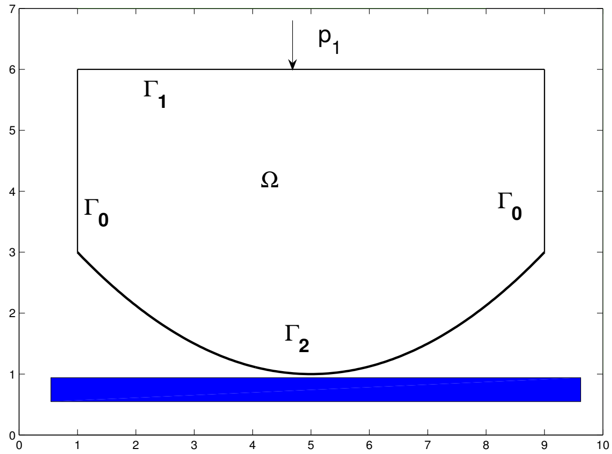


Figure 1. Initial Domain Ω.

$f(x) = (f_1(x),\, f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion $\Gamma_1$ of the boundary $\Gamma$. We assume that the body is clamped along the portion $\Gamma_0$ of the boundary $\Gamma$, and that the contact conditions are prescribed on the portion $\Gamma_2$, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i,j = 0,1,2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i,j = 1,2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i,j,k,l = 1,2$

$$\sigma_{ij}(u(x)) = a_{ijkl}(x) e_{kl}(u(x)). \tag{1}$$

It is assumed that elasticity coefficients $a_{ijkl}(x)$, $i,j,k,l = 1,2$, satisfy (see Haslinger and Mäkinen, 2003) usual symmetry, boundedness and ellipticity conditions, i.e.,

$$a_{ijkl}(x) \in L^\infty(\Omega), \quad a_{ijkl} = a_{jikl} = a_{klij}, \tag{2}$$

$$\exists \alpha_1 > 0,\ \alpha_0 > 0 \ :\ \alpha_0 t_{ij} t_{ij} \leq a_{ijkl}(x) t_{ij} t_{kl} \leq \alpha_1 t_{ij} t_{kl}, \tag{3}$$

for almost all $x \in \Omega$, for all symmetric $2 \times 2$ matrices $t_{ij}$, $i,j = 1,2$. We use here and throughout the paper the summation convention over repeated indices (see Haslinger and Mäkinen, 2003). The strain $e_{kl}(u(x))$, $k,l = 1,2$, is defined by:

$$e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)) \quad u_{k,l}(x) \stackrel{def}{=} \frac{\partial u_k(x)}{\partial x_l}. \tag{4}$$

The stress field $\sigma$ satisfies the system of equations (see Haslinger and Mäkinen, 2003)

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad x \in \Omega, i, j = 1, 2, \tag{5}$$

where $\sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}$, $i, j = 1, 2$. The following boundary conditions are imposed

$$u_i(x) = 0 \quad \text{on} \quad \Gamma_0, \;\; i = 1, 2, \tag{6}$$

$$\sigma_{ij}(x)n_j = p_i \quad \text{on} \quad \Gamma_1, \;\; i, j = 1, 2, \tag{7}$$

$$u_N \leq 0, \;\; \sigma_N \leq 0, \;\; u_N \sigma_N = 0 \quad \text{on } \Gamma_2, \tag{8}$$

$$\mid \sigma_T \mid \leq 1, \quad u_T \sigma_T + \mid u_T \mid = 0 \;\; \text{on } \Gamma_2, \tag{9}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary $\Gamma$. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i, j = 1, 2$, denote the normal components of displacement $u$ and stress $\sigma$, respectively. The tangential components of displacement $u$ and stress $\sigma$ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. $\mid u_T \mid$ denotes the Euclidean norm in $R^2$ of the tangent vector $u_T$.

### 2.1. Variational formulation of the contact problem

Let us formulate the contact problem (5) - (9) in the variational form. Denote by $V_{sp}$ and $K$ the space and set of kinematically admissible displacements:

$$V_{sp} = \{z \in [H^1(\Omega)]^2 \; : \; z_i = 0 \text{ on } \Gamma_0, \; i = 1, 2\}, \tag{10}$$
$$K = \{z \in V_{sp} \; : \; z_N \leq 0 \;\; \text{on} \;\; \Gamma_2\}.$$

$H^1(\Omega)$ denotes Sobolev space (see Haslinger and Mäkinen, 2003; Sokołowski and Zolésio, 1992) and $[H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega)$. Denote also by $\Lambda$ the set

$$\Lambda = \{\zeta \in L^2(\Gamma_2) \; : \mid \zeta \mid \leq 1\}. \tag{11}$$

Let $f \in [L^2(\Omega)]^2$ and $p \in [L^2(\Gamma_2)]^2$ be given. Variational formulation of problem (5) - (9) has the form: *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying for* $i, j, k, l = 1, 2$

$$\int_\Omega a_{ijkl} e_{ij}(u) e_{kl}(\varphi - u) dx - \int_\Omega f_i(\varphi_i - u_i) dx -$$
$$\int_{\Gamma_1} p_i(\varphi_i - u_i) ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T) ds \geq 0 \;\; \forall \varphi \in K, \tag{12}$$

$$\int_{\Gamma_2} (\zeta - \lambda) u_T ds \leq 0 \quad \forall \zeta \in \Lambda. \tag{13}$$

From Hlavaček et al. (1986), Theorem 5.44, p. 197, it results that problem (1) - (9) possesses a unique solution $(u, \lambda) \in K \times \Lambda$. Function $\lambda$ is interpreted as a Lagrange multiplier and is equal to tangent stress along the boundary $\Gamma_2$, i.e., $\lambda = \sigma_{T|\Gamma_2}$ (see Haslinger and Mäkinen, 2003; Sokołowski and Zolésio, 1992).

