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Abstract: The paper concerns the existence of weak solutions to the Cahn-Hilliard-Gurtin system coupled with nonstationary elasticity. The system describes phase separation process in elastically stressed material. It generalizes the Cahn-Hilliard equation by admitting a more general structure and by coupling diffusive and elastic effects. The system is studied with the help of a singularly perturbed problem which has the form of a well-known phase field model coupled with elasticity. The established existence results are restricted to the homogeneous problem with gradient energy tensor and elasticity tensor independent of the order parameter.

Keywords: Cahn-Hilliard-Gurtin model, phase separation, elasticity system, existence of weak solutions.

1. Introduction

In this paper we study the existence of solutions to the Cahn-Hilliard system coupled with elasticity, which has been proposed by Gurtin (1996). Such system generalizes the classical Cahn-Hilliard equation by admitting a more general structure and at the same time accounts for a deformation of the material. The system describes phase separation process in a binary deformable alloy quenched below a certain critical temperature. From the materials science literature it is known that elastic effects strongly influence the microstructure evolution in phase separation process, especially in its later stages (coarsening), see reviews by Fratzl, Penrose and Lebowitz (1999), Fried and Gurtin (1999), and numerical simulations in Garcke, Rumpf and Weikard (2001), Leo, Lowengrub and Jou (1998), Dreyer and Müller (2001). The important factors are the elastic anisotropy and heterogeneity as well as the impact of external body forces. In particular, the elastic fields can be used to control and stabilize the coarsening process and thereby influence the material properties, see Leo, Lowengrub and Jou (1998).

The Cahn-Hilliard models accounting for elastic effects have been first derived on the basis of variational arguments by Larché and Cahn (1982, 1985, 1992) and Onuki (1989). Having in mind several objections to variational derivations Gurtin (1996) proposed a thermodynamical theory which relies on the fundamental balance laws in conjunction with an auxiliary balance law for the microforces and a mechanical version of the second law. Gurtin's theory generalizes the Cahn-Hilliard equation to the following system

$$\begin{aligned} &\chi_t - \nabla\cdot(\mathbf{M}\nabla\mu + \mathbf{h}\chi_t) = 0, \\ &\mu - \mathbf{g}\cdot\nabla\mu = -\nabla\cdot(\mathbf{\Gamma}\nabla\chi) + \Psi'(\chi) + \beta\chi_t \end{aligned} \tag{1}$$

defined on a spatial domain $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}$, where χ is the scalar order parameter, μ is the chemical potential, $\chi_t = \partial\chi/\partial t$, $\Psi(\chi)$ is a double-well potential, whose wells characterize the phases of the material, $\mathbf{M}$ is a positive definite mobility matrix (special case $\mathbf{M} = m\mathbf{I}$, $m > 0$ constant), $\mathbf{\Gamma}$ is a positive definite gradient matrix (special case $\mathbf{\Gamma} = \gamma\mathbf{I}$, $\gamma > 0$ constant), $\beta > 0$ is the viscosity coefficient, $\mathbf{h}$ and $\mathbf{g}$ are given vectors. The quantities $\mathbf{M}, \beta, \mathbf{h}, \mathbf{g}$ can in general depend on $\chi, \nabla\chi, \chi_t, \mu, \nabla\mu$, and are subject to the condition

$$\mathbf{X}\cdot\begin{bmatrix}\mathbf{M} & \mathbf{h}\\ \mathbf{g}^T & \beta\end{bmatrix}\mathbf{X} \geq 0 \quad \forall\ \mathbf{X} := (\nabla\mu, \chi_t) \in \mathbb{R}^n\times\mathbb{R}$$
$$\text{and } (\chi, \nabla\chi, \mu) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}.$$

Equation $(1)_1$ represents the mass balance and $(1)_2$ the microforce balance. The system (1) differs from the Cahn-Hilliard equation by the presence of the coupling terms with vectors $\mathbf{h}$ and $\mathbf{g}$. The physical interpretation of these terms in the framework of Gurtin's theory is given in Section 2. In the case of $\mathbf{h} = \mathbf{0}$, $\mathbf{g} = \mathbf{0}$ and $\beta = 0$ the system reduces to the classical Cahn-Hilliard equation while in the case of $\mathbf{h} = \mathbf{0}$, $\mathbf{g} = \mathbf{0}$ and $\beta > 0$ to the viscous Cahn-Hilliard equation. Such equations have been extensively studied in the mathematical literature (for recent survey see, e.g., Miranville, 2003).

The elastic effects are taken into account by coupling (1) with the linear momentum balance (see Gurtin, 1996)

$$\mathbf{u}_{tt} - \nabla\cdot(\mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))) = \mathbf{b} \tag{2}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linearized strain tensor, $\overline{\boldsymbol{\varepsilon}}(\chi)$ is the eigenstrain, and $\mathbf{A}(\chi)$ is the elasticity tensor. Since the mechanical equilibrium is usually attained on a much faster time scale than diffusion, a quasi-stationary approximation of (2) obtained by neglecting the inertial term $\mathbf{u}_{tt}$ is often applied.