### 2.2. Structural optimization problem

Before formulating a structural optimization problem for system (12) - (13) let us introduce the set $U_{ad}$ of admissible domains. Denote by $Vol(\Omega)$ the volume of the domain $\Omega$ equal to

$$Vol(\Omega) = \int_{\Omega} dx. \tag{14}$$

Domain $\Omega$ is assumed to satisfy the volume constraint of the form

$$Vol(\Omega) - const_0 \leq 0, \tag{15}$$

where the constant $const_0 > 0$ is given. Note that in the case of shape optimization of problem (12) - (13) the optimized domain $\Omega$ is assumed to satisfy equality volume condition, i.e., (15) is assumed to be satisfied as equality. In the case of topology optimization $const_0$ is assumed to be the initial domain volume and (15) is satisfied (see Sokołowski and Zolésio, 2004) in the form $Vol(\Omega) = r_{fr}\ const_0$ with $r_{fr} \in (0,1)$. Therefore, the set $U_{ad}$ has the following form

$$U_{ad} = \{\Omega \ :\ E \subset \Omega \subset D \subset R^2 \ :\ \Omega \text{ is Lipschitz continuous,} \\ \Omega \text{ satisfies condition (15),} \quad P_D(\Omega) \leq const_1\}, \tag{16}$$

where

$$P_D(\Omega) = \int_{\Gamma} dx, \tag{17}$$

is a perimeter of a domain $\Omega$ in $D$ (see Delfour and Zolésio, 2001; Sokołowski and Zolésio, 1992, p. 126). The perimeter constraint is added in (16) to ensure the compactness of the set $U_{ad}$ in the $L^2$ topology of characteristic functions as well as the existence of optimal domains. The constant $const_1 > 0$ is assumed to exist. The set $U_{ad}$ is assumed to be nonempty.

In order to define a cost functional we shall also use the following set $M^{st}$ of auxiliary functions

$$M^{st} = \{\phi = (\phi_1, \phi_2) \in [H^1(D)]^2 : \phi_i \leq 0 \ \text{ on } D,\ i=1,2,\ \| \phi \|_{[H^1(D)]^2} \leq\ 1\}, \tag{18}$$

where the norm is $\| \phi \|_{[H^1(D)]^2} \stackrel{def}{=} (\sum_{i=1}^{2} \| \phi_i \|^2_{H^1(D)})^{1/2}$.

The cost functional approximating the normal contact stress on the contact boundary $\Gamma_2$ is chosen as equal to (see Myśliński, 2006)

$$J_{\phi}(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\phi_N(x)ds. \tag{19}$$

This cost functional depends on the auxiliary given bounded function $\phi(x) \in M^{st}$. Here, $\sigma_N$ and $\phi_N$ are the normal components of the stress field $\sigma$ corresponding to a solution $u$ satisfying system (12) - (13) and the function $\phi$, respectively.

Consider the following structural optimization problem: *for a given function* $\phi \in M^{st}$*, find a domain* $\Omega^\star \in U_{ad}$ *such that*

$$J_\phi(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\phi(u(\Omega)). \tag{20}$$

The existence of an optimal domain $\Omega^\star \in U_{ad}$ follows by standard arguments (for details see Delfour and Zolésio, 2001, Theorem 5.9, p. 135, and Sokołowski and Zolésio, 1992).

## 3. Necessary optimality condition

Consider the variations of the cost functional (19) resulting both from the variations of the boundary $\Gamma$ of the domain $\Omega$ and from the nucleation of an internal small hole inside domain $\Omega$.

Let $\tau$ be a given parameter such that $0 \leq \tau < \tau_0$, with $\tau_0$ prescribed, and $V = V(x,\tau)$, $x \in \Omega$, be a given admissible velocity field. The set of admissible velocity fields $V$ consists of vector fields regular enough ($C^k$ class, $k \geq 1$, for details see Sokołowski and Zolésio, 1992) with respect to $x$ and $\tau$ and such that on the boundary $\partial D$ of $D$ either $V = 0$ at singular points of this boundary or normal component $V \cdot n$ of $V$ equals to $V \cdot n = 0$ at points of this boundary where the outward unit normal field $n$ exists. Therefore, the perturbations of the boundary $\Gamma$ of the domain $\Omega$ are governed by the transformation $T(\tau, V) : \bar{D} \to \bar{D}$ (see Sokołowski and Zolésio, 1992) and

$$\Omega_\tau = T(\tau, V)(\Omega).$$

Since only small perturbations of $\Omega$ are considered, this transformation can have the form of perturbation of the identity operator $I$ in $R^2$. An example of such transformation is $T(\tau, \tilde{V}) = I + \tau\tilde{V}$, where $\tilde{V}$ denotes a smooth vector field defined on $R^2$ (see Sokołowski and Zolésio, 1992). Therefore

$$\Omega_\tau = T(\tau, \tilde{V})(\Omega) = (I + \tau\tilde{V})(\Omega). \tag{21}$$