The equations (1), (2) constitute the Cahn-Hilliard-Gurtin system coupled with elasticity, which is the subject of our study. We mention that in Pawłow (2005) it has been shown that it is possible to reconstruct Gurtin's theory by using the approach based on the fundamental balance laws and the entropy inequality with multipliers. It turns out that the differential equation for the multiplier of the mass balance can be identified with the microforce balance of Gurtin's theory. We add that the approach of Gurtin allows taking into account thermal effects, see e.g. Miranville and Schimperna (2005). Similarly, the multipliers-based approach allows also to account for such effects. Generalized nonisothermal Cahn-Hilliard models in elastic solids will be the subject of a future work.

In order to place our study in the present theory of Cahn-Hilliard systems coupled with elasticity we review first the known results.

Recently, Dreyer and Müller (2000, 2001) have extensively studied the modeling aspects of binary tin-lead solders. They proposed a specific system of equations which falls into Gurtin's framework, and have examined it by numerical computations for experimental data.

In Garcke (2000, 2003) the Cahn-Hilliard system with a multicomponent order parameter coupled with the quasi-stationary elasticity has been analysed mathematically. The existence result has been obtained in a general case of heterogeneous elasticity, i.e., the order parameter-dependent elasticity tensor $\mathbf{A} = \mathbf{A}(\chi)$. The order parameter-dependence of the elasticity tensor introduces a nonlinear coupling between the equations and makes the analysis much more complicated. We underline that the existence result in Garcke (2000, 2003) is based on the monotonicity argument for the quasi-stationary elasticity equation. Such an argument does not extend to the nonstationary case.

An even more difficult, but physically more adequate, multicomponent Cahn-Hilliard system with logarithmic free energy, coupled with elasticity has been recently studied in Garcke. As in Garcke (2003), due to the quasi-stationary elasticity equation, a higher integrability result for the strain has been established which allowed to consider order parameter dependent elasticity tensor.

In Bonetti et al. (2002) the physical model proposed by Dreyer and Müller (2000, 2001) has been studied. For a system with heterogeneous, quasi-stationary elasticity the existence and uniqueness results have been obtained in case of single dimension (1-D) and for homogeneous elasticity in case of 2-D. In contrast to the previous works the framework of Bonetti et al. (2002) refers to a non-differentiable free energy involving the indicator function of a closed interval within which the order parameter is forced to attain its values. Besides, the order parameter-dependence of the gradient coefficient $\gamma = \gamma(\chi)$ is there taken into account, with certain structural simplifications suggested in Dreyer and Müller (2000, Appendix). We mention also the paper by Bonetti, Dreyer and

Schimperna (2003) where uncoupled, constrained Cahn-Hilliard equation with additional nonlinear terms imitating the elastic effects has been examined.

Various variants of the Cahn-Hilliard-Gurtin system without and with elasticity have been extensively studied by Miranville and associates (see Carrive, Miranville and Piétrus, 2000; Carrive et al., 1998, 1999; Miranville, 1999, 2000, 2001a, 2001b, 2003) from the point of view of the existence, uniqueness and long time behaviour of the solutions. In all these papers it has been assumed that the gradient matrix is isotropic $\mathbf{\Gamma} = \gamma \mathbf{I}$ with constant $\gamma > 0$, the mobility matrix $\mathbf{M}$ is constant, and in case of a coupled system that the elasticity tensor $\mathbf{A}$ is constant.

In Miranville, Piétrus and Rakotoson (1998) the viscous Cahn-Hillard equation ($\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = const > 0$) without elasticity has been studied, and in Miranville (2001a) coupled with quasi-stationary or nonstationary elasticity.

In Carrive et al. (1999) the classical Cahn-Hilliard equation ($\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = 0$) coupled with stationary, isotropic elasticity has been considered. The analysis in that paper is based on the fact that in such a case the equation for the order parameter is independent of the displacement $\mathbf{u}$. A more general case without isotropy assumption has been investigated in Carrive, Miranville, Piétrus (2000).

The Cahn-Hilliard-Gurtin system (1) without elasticity in a special case of $\mathbf{h} = \mathbf{0}$ and constant vector $\mathbf{g} \neq \mathbf{0}$ has been studied in Miranville (1999), and in the case of constant vectors $\mathbf{g} \neq \mathbf{0}$ and $\mathbf{h} \neq \mathbf{0}$ under periodic boundary conditions in Miranville (2003).

In Miranville (2000, 2001b) the Cahn-Hilliard-Gurtin system ($\mathbf{g} \neq \mathbf{0}$, $\mathbf{h} \neq \mathbf{0}$) coupled with quasi-stationary elasticity has been analysed. The considerations in Miranville (2000) make use of the fact that in case of quasi-stationary elasticity equations for the order parameter and the displacement can be uncoupled.

In Miranville (2001b) the problem has been studied under geometry assumptions and a special structure of vectors $\mathbf{g}$ and $\mathbf{h}$. Namely, the domain has been assumed to be a two-dimensional (2-D) or a three-dimensional (3-D) parallelepiped, and mixed periodic-Neumann boundary conditions have been imposed. The vectors $\mathbf{g}$ and $\mathbf{h}$ have been assumed constant with vanishing components in x_2- direction in 2-D case and vanishing components in x_3-direction in 3-D case.

We point out that in the above mentioned papers by Miranville and associates the system (1) has been reformulated as a single equation for the order parameter χ. This is in contrast to our approach in which we treat χ and μ as independent variables.

The goal of the present paper is to study the existence of solutions to the Cahn-Hilliard-Gurtin system coupled with elasticity in the following cases that have not been, or have been only partially, addressed in previous works:

(i) The presence of the coupling terms with vectors $\mathbf{g}$ and $\mathbf{h}$;
(ii) The nonstationary elasticity equation. The vanishing-inertial term analysis, i.e., examination of the time re-scaling limit to the quasi-stationary

problem;

(iii) The mobility tensor $\mathbf{M}(\chi)$ depending on the order parameter (anisotropic, heterogeneous diffusion);

(iv) The gradient tensor $\mathbf{\Gamma}(\chi)$ in free energy depending on the order parameter (anisotropic, heterogeneous interfacial structure);

(v) The elasticity tensor $\mathbf{A}(\chi)$ depending on the order parameter (anisotropic, heterogeneous elasticity).

We add also that as a by-product of our analysis we obtain

(vi) the existence result for a phase-field model coupled with elasticity and its convergence to the Cahn-Hilliard-Gurtin system with elasticity.

We point out that in the present paper the dependencies of the gradient and elasticity tensors on the order parameter are considered only for the Faedo-Galerkin approximations but not in passing to the limit (see below for a detailed description of the results).

We add also that our considerations are restricted to a scalar, unconstrained order parameter. More advanced models should take into account the constraints on the order parameter like in Bonetti et al. (2002).

We formulate now the initial-boundary-value *problem* (P) we deal with:

$$\begin{aligned} &\mathbf{u}_{tt} - \nabla\cdot W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi) = \mathbf{b}, && \text{in } \Omega^T = \Omega\times(0,T),\\ &\mathbf{u}|_{t=0} = \mathbf{u}_0,\ \mathbf{u}_t|_{t=0} = \mathbf{u}_1, && \text{in } \Omega,\\ &\mathbf{u} = \mathbf{0}, && \text{on } S^T = S\times(0,T), \end{aligned} \tag{3}$$

$$\begin{aligned} &\chi_t - \nabla\cdot(\mathbf{M}(\chi)\nabla\mu + \mathbf{h}\chi_t) = 0, && \text{in } \Omega^T,\\ &\chi|_{t=0} = \chi_0, && \text{in } \Omega,\\ &\mathbf{n}\cdot(\mathbf{M}(\chi)\nabla\mu + \mathbf{h}\chi_t) = 0, && \text{on } S^T, \end{aligned} \tag{4}$$

$$\begin{aligned} &\mu - \mathbf{g}\cdot\nabla\mu + \nabla\cdot(\mathbf{\Gamma}(\chi)\nabla\chi) - \frac{1}{2}\nabla\chi\cdot\mathbf{\Gamma}'(\chi)\nabla\chi\\ &\quad - \Psi'(\chi) - W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi) - \beta\chi_t = 0, && \text{in } \Omega^T,\\ &\mathbf{n}\cdot(\mathbf{\Gamma}(\chi)\nabla\chi) = 0, && \text{on } S^T, \end{aligned} \tag{5}$$

where $W(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)$ is given by

$$W(\boldsymbol{\varepsilon}(\mathbf{u}),\chi) = \frac{1}{2}(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))\cdot\mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)), \tag{6}$$

so

$$\begin{aligned} &W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi) = \mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)),\\ &W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi) = -\overline{\boldsymbol{\varepsilon}}'(\chi)\cdot\mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))\\ &\qquad\qquad + \frac{1}{2}(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))\cdot\mathbf{A}'(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)). \end{aligned} \tag{7}$$

In the quasi-stationary version of (P) the elasticity system (3) is replaced by the elliptic problem

$$-\nabla \cdot W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}), \chi) = \mathbf{b} \quad \text{in } \Omega^T, \qquad (8)$$
$$\mathbf{u} = \mathbf{0}, \quad \text{on } S^T.$$

In the above, $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is a bounded domain with a smooth boundary S, occupied by a solid body in a reference configuration, with constant mass density $\rho = 1$; $\mathbf{n}$ denotes the outward unit normal to S; $T > 0$ is an arbitrary fixed time.

The unknown variables are the fields of the displacement $\mathbf{u} : \Omega^T \to \mathbb{R}^n$, the scalar order parameter $\chi : \Omega^T \to \mathbb{R}$, and the chemical potential difference between the components (shortly referred to as the chemical potential) $\mu : \Omega^T \to \mathbb{R}$. In case of a binary a-b alloy the order parameter is related to the volumetric fraction of one of the two phases, characterized by different crystalline structures of the components, for example $\chi = 0$ corresponds to phase a and $\chi = 1$ to phase b. The second order symmetric tensor

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

denotes the linearized strain (for simplicity we write $\boldsymbol{\varepsilon}$ instead of $\boldsymbol{\varepsilon}(\mathbf{u})$), $\mathbf{b} : \Omega^T \to \mathbb{R}^n$ is the external body force.