The topology variations of geometrical domains are defined in Sokołowski and Żochowski (2004) as functions of a small parameter $\rho$ such that $0 < \rho < R$, where $R > 0$ is given. These variations are based on the creation of a small hole

$$B(x,\rho) = \{z \in R^2 : | x - z | < \rho\}$$

of radius $\rho$ at a point $x \in \Omega$ in the interior of the domain $\Omega$. The Neumann boundary conditions are prescribed on the boundary $\partial B$ of the hole. Denote by $\Omega_\rho = \Omega \setminus \overline{B(x,\rho)}$ the perturbed domain.

In order to take into account these shape and topology perturbations, in Sokołowski and Żochowski (2003) the notion of the domain differential of the domain functional has been introduced. The domain differential $DJ_\phi(\Omega; V, x_0)$

of the cost functional (19) at domain $\Omega \subset R^2$ in direction of the velocity field $V$ and at point $x_0 \in \Omega$ is defined as

$$DJ_\phi(\Omega; V, x_0)(\tau, \rho) = \tau dJ_\phi(u(\Omega), V) + \pi\rho^2 TJ_\phi(u(\Omega), x_0), \tag{22}$$

where $dJ_\phi(u(\Omega), V)$ and $TJ_\phi(u(\Omega), x_0)$ denote shape and topological derivatives of the cost functional (19), respectively. For definitions of these derivatives, see Sokołowski and Zolésio (1992), Sokołowski and Żochowski (2004). These derivatives depend on the solution $u = u(\Omega)$ to the state system (12) - (13). Parameter $\tau > 0$ and radius $\rho > 0$ of the hole describe perturbation of the boundary of domain $\Omega$ and hole nucleation, respectively. This differential completely characterizes the variation of the cost functional $J_\phi(\Omega)$ with respect to the simultaneous shape and topology perturbations (for details, see Sokołowski and Żochowski, 2003).

In Myśliński (2006), using the material derivative approach from Sokołowski and Zolésio (1992), the Euler derivative $dJ_\phi(\Omega, V)$ of the cost functional (19) has been calculated. Recall from Myśliński (2006) the form of this Euler derivative:

$$\begin{aligned} dJ_\phi(u(\Omega); V) = &\int_\Gamma (\sigma_{ij} e_{kl}(\phi + p^{adt}) - f \cdot \phi) V(0) \cdot n ds - \\ &\int_{\Gamma_1} [\frac{\partial(p \cdot (p^{adt} + \phi))}{\partial n} + \kappa p \cdot (p^{adt} + \phi)] V(0) \cdot n ds + \\ &\int_{\Gamma_2} [\lambda(p_T^{adt} + \phi_T) + q^{adt} u_T] \kappa V(0) \cdot n ds, \end{aligned} \tag{23}$$

where $i, j, k, l = 1, 2$, $V(0) = V(x, 0)$, the displacement $u \in V_{sp}$ and the stress $\lambda \in \Lambda$ satisfy the state system (12) - (13). $\kappa$ denotes the mean curvature of the boundary $\Gamma$. The adjoint functions $p^{adt} \in K_1$ and $q^{adt} \in \Lambda_1$ satisfy for $i, j, k, l = 1, 2$, the following system

$$\int_\Omega a_{ijkl} e_{ij}(\phi + p^{adt}) e_{kl}(\varphi) dx + \int_{\Gamma_2} q^{adt} \varphi_T ds = 0, \ \forall \varphi \in K_1, \tag{24}$$

and

$$\int_{\Gamma_2} \zeta(p_T^{adt} + \phi_T) ds = 0, \ \ \forall \zeta \in \Lambda_1, \tag{25}$$

where the cones $K_1$ and $\Lambda_1$ are given by (see Myśliński, 2006, or Sokołowski and Zolésio, 1992)

$$K_1 = \{\xi \in V_{sp} \ : \ \xi_N = 0 \ \text{ on } A^{st} \},$$
$$\Lambda_1 = \{\zeta \in L^2(\Gamma_2) \ : \ \zeta(x) = 0 \text{ on } B_1 \cup B_2 \cup B_1^+ \cup B_2^+\},$$

while the coincidence set $A^{st} = \{x \in \Gamma_2 \ : \ u_N = 0\}$. Moreover, $B_1 = \{x \in \Gamma_2 : \lambda(x) = -1\}$, $B_2 = \{x \in \Gamma_2 : \lambda(x) = +1\}$, $\tilde{B}_i = \{x \in B_i : u_N(x) = 0\}$, $i = 1, 2$, $B_i^+ = B_i \setminus \tilde{B}_i$, $i = 1, 2$.