The free energy density underlying the system (3)-(5) has the Landau-Ginzburg-Cahn-Hilliard form accounting for the elastic effects

$$f(\boldsymbol{\varepsilon}, \chi, \nabla\chi) = W(\boldsymbol{\varepsilon}, \chi) + \Psi(\chi) + \frac{1}{2}\nabla\chi \cdot \boldsymbol{\Gamma}(\chi)\nabla\chi, \qquad (9)$$

where $W(\boldsymbol{\varepsilon}, \chi)$ is the homogeneous elastic energy, $\Psi(\chi)$ is the exchange energy, and the last term with a symmetric, positive definite tensor $\boldsymbol{\Gamma}(\chi) = (\Gamma_{ij}(\chi))$ is the order parameter gradient energy.

The standard form of the elastic energy $W(\boldsymbol{\varepsilon}, \chi)$ is given by (6) where $\mathbf{A}(\chi) = (A_{ijkl}(\chi))$ is the fourth order elasticity tensor depending on the order parameter, and $\overline{\boldsymbol{\varepsilon}}(\chi) = (\overline{\varepsilon}_{ij}(\chi))$ is the symmetric stress free strain (eigenstrain).

The exchange energy $\Psi(\chi)$ characterizes the energetic favorability of the individual phases a and b. The standard form is a double-well potential with equal minima at $\chi = 0$ and $\chi = 1$:

$$\Psi(\chi) = \frac{1}{2}\chi^2(1-\chi)^2. \qquad (10)$$

Furthermore, $\mathbf{M}(\chi) = (M_{ij}(\chi))$ is the mobility matrix, $\beta \geq 0$ is the diffusional viscosity, and the vectors $\mathbf{g} = (g_i)$, $\mathbf{h} = (h_i)$ represent the coupling effects; for usual isotropic materials $\mathbf{g} = \mathbf{0}$ and $\mathbf{h} = \mathbf{0}$.

By thermodynamical consistency the coefficients matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{M} & \mathbf{h} \\ \mathbf{g}^T & \beta \end{bmatrix} \tag{11}$$

has to satisfy the condition

$$\mathbf{X} \cdot \mathbf{B}\mathbf{X} \geq 0 \quad \forall\ \mathbf{X} = (\nabla\mu, \chi_t) \in \mathbb{R}^n \times \mathbb{R}. \tag{12}$$

If $\mathbf{B}$ is independent of $\mathbf{X}$ then (12) means the positive semi-definiteness of $\mathbf{B}$. In general, $\mathbf{M}, \mathbf{g}, \mathbf{h}, \beta$ can depend on $\nabla\mu, \chi_t, \boldsymbol{\varepsilon}, \chi$. Here we shall assume that $\mathbf{M} = \mathbf{M}(\chi)$ is positive definite, $\beta \geq 0$ is a constant, vectors $\mathbf{g}$ and $\mathbf{h}$ are constant, and the coefficients matrix $\mathbf{B}$ is positive definite in the sense that there exist constants $\underline{c}_M^* > 0$ and $\underline{c}_\beta^* > 0$ such that

$$\mathbf{X} \cdot \mathbf{B}\mathbf{X} \geq \underline{c}_M^* |\nabla\mu|^2 + \underline{c}_\beta^* |\chi_t|^2 \quad \forall\ \ \mathbf{X} = (\nabla\mu, \chi_t) \in \mathbb{R}^n \times \mathbb{R}. \tag{13}$$

We point out that (13) represents one of the two main structural postulates we impose in this paper. The second postulate requires the following lower bound for the free energy

$$f(\boldsymbol{\varepsilon}, \chi, \nabla\chi) \geq c(|\boldsymbol{\varepsilon}|^2 + |\chi|^r + |\nabla\chi|^2) - c, \tag{14}$$

where $r > 2$ and $c > 0$ are constants. Under assumptions formulated in Section 3 the Landau-Ginzburg free energy (9) will be shown to satisfy condition (14). The structure bounds (13) and (14) are used for deriving energy estimates (see Lemma 4.2).

Finally, we mention that the functions $\mathbf{u}_0 : \Omega \to \mathbb{R}^n$, $\mathbf{u}_1 : \Omega \to \mathbb{R}^n$, $\chi_0 : \Omega \to \mathbb{R}$ denote the initial conditions for the displacement, velocity and the order parameter. The boundary conditions in (3)-(5) represent respectively the prescribed displacement, the mass isolation and the natural boundary condition associated with Landau-Ginzburg free energy (9). The homogeneous Dirichlet boundary condition for the displacement is assumed for the sake of simplicity. The results of the paper can be extended to other boundary conditions (see Remark 3.2).

To analyse problem (P) in a general case (with coupling terms $\mathbf{g}$, $\mathbf{h}$) we introduce first a singularly perturbed problem $(P)^\nu$ with a parameter $\nu \in (0, 1]$ which we let to decrease to zero. In this case we have to assume that the viscosity coefficient is a positive constant $\beta > 0$. The special case with vectors $\mathbf{g} = \mathbf{h} = \mathbf{0}$ and the viscosity coefficient $\beta = 0$ (standard Cahn-Hilliard case) can be analysed directly without the use of $(P)^\nu$.