The formulae for topological derivatives of cost functionals for plane elasticity systems or contact problems are provided, for instance, in Fulmański et al. (2007), Garreau et al. (2001), Novotny et al. (2005), Sokołowski and Żochowski (2003, 2005). Using the methodology from Sokołowski and Żochowski (2004) as well as the results of differentiability of solutions to variational inequalities (see Sokołowski and Zolésio, 1992), we can calculate the formulae of the topological derivative $TJ_\phi(\Omega; x_0)$ of the cost functional (19) at a point $x_0 \in \Omega$. This derivative is equal to (see Myśliński, 2008)

$$TJ_\phi(u(\Omega), x_0) = -[f(\phi + w^{adt}) + \frac{1}{E}(a_u a_{w^{adt}+\phi} + 2b_u b_{w^{adt}+\phi} \cos 2\delta)]_{|x=x_0} - \int_{\Gamma_2} (s^{adt} u_T + \lambda(w_T^{adt} + \phi_T))\kappa ds, \tag{26}$$

where $a_{\tilde{\beta}} = \sigma_I(\tilde{\beta}) + \sigma_{II}(\tilde{\beta})$, $b_{\tilde{\beta}} = \sigma_I(\tilde{\beta}) - \sigma_{II}(\tilde{\beta})$, and either $\tilde{\beta} ='' u''$ or $\tilde{\beta} ='' w^{adt} + \phi''$, $\sigma_I(u)$ and $\sigma_{II}(u)$ denote principal stresses for the displacement $u$, and $\delta$ is the angle between principal stresses directions (see Sokołowski and Żochowski, 2004). Constant $E$ denotes the Young modulus. The dependence of tangent displacement and stress functions on $\rho$ along $\Gamma_2$ is assumed. The adjoint state $(w_\rho^{adt}, s_\rho^{adt}) \in K_1 \times \Lambda_1$ satisfies the system (24) - (25) in domain $\Omega_\rho$ rather than $\Omega$ and $w_{\rho\ |\rho=0}^{adt} = w^{adt}(x_0)$. Using standard arguments as in Sokołowski and Zolésio (1992) we can show

Lemma 3.1 *Let $\Omega^\star \in U_{ad}$ be an optimal solution to the problem (20). Then, there exist Lagrange multipliers $\mu_1 \in R$, $\mu_1 \geq 0$, associated with the volume constraint (15), and $\mu_2 \in R$, $\mu_2 \geq 0$, associated with the finite perimeter constraint (16), such that for all admissible vector fields $V$, for all admissible pairs $(\rho, \tau)$ of parameters and for all $x_0 \in \Omega^\star$ and such that all perturbations $\delta\Omega \in U_{ad}$ of domain $\Omega \in U_{ad}$ satisfy $E \subset \Omega \cup \delta\Omega \subset D$, at any optimal solution $\Omega^\star \in U_{ad}$ to the shape and topology optimization problem (20), the following conditions are satisfied:*

$$DJ_\phi(u(\Omega^\star); V, x_0)(\tau, \rho) + \mu_1 \int_{\Gamma^\star} V(0) \cdot n ds + \mu_2 dP_D(\Omega^\star; V) \geq 0, \tag{27}$$

$$(\mu_1^\sim - \mu_1)(\int_{\Omega^\star} dx - const_0) \leq 0, \forall \mu_1^\sim \in R, \ \ \mu_1^\sim \geq 0, \tag{28}$$

$$(\mu_2^\sim - \mu_2)(P_D(\Omega^\star) - const_1) \leq 0, \ \ \forall \mu_2^\sim \in R, \ \ \mu_2^\sim \geq 0, \tag{29}$$

*where $u(\Omega^\star)$ denotes the solution to (12) - (13) in the domain $\Omega^\star$, $\Gamma^\star = \partial\Omega^\star$, the domain differential $DJ_\phi(u(\Omega^\star); V, x_0)(\tau, \rho)$ is given by (22) and $dP_D(\Omega; V)$ denotes the derivative of the finite perimeter functional $P_D(\Omega)$ (see Allaire et al., 2004, Fulmański et al., 2007, Sokołowski and Zolésio, 1992, p. 126). The given constant $const_0 > 0$ and constant $const_1 > 0$ are the same as in (16).*

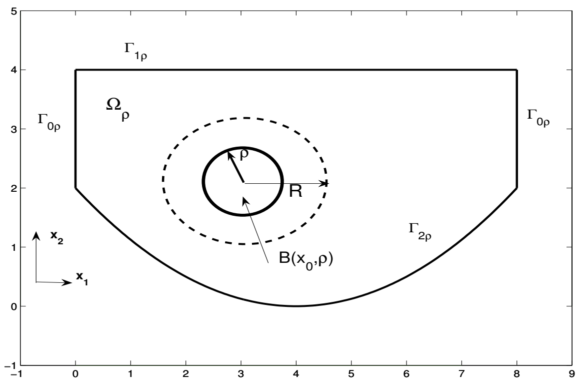


Figure 2. Level set function Φ.

## 4. Level set shape representation

In the paper the level set method (see Osher and Fedkiw, 2003) is employed to solve numerically the structural problem (20). Consider the evolution of a domain $\Omega$ under a velocity field $V$. Let $t > 0$ denote the time variable. Under the mapping $T(t, V)$ domain $\Omega$ is transformed into domain $\Omega_t$, given by (21).

Let us denote by $\Omega_t^-$ and $\Omega_t^+$ the interior and the outside of the domain $\Omega_t$, respectively. The domain $\Omega_t$ and its boundary $\partial\Omega_t$ are determined by a function $\Phi = \Phi(x, t) \; : \; R^2 \times [0, t_0) \to R$ (see Fig. 2) such that

$$\begin{cases} \Phi(x,t) \; = \; 0, & \text{if } \; x \in \partial\Omega_t, \\ \Phi(x,t) \; < \; 0, & \text{if } \; x \in \Omega_t^-, \\ \Phi(x,t) \; > \; 0, & \text{if } \; x \in \Omega_t^+, \end{cases} \tag{30}$$

i.e., the boundary $\partial\Omega_t$ is the level curve of the function $\Phi$. Recall from Osher and Fedkiw (2003) that the gradient of the implicit function is defined as $\nabla\Phi = (\frac{\partial\Phi}{\partial x_1}, \frac{\partial\Phi}{\partial x_2})$, the local unit outward normal $n$ to the boundary is equal to $n = \frac{\nabla\Phi}{|\nabla\Phi|}$, the mean curvature $\kappa = \nabla \cdot n$. In the level set approach the Heaviside function $H(\Phi)$ and the Dirac function $\delta(\Phi)$ are used to transform integrals from domain $\Omega$ into domain $D$. These functions are defined as

$$H(\Phi) = 1 \;\text{ if }\; \Phi \geq 0, \;\; H(\Phi) = 0 \;\text{ if }\; \Phi < 0, \tag{31}$$

$$\delta(\Phi) = H'(\Phi), \;\; \delta(x) = \delta(\Phi(x)) \mid \nabla\Phi(x) \mid, \;\; x \in D. \tag{32}$$

Assume that admissible velocity field $V$ is known for every point $x = x(t)$ lying on the boundary $\partial\Omega_t$, i.e., at this point $\Phi(x(t), t) = 0$. Differentiating this latter equation with respect to $t$ and using the formula for the gradient $\nabla\Phi$ of the function $\Phi$ with respect to $x$ we obtain the equation governing the evolution of the interface $\partial\Omega_t$ in $D \times [0, t_0]$ in the following form (see Osher and Fedkiw, 2003)

$$\Phi_t(x,t) + V(x,t)\cdot n \mid \nabla\Phi(x,t) \mid = 0, \;\; \Phi(x,0) = \Phi_0(x), \tag{33}$$

where $\Phi_t$ denotes a partial derivative of $\Phi$ with respect to the time variable $t$, $V(x,t)\cdot n$ is a normal component of $V$ on the interface and $\Phi_0(x)$ is a given function. Equation (33) is known in literature as the Hamilton - Jacobi equation.

### 4.1. Structural optimization problem in domain $D$

Using the level set function (30), as well as functions (31) and (32), the structural optimization problem (20) may be reformulated in terms of function $\Phi$ in the following way: *for a given function $\phi \in M^{st}$, find function $\Phi$ such that*

$$J_\phi(u(\Phi^\star)) = \min_{\Phi \in U_{ad}^\Phi} J_\phi(u(\Phi)) \tag{34}$$

where

$$J_\phi(u(\Phi)) = \int_D \sigma_N(u)\phi_N(x)\delta(\Phi) \mid \nabla\Phi \mid ds, \tag{35}$$

$$U_{ad}^\Phi = \{\Phi \; : \; \Phi \text{ satisfies (30)}, \quad Vol(\Phi) \le const_0, \quad P_D(\Phi) \le const_1\},$$
$$Vol(\Phi) = \int_D H(\Phi)dx, \quad P_D(\Phi) = \int_D \delta(\Phi) \mid \nabla\Phi \mid dx. \tag{36}$$

Moreover, the pair $(u,\lambda) \in K \times \Lambda$ satisfies the system

$$\int_D a_{ijkl}e_{ij}(u)e_{kl}(\varphi - u)H(\Phi)dx -$$
$$\int_D f_i(\varphi_i - u_i)H(\Phi)dx - \int_D p_i(\varphi_i - u_i)\delta(\Phi) \mid \nabla\Phi \mid dx +$$
$$\int_D \lambda(\varphi_T - u_T)\delta(\Phi) \mid \nabla\Phi \mid dx \ge 0 \;\; \forall\varphi \in K, \tag{37}$$

$$\int_D (\zeta - \lambda)u_T\delta(\Phi) \mid \nabla\Phi \mid dx \le 0 \quad \forall\zeta \in \Lambda, \tag{38}$$

where $i,j,k,l = 1,2$ and $V_{sp}$, $K$ and $\lambda$ are defined by (10) and (11), respectively, on domain $D$ rather than $\Omega$.

## 5. Level set based numerical algorithm

The topological derivative can provide better prediction of the structure topology with different levels of material volume than the method based on updating the shape of initial structure, containing many regularly distributed holes (see

Allaire et al., 2004, Wang et al., 2003). Our approach is based on the application of the topological derivative to predict the structure topology and substitute material according to the material volume constraint and then to optimize the structure topology by merging the unreasonable material interfaces and changing the shape of material boundary. For the sake of simplicity, in the description of the algorithm we omit the bounded perimeter constraint in (16). Therefore, the level set method combined with the shape or topological derivatives results in the following conceptual algorithm (A1) for solving the structural optimization problem (20):

Step 1: Choose: a computational domain $D$ such that $\Omega \subset D$, an initial level set function $\Phi^0 = \Phi_0$ representing $\Omega^0 = \Omega$, function $\phi \in M^{st}$, parameters $r^0$, $\varepsilon_1$, $\varepsilon_2$, $q$, $r_{fr} \in (0,1)$. Set $m_0 = Vol(\Omega^0)$, $\tilde{\mu}_1^0 = \mu_1^0 = 0$, $k = n = 0$.