We formulate now *problem* $(P)^\nu$

$$\begin{aligned} &\mathbf{u}_{tt}^\nu - \nabla \cdot W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}^\nu), \chi^\nu) = \mathbf{b}, && \text{in } \Omega^T, \\ &\mathbf{u}^\nu|_{t=0} = \mathbf{u}_0,\ \mathbf{u}_t^\nu|_{t=0} = \mathbf{u}_1, && \text{in } \Omega, \\ &\mathbf{u}^\nu = \mathbf{0}, && \text{on } S^T, \end{aligned} \tag{15}$$

$$\begin{aligned}
&\nu\mu^\nu_t + \chi^\nu_t - \nabla\cdot(\mathbf{M}(\chi^\nu)\nabla\mu^\nu + \mathbf{h}\chi^\nu_t) = 0, && \text{in } \Omega^T, && (16)\\
&\mu^\nu|_{t=0} = \mu_0,\ \chi^\nu|_{t=0} = \chi_0, && \text{in } \Omega,\\
&\mathbf{n}\cdot(\mathbf{M}(\chi^\nu)\nabla\mu^\nu + \mathbf{h}\chi^\nu_t) = 0, && \text{on } S^T,
\end{aligned}$$

$$\begin{aligned}
&\mu^\nu - \mathbf{g}\cdot\nabla\mu^\nu + \nabla\cdot(\boldsymbol{\Gamma}(\chi^\nu)\nabla\chi^\nu) - \frac{1}{2}\nabla\chi^\nu\cdot\boldsymbol{\Gamma}'(\chi^\nu)\nabla\chi^\nu && && (17)\\
&\quad - \Psi'(\chi^\nu) - W_{,\chi^\nu}(\boldsymbol{\varepsilon}(\mathbf{u}^\nu),\chi^\nu) - \beta\chi^\nu_t = 0, && \text{in } \Omega^T,\\
&\mathbf{n}\cdot(\boldsymbol{\Gamma}(\chi^\nu)\nabla\chi^\nu) = 0, && \text{on } S^T,
\end{aligned}$$

where the data are as in (P) and $\mu_0 \in L_2(\Omega)$ is given.

It should be pointed out that in the case of $\beta > 0$ the problem $(P)^\nu$ has the structure of the well-known phase field model of solidification coupled with elasticity. In this context μ can be identified with temperature and χ with the phase ratio. In view of such a correspondence the existence results for $(P)^\nu$ and its singular limits for $\nu \to 0$ are of an independent theoretical interest. We mention that similar limits $\nu, \beta \to 0$ for phase field systems without elasticity have been studied by several authors, e.g., Laurençot (1994), Stoth (1995).

We consider also a time re-scaled problem $(P)^\alpha, \alpha \in (0,1]$, which has the form of problem (P) with the term $\mathbf{u}_{tt}$ in elasticity equation (3) replaced by $\alpha\mathbf{u}_{tt}$. By letting the parameter α to decrease to zero we shall establish the existence of solutions to the Cahn-Hilliard system (4), (5) coupled with quasi-stationary elasticity (8).

The main results of the present paper concern the existence of weak solutions to problems $(P)^\nu$ and (P) in the homogeneous case with constant tensors $\boldsymbol{\Gamma}$ and $\mathbf{A}$. The problems are studied by means of the Faedo-Galerkin approximation. We point out that the existence results for the approximate problems refer to the heterogeneous case. The restriction to constant tensors $\boldsymbol{\Gamma}$ and $\mathbf{A}$ is needed only at the stage of passing to the limit in the approximate problems. The origin of the difficulties are the terms $\nabla\chi\cdot\boldsymbol{\Gamma}'(\chi)\nabla\chi$ and $(\boldsymbol{\varepsilon}(\mathbf{u})-\overline{\boldsymbol{\varepsilon}}(\chi))\cdot\mathbf{A}'(\chi)(\boldsymbol{\varepsilon}(\mathbf{u})-\overline{\boldsymbol{\varepsilon}}(\chi))$ in the weak formulation of (5).

In a separate paper (see Pawłow and Zajaczkowski, 2006) we shall apply the same Faedo-Galerkin approximations to prove the existence of measure-valued solutions to problem (P) in the heterogeneous case. The idea of such solutions originates from the papers by Neustupa (1993), Kröner and Zajączkowski (1996) where it has been applied to the Euler and Navier-Stokes equations.

The paper is organized as follows. In Section 2 we present a thermodynamical basis for problem (P). In particular we give a general scheme of deriving energy estimates, which later is used in the analysis of the Faedo-Galerkin approximations. In Section 3 we formulate the assumptions and state the main results on the existence of weak solutions for the following problems in homogeneous case: $(P)^\nu$ (Theorem 3.1), (P) (Theorem 3.2), (P) in the special case $\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = 0$ (Theorem 3.3), (P) in the quasi-stationary case and $\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = 0$ (Theorem 3.4). In Section 4 we study the Faedo-Galerkin approximations for $(P)^\nu$. In Section 5 we study the Faedo-Galerkin approximations for (P) in the special case $\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = 0$. Sections 6-9 contain the proofs of Theorems 3.1-3.4.