Step 2: Calculate the solution $(u^n, \lambda^n)$ to the state system (37) - (38).

Step 3: Calculate the solution $((w^{adt})^n, (s^{adt})^n)$ to the adjoint system (24) - (25) as well as the topological derivative $TJ_\phi(\Omega^n, x)$ of the cost functional (19) given by (26).

Step 4: For given $\tilde{\mu}_1^n$ set $\Omega^{n+1} = \{x \in \Omega^n : TJ_\phi(\Omega, x) \geq \chi_{n+1}\}$, where $\chi_{n+1}$ is chosen in such a way that $Vol(\Omega^{n+1}) = m_{n+1}$, $m_{n+1} = qm_n$. Fill the void part $D \setminus \Omega^{n+1}$ with a very weak material with Young modulus $E^w = 10^{-5}E$. Update $\tilde{\mu}_1^{n+1} = \tilde{\mu}_1^n + r^n(Vol_1^{giv})$, $r^n > 0$ and $Vol_1^{giv} = Vol(\Omega^{n+1}) - r_{fr} const_0$.
If $\mid \tilde{\mu_1}^{n+1} - \tilde{\mu_1}^n \mid \leq \varepsilon_1$ then set $\Omega^k = \Omega^{n+1}$ and go to Step 5.
Otherwise set $n = n + 1$, go to Step 2.

Step 5: Calculate the solution $((p^{adt})^k, (q^{adt})^k)$ to the adjoint system (24) - (25). Calculate the shape derivative $dJ_\phi(u(\Omega^k))$ of the cost functional (19) given by (23).

Step 6: For given $\mu_1^k$ solve the level set equation (33) to calculate the level set function $\Phi^{k+1}$.

Step 7: Set $\Omega^{k+1}$ equal to the zero level set of function $\Phi^{k+1}$. Calculate $\mu_1^{k+1} = \mu_1^k + r^k(Vol(\Omega^{k+1}) - Vol_1^{giv})$, $r^k > 0$.
If $\mid \mu_1^{k+1} - \mu_1^k \mid \leq \varepsilon_2$ then Stop.
Otherwise set $k = k + 1$, $\Omega^n = \Omega^{k+1}$ and go to Step 2.

State and adjoint systems, (12) - (13) and (24) - (25), respectively, are discretized using finite element method (see Haslinger and Mäkinen, 2003). Displacement and stress functions in the state system (12) - (13) are approximated by piecewise bilinear functions in domain $D$ and piecewise constant functions on the boundary $\Gamma_2$, respectively. Similar approximation is used to discretize the adjoint system (24) - (25). These systems are solved using the primal-dual algorithm with active set strategy described in Stadler (2004). In the level set approach these state and adjoint systems are transfered from domain $\Omega$ into fixed hold–all domain $D$ using the regularized Heaviside and Dirac functions.

## 6. Radial basis functions

A radial basis function is a continuous univariate function that has been radialized by composition with the Euclidean norm (see Douan, 2008; Larsson and Forenberg, 2003; Wang et al., 2007). Radial basis functions (RBF) $\varphi : R^{+} \to R$, $\varphi(0) \geq 0$ are radially symmetric functions centered at a particular point $x_i \in D$, $i = 1, ..., N$:

$$\varphi_i(x) = \varphi(\mid x - x_i \mid), \quad x_i \in D. \tag{39}$$

Among many classes of RBF, multiquadric RBF proved to be numerically efficient (see Larsson and Forenberg, 2003). They can be written as

$$\varphi_i(x) = \sqrt{(x - x_i)^2 + \varepsilon_i^2}, \tag{40}$$

where $\varepsilon_i$ is the free shape parameter usually assumed to be constant. In general, small shape parameters produce the most accurate results, but are also associated with a poorly conditioned interpolation matrix. Optimal shape parameter should ensure maximal accuracy while maintaining numerical stability. However, determining the optimal shape parameter is still an open question. Most applications of the multiquadrics use experimental tuning parameters or expensive optimization techniques to evaluate the optimum shape parameter (for details see Larsson and Forenberg, 2003).

Recently, the RBF methods have emerged as an important type of methods for the numerical solution of partial differential equations based on a scattered data interpolation problem. Let $\tilde{D}$ be a finite distinct set of points in $D \subset R^2$, which are traditionally called centers in the language of RBFs. The idea is to use linear combinations of translates of one function $\varphi(\cdot)$ of one real variable, which is centered in $x_i \in \tilde{D}$, $i = 1, ..., N$, to approximate a function $f$ as

$$s(x) = \sum_{i=1}^{N} \alpha_i \varphi_i(x) + p(x), \tag{41}$$

where the linear polynomial

$$p(x) = p_0 + p_1\, x_1 + p_2\, x_2, \tag{42}$$

is added to account for the linear and constant portions of $f(x)$ and to ensure the unique solution to the interpolation equation. The introduction of polynomial $p(x)$ implies additional constraints on coefficients of the form

$$\sum_{i=1}^{N} \alpha_i = 0, \quad \sum_{i=1}^{N} \alpha_i x_{1i} = 0, \quad \sum_{i=1}^{N} \alpha_i x_{2i} = 0. \tag{43}$$