We use the following notations:

- $\mathbf{x} \in \mathbb{R}^n, n = 2$ or $n = 3$, the material point, $f_{,i} = \frac{\partial f}{\partial x_i}, f_t = \frac{df}{dt}$ the material space and time derivatives,
- $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,\dots,n}$, $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi) = (\frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial \varepsilon_{ij}})_{i,j=1,..,n}$, $W_{,\chi}(\boldsymbol{\varepsilon}, \chi) = \frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial \chi}$,
- $\mathbf{\Gamma}'(\chi) = (\Gamma'_{ij}(\chi))_{i,j=1,\dots,n}$, $\Gamma'_{ij}(\chi) = \frac{d\Gamma_{ij}(\chi)}{d\chi}$.

For simplicity, whenever there is no danger of confusion, we omit the arguments $(\boldsymbol{\varepsilon}, \chi)$. The specification of tensor indices is omitted as well.

Vector and tensor-valued mappings are denoted by bold letters.

The summation convention over repeated indices is used, as well as the notation:

- for vectors $\mathbf{a} = (a_i), \tilde{\mathbf{a}} = (\tilde{a}_i)$, and tensors $\mathbf{B} = (B_{ij})$, $\tilde{\mathbf{B}} = (\tilde{B}_{ij})$, $\mathbf{A} = (A_{ijkl})$ we write $\mathbf{a} \cdot \tilde{\mathbf{a}} = a_i \tilde{a}_i$, $\mathbf{B} \cdot \tilde{\mathbf{B}} = B_{ij} \tilde{B}_{ij}$, $\mathbf{AB} = (A_{ijkl} B_{kl})$, $\mathbf{BA} = (B_{ij} A_{ijkl})$,
- $|\mathbf{a}| = (a_i a_i)^{\frac{1}{2}}$, $|\mathbf{B}| = (B_{ij} B_{ij})^{\frac{1}{2}}$,
- ∇ and $\nabla \cdot$ denote the gradient and the divergence operators with respect to the material point $\mathbf{x} \in \mathbb{R}^n$. For divergence of a tensor field we use the convention of the contraction over the last index $\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{x}) = (\varepsilon_{ij/j}(\mathbf{x}))$.

We use the standard Sobolev spaces notation $H^m(\Omega) = \mu_2^m(\Omega)$ for $m \in \mathbb{N}$. For simplicity we write

- $\mathbf{L}_2(\Omega) = (L_2(\Omega))^n$, $\mathbf{V}_0 = \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^n$, $n = 2$ or 3,
- $(\cdot, \cdot)_{L_2(\Omega)}$, $(\cdot, \cdot)_{\mathbf{L}_2(\Omega)}$ denote the scalar products in $L_2(\Omega)$ and $\mathbf{L}_2(\Omega)$.
- We denote by V' the dual space of $V = H^1(\Omega)$ and by $< \cdot, \cdot >_{V',V}$ the duality pairing between V and V'.
- Similarly $\mathbf{V}_0'$ denotes the dual space of $\mathbf{V}_0$ and $< \cdot, \cdot >_{\mathbf{V}_0', \mathbf{V}_0}$ the duality pairing between $\mathbf{V}_0$ and $\mathbf{V}_0'$.

Throughout the paper c denotes a generic positive constant different in various instances, in general depending on time horizon T.

2. The thermodynamical basis

We outline the thermodynamical derivation of system (3)-(5), presented in detail in Pawłow (2005), and next compare it with Gurtin's framework. The approach is based on the second law of thermodynamics in the form of the entropy principle according to I. Müller and I. S. Liu, which leads to the evaluation of the entropy inequality with multipliers (see Müller, 1985). The application of this approach to the phase separation process of our concern requires a procedure which can be summarized in the following three steps.

In the first step we consider the balance laws of linear momentum and mass

$$\begin{aligned} \mathbf{u}_{tt} - \nabla \cdot \mathbf{S} &= \mathbf{b}, \\ \chi_t + \nabla \cdot \mathbf{j} &= 0, \end{aligned} \tag{18}$$

where $\mathbf{S}$ and $\mathbf{j}$ are the referential stress tensor and the mass flux. They are assumed to be given by the constitutive equations

$$\mathbf{S} = \hat{\mathbf{S}}(Y), \quad \mathbf{j} = \hat{\mathbf{j}}(Y)$$

with the constitutive set

$$Y := \{\boldsymbol{\varepsilon}, \chi, \nabla\chi, ..., \nabla^M \chi, \chi_t\}, \quad M \in \mathbb{N}, \quad M \geq 2,$$

which is relevant for the problem under consideration. Distinctive elements in this set are variables representing higher gradients of the order parameter and its time derivative. The presence of such variables is characteristic for theories involving free energies of Landau-Ginzburg type. In accordance with the principle of equipresence we assume that the constitutive quantities $\mathbf{S}$ and $\mathbf{j}$ are defined on the same set of variables.