The most attractive feature of the RBF methods is that the location of the centers can be chosen arbitrarily in the domain of interest. The interpolation problem is to find expansion coefficients $\alpha_i$, $i = 1, ..., N$, so that for a given data $f_{\tilde{D}}$:

$$s_{\tilde{D}} = f_{\tilde{D}},$$

i.e., at knot locations $x_i$

$$s(x_i) = f_i, \quad f_i \text{ are known at knot locations.} \tag{44}$$

These coefficients are obtained by solving the linear system

$$H\ \alpha = f, \tag{45}$$

where the interpolation matrix $H$ and vectors $\alpha$ and $f$ are given by:

$$H = \begin{bmatrix} A & P \\ P^T & O \end{bmatrix}, \tag{46}$$

$$\alpha = \begin{bmatrix} \alpha_1 & \dots & \alpha_N & p_0 & p_1 & p_2 \end{bmatrix}^T, \quad f = \begin{bmatrix} f_1 & \dots & f_N & 0 & 0 & 0 \end{bmatrix}^T.$$

The following matrix and vector notation is here employed:

$$A = \begin{bmatrix} \varphi_1(x_1) & \dots & \varphi_N(x_1) \\ \vdots & \ddots & \vdots \\ \varphi(x_N) & \dots & \varphi_N(x_N) \end{bmatrix}, \quad P = \begin{bmatrix} 1 & x_1 & x_2 \\ \vdots & \vdots & \vdots \\ 1 & x_{1N} & x_{2N} \end{bmatrix}.$$

The solvability of system (44) or linear system (45) is ensured for the class of conditionally positive definite basis functions. Recall from Douan (2008) or Iske (2003):

DEFINITION 6.1 *A continuous radial basis function $\varphi : [0, \infty) \to R$ is said to be conditionally positive definite of order* 2 *on $R^2$ iff the following condition is satisfied:*

$$\exists \gamma > 0 \quad y^T A y \geq \gamma \parallel y \parallel_{R^N} \quad \forall y \in R^N \ \textit{ such that } \ Py = 0 \tag{47}$$

*for all possible choices of finite point sets $\tilde{D} \subset R^2$.*

THEOREM 6.1 *Assume $\varphi$ is conditional positive definite radial basis function and the following condition is satisfied:*

$$\textit{rank}\, P = 3 \leq N. \tag{48}$$

*Then the interpolation problem (44) has a unique solution s in the form of (41).*

From conditions (47) and (48) follows the invertibility (see Douan, 2008) of the multiquadric collocation matrix $H$. Therefore, we have from (45):

$$\alpha = H^{-1} f, \quad \text{and} \ \ f(x) = \varphi^T(x)\alpha, \tag{49}$$

where

$$\varphi(x) = \begin{bmatrix} \varphi_1(x) & \dots & \varphi_N(x) & 1 & x_1 & x_2 \end{bmatrix}^T.$$

### 6.1. Approximation of the Hamilton - Jacobi equation

We use RBF as shape functions to approximate and solve the Hamilton - Jacobi equation (33). Assuming time dependence of function $\Phi$ we have

$$\Phi(x,t) = \phi^T(x)\alpha(t) \tag{50}$$

and Hamilton - Jacobi equation (33) takes the form

$$\phi^T \frac{d\alpha}{dt} + v_n \mid (\nabla\phi)^T \alpha \mid = 0, \tag{51}$$

where

$$\mid (\nabla\phi)^T \alpha \mid = \left[ \; (\tfrac{\partial \phi^T}{\partial x_1}\alpha)^2 + (\tfrac{\partial \phi^T}{\partial x_2}\alpha)^2 \; \right],$$

$$\frac{\partial \phi}{\partial x_1} = \left[ \begin{array}{cccccc} \frac{\partial \phi_1}{\partial x_1} & \dots & \frac{\partial \phi_N}{\partial x_1} & 0 & 1 & 0 \end{array} \right],$$

$$\frac{\partial \phi}{\partial x_2} = \left[ \begin{array}{cccccc} \frac{\partial \phi_1}{\partial x_2} & \dots & \frac{\partial \phi_N}{\partial x_2} & 0 & 0 & 1 \end{array} \right].$$

Additionally,

$$\sum_{i=1}^{N} \dot{\alpha}_i = 0, \;\; \sum_{i=1}^{N} \dot{\alpha}_i x_{1i} = 0, \;\; \sum_{i=1}^{N} \dot{\alpha}_i x_{2i} = 0.$$

Therefore, we obtain a system of ODEs written in the form:

$$H \frac{d\alpha}{dt} + B(\alpha) = 0, \tag{52}$$

where $v_n^e$ denotes normal velocity $v_n$ extended from the boundary $\partial\Omega$ in $D$ and

$$B(\alpha) = \left[ \begin{array}{c} v_n^e \mid \nabla\phi^T(x_1))\alpha \mid \\ \vdots \\ v_n^e \mid \nabla\phi^T(x_N))\alpha \mid \\ 0 \\ 0 \\ 0 \end{array} \right].$$

The approximation of Hamilton - Jacobi equation using RBF leads to a system of ODEs possessing a unique solution. Moreover, since $\mid \nabla\Phi \mid \neq 0$ for most points of the interface $\Phi = 0$, this approach leads to smooth level set evolution and makes unnecessary reinitialization. First order forward Euler scheme is used to solve the resulting ODEs. The choice of time step size $\triangle t$ and the total number $N$ of RBF knots is determined by the requirement of decreasing the cost functional value while remaining in the feasible region rather than by solving Hamilton - Jacobi equation accurately. It also implies that maximum principle and CFL stability condition will not be strictly satisfied. Note that RBF approximation may be also interpreted as the collocation formulation of the method of lines.