In the second step we postulate the free energy inequality with multipliers which in the isothermal case has the form

$$f_t + \nabla \cdot \boldsymbol{\Phi} - \mathbf{S} \cdot \boldsymbol{\varepsilon}(\mathbf{u}_t) + \boldsymbol{\lambda}_{\mathbf{u}} \cdot (\mathbf{u}_{tt} - \nabla \cdot \mathbf{S}) + \lambda(\chi_t + \nabla \cdot \mathbf{j}) \leq 0$$

for all fields $\mathbf{u}$, χ. Here f is the free energy, $\boldsymbol{\Phi}$ is the free energy flux and $\boldsymbol{\lambda}_{\mathbf{u}}$, λ are multipliers conjugated, respectively, with balances $(18)_1$ and $(18)_2$. Again, in consistency with equipresence, we assume that the quantities f, $\boldsymbol{\Phi}$, $\boldsymbol{\lambda}_{\mathbf{u}}$ and λ depend on the same constitutive set

$$f = \hat{\mathbf{f}}(Y), \quad \boldsymbol{\Phi} = \hat{\boldsymbol{\Phi}}(Y), \quad \boldsymbol{\lambda}_{\mathbf{u}} = \hat{\boldsymbol{\lambda}}_{\mathbf{u}}(Y), \quad \lambda = \hat{\lambda}(Y).$$

Next, making no assumptions on the multipliers $\boldsymbol{\lambda}_{\mathbf{u}}$, λ, we exploit the above free energy inequality by using appropriately arranged algebraic operations. As a result we conclude a collection of algebraic restrictions on the constitutive equations, in particular that the constitutive dependence of f is restricted to

$$f = \hat{f}(\boldsymbol{\varepsilon}, \chi, \nabla\chi),$$

the stress tensor $\mathbf{S}$ satisfies

$$\mathbf{S} - f_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi, \nabla\chi) = \mathbf{0}, \tag{19}$$

and the multiplier $\boldsymbol{\lambda}_{\mathbf{u}} = \mathbf{0}$. In addition, we obtain algebraic versions of the differential equation for λ and of the residual (dissipation) inequality.

In the third step we presuppose that the multiplier λ can be treated as an additional independent variable. Then, regarding the algebraic restrictions obtained in the previous step, we deduce an extended system of equations including in addition to the balance laws (18 the differential equation for the multiplier

$$-\lambda - \frac{\partial f}{\partial \chi}(\boldsymbol{\varepsilon}, \chi, \nabla\chi) + a = 0 \tag{20}$$

where

$$\frac{\delta f}{\delta \chi}(\boldsymbol{\varepsilon}, \chi, \nabla\chi) = f_{,\chi}(\boldsymbol{\varepsilon}, \chi, \nabla\chi) - \nabla \cdot f_{,\nabla\chi}(\boldsymbol{\varepsilon}, \chi, \nabla\chi)$$

denotes the first variation of f with respect to χ, and a is a scalar field. In view of the resemblance of (20) with the classical definition of the chemical potential, we identify the negative of the multiplier λ with the chemical potential

$$\mu = -\lambda$$

The quantities $\mathbf{j}$ in $(18)_2$ and a in (20) are subject to the dissipation inequality

$$-\mathbf{X} \cdot \mathbf{J}(\mathbf{X}; \boldsymbol{\omega}) = -(\nabla\mu \cdot \mathbf{j} + \chi_t a) \geq 0 \quad \forall(\mathbf{X}; \boldsymbol{\omega}), \tag{21}$$

where $\mathbf{X} := (\nabla\mu, \chi_t)$ is a thermodynamical flux, and $\boldsymbol{\omega} := (\boldsymbol{\varepsilon}, \chi, \nabla\chi, \nabla^2\chi)$ is a vector of state variables. According to Gurtin (1996, Appendix B) a general solution of inequality (21) is given by

$$\mathbf{J}(\mathbf{X}, \omega) = -\mathbf{B}(\mathbf{X}, \omega)\mathbf{X} \tag{22}$$

with the matrix $\mathbf{B}$ consistent with (11), (12). Hence,

$$\begin{aligned} \mathbf{j} &= -(\mathbf{M}\nabla\mu + \mathbf{h}\chi_t), \\ a &= -(\mathbf{g} \cdot \nabla\mu + \beta\chi_t). \end{aligned} \tag{23}$$

Combining relations (18)-(20), (23) and assuming that $f(\boldsymbol{\varepsilon}, \chi, \nabla\chi)$ is given by (9), we arrive at the field equations in (3)-(5).

It is easy to check that the introduced system of balance laws (18) with constraints (19), (20) and subjected to (21) satisfies the following free energy inequality which assures its thermodynamical consistency

$$\begin{aligned} &\frac{d}{dt}\Big(f(\boldsymbol{\varepsilon}, \chi, \nabla\chi) + \frac{1}{2}|\mathbf{u}_t|^2\Big) + \nabla \cdot (-\mathbf{u}_t\mathbf{S} + \mu\mathbf{j} - f_{,\nabla\chi}\chi_t) \\ &\quad + \boldsymbol{\Lambda}_{\mathbf{u}} \cdot (\mathbf{u}_{tt} - \nabla \cdot \mathbf{S}) \\ &\quad + \Lambda_\chi(\chi_t + \nabla \cdot \mathbf{j}) \\ &\quad + \Lambda_\mu(\mu - f_{,\chi} + \nabla \cdot f_{,\nabla\chi} + a) \\ &\quad + \boldsymbol{\Lambda}_{\mathbf{S}} \cdot (\mathbf{S} - f_{,\boldsymbol{\varepsilon}}) \\ &= \nabla\mu \cdot \mathbf{j} + \chi_t a \leq 0 \quad \text{for all fields } \mathbf{u}, \chi, \mu, \end{aligned} \tag{24}$$

where

$$\mathbf{\Lambda_u} := -\mathbf{u}_t,\ \Lambda_\chi := -\mu,\ \Lambda_\mu := \chi_t,\ \mathbf{\Lambda_S} := \boldsymbol{\varepsilon}_t \tag{25}$$

are multipliers conjugated, respectively, with the linear momentum balance, the mass balance, and the equations for the chemical potential and the stress.