### 6.2. Extension of normal velocity

Since the normal velocity $V \cdot n$ in (33) is prescribed only on the boundary $\Gamma$, in order to solve (33) we have to extend it to the whole domain $D$. The extended normal velocity $v_n^e q = q(x,t) : D \times [0, t_0]$ is calculated as a solution up to the stationary state of the following equation:

$$q_t + S(\Phi)\frac{\nabla\Phi}{\mid \nabla\Phi \mid} = 0 \text{ in } D \times [0, t_0], \quad q(x,0) = p(x,0) \;\; x \in D \tag{53}$$

where $p(x,t) = V \cdot n$ on the boundary $\Gamma$ and 0 elsewhere. Function $S(\Phi)$ denotes an approximation of the sign function (see Myśliński, 2004).

## 7. Numerical methods and example

The discretized structural optimization problem (20) is solved numerically. The numerical algorithms described in the previous sections have been used. The algorithm is programmed in Matlab environment. As an example a body occupying 2D domain

$$\Omega = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \;\wedge\; 0 < v(x_1) \leq x_2 \leq 4\}, \tag{54}$$

is considered. The boundary $\Gamma$ of the domain $\Omega$ is divided into three pieces

$$\begin{aligned} \Gamma_0 &= \{(x_1, x_2) \in R^2 : x_1 = 0, 8 \;\wedge\; 0 < v(x_1) \leq x_2 \leq 4\}, \\ \Gamma_1 &= \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \;\wedge\; x_2 = 4\}, \\ \Gamma_2 &= \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \;\wedge\; v(x_1) = x_2\}. \end{aligned} \tag{55}$$

The domain $\Omega$ and the boundary $\Gamma_2$ depend on the function $v$. This function is the variable subject to shape optimization. The initial position of the boundary $\Gamma_2$ is given as in Myśliński (2006, 2008). The computations are carried out for the elastic body characterized by the Poisson's ratio $\nu = 0.29$, and the Young modulus $E = 2.1 \cdot 10^{11} N/m^2$. The body is loaded by boundary traction $p_1 = 0$, $p_2 = -5.6 \cdot 10^6 N$ along $\Gamma_1$, body forces $f_i = 0$, $i = 1, 2$. Auxiliary function $\phi$ is selected as piecewise constant (or linear) on $D$ and is approximated by a piecewise constant (or bilinear) functions. The computational domain $D = [0, 8] \times [0, 4]$ is selected. Domain $D$ is discretized with a fixed rectangular mesh of $24 \times 12$.

Fig. 3 presents the optimal domain obtained by solving topological and shape optimization problem (20) in the computational domain $D$ using algorithm (A1) and employing the optimality condition (27) - (29). The holes denoted by dotted lines appear in the central part of the body and near the fixed edges. Although the shape of the optimal contact boundary $\Gamma_2$ is similar to the optimal shape obtained in a case of shape optimization only (see Myśliński, 2006), but the obtained shape of the boundary $\Gamma_2$ is not so strongly changed compared to the

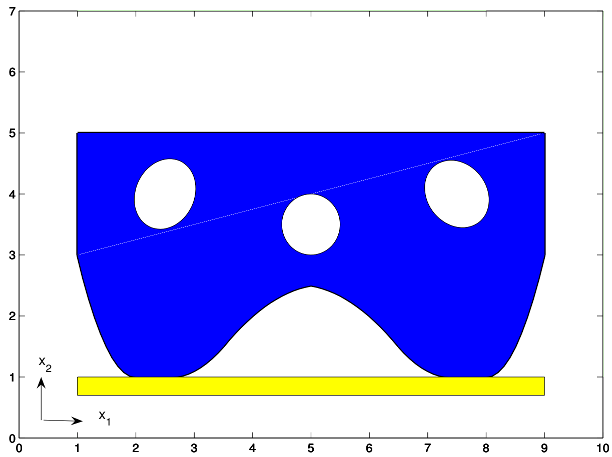


Figure 3. Simultaneous topology and shape optimization - optimal domain $\Omega^{\star}$.

initial one as the optimal shape obtained in the case of shape optimization only. The obtained normal contact stress is almost constant along the optimal shape boundary and has been significantly reduced compared to the initial one.

## 8. Conclusions

The structural optimization problem for the elastic contact problem with the prescribed friction is solved numerically in the paper. The topological derivative method as well as the level set approach combined with the shape gradient method are used. The prescribed friction term complicates both the form of the gradients of the cost functional as well as the numerical process. The numerical results obtained appear to be in accordance with physical reasoning. They indicate that the proposed numerical algorithm allows for significant improvements in the structure from one iteration to the subsequent one.

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