As an immediate consequence of (24) it follows that the solutions of balance laws (18) with constraints (19), (20) satisfy the following energy identity

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (f(\boldsymbol{\varepsilon},\chi,\nabla\chi) + \frac{1}{2}|\mathbf{u}_t|^2)\ dx + \\ &\quad \int_S [-(\mathbf{Sn})\cdot\mathbf{u}_t + \mu\mathbf{n}\cdot\mathbf{j} - \chi_t\mathbf{n}\cdot\ f_{,\nabla\chi}]\ dS \\ &= \int_\Omega (\nabla\mu\cdot\mathbf{j} + \chi_t a)\ dx + \int_\Omega \mathbf{b}\cdot\mathbf{u}_t\ dx. \end{aligned} \tag{26}$$

We point on two important consequences of the identity (26). The first one is concerned with the general thermodynamical property known as the Lyapunov relation. Namely, in view of the dissipation inequality (21), if the external force $\mathbf{b} = \mathbf{0}$, and if the boundary conditions on S imply that

$$(\mathbf{Sn})\cdot\mathbf{u}_t = 0,\ \mu\ \mathbf{n}\cdot\mathbf{j} = 0,\ \chi_t\mathbf{n}\cdot f_{,\nabla\chi} = 0, \tag{27}$$

then the Lyapunov relation follows from (26):

$$\frac{d}{dt}\int_\Omega (f(\boldsymbol{\varepsilon},\chi,\nabla\chi) + \frac{1}{2}|\mathbf{u}_t|^2)\ dx \leq 0. \tag{28}$$

It states that the total energy is non-increasing on solutions paths. We note that the boundary conditions in system (3)-(5) are consistent with (27). The second consequence of (26) which is of key mathematical importance are energy estimates. They result from (26) under the structural assumption of the free energy bound (14), and the positive definiteness (13) of the matrix $\mathbf{B}$. The presented above general scheme of deriving energy estimates will be used in Section 4.

Finally we comment on the relations with Gurtin's (1996) framework. The system (18) with constraints (19), (20) and subjected to the inequality (21) coincides (up to neglecting the term $\mathbf{u}_{tt}$ in (18)) with equations resulting from Gurtin's theory (see Gurtin (1996), Sections 3, 4). We point out that in Gurtin's theory the underlying laws are the linear momentum and the mass balance given by (18), and in additon the following microforce balance

$$\nabla\cdot\boldsymbol{\xi} + \pi + \gamma = 0 \tag{29}$$

where $\boldsymbol{\xi}$ is the microstress, π is the internal microforce and γ is an external microforce. By assuming as constitutive variables $(\boldsymbol{\varepsilon},\chi,\nabla\chi,\chi_t,\mu,\nabla\mu)$ (we use our notation) and applying a mechanical version of the second law the following relations have been obtained in Gurtin (1996):

$$\begin{aligned}
&f = f(\boldsymbol{\varepsilon}, \chi, \nabla\chi), \quad \mathbf{S} = f_{,\boldsymbol{\varepsilon}}, \quad \xi = f_{,\nabla\boldsymbol{\varepsilon}},\\
&\pi = \mu - f_{,\chi} + \pi_{dis},\\
&j = -(\mathbf{M}\nabla\mu + \mathbf{h}\chi_t),\\
&\pi_{dis} = -(\mathbf{g}\cdot\nabla\mu + \beta\chi_t)
\end{aligned}$$

where the coefficients $\mathbf{M}, \mathbf{h}, \mathbf{g}, \beta$ comply with (11), (12). The above relations show that equation (20) for $\mu = -\lambda$ can be interpreted as a microforce balance while the quantity a as a dissipative part of the internal microforce.

3. The assumptions and the main results

We list the assumptions under which the Faedo-Galerkin approximations of problems $(P)^\nu$ and (P) are studied. These assumptions refer to the heterogeneous case involving tensors $\boldsymbol{\Gamma}(\chi)$ and $\mathbf{A}(\chi)$ depending on χ. The existence results for the original problems $(P)^\nu$ and (P) will be proved only in case of constant tensors $\boldsymbol{\Gamma}$ and $\mathbf{A}$.

(A1) $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is a bounded domain with C^1 boundary S.
The following assumptions concern the components of the Landau-Ginzburg free energy

$$f(\boldsymbol{\varepsilon}, \chi, \nabla\chi) : \mathcal{S}^2 \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

given by (9), where $\mathcal{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^n$.

We assume that

(A2) The elasticity tensor $\mathbf{A}(\chi) = (A_{ijkl}(\chi)) : \mathcal{S}^2 \to \mathcal{S}^2$ is a linear mapping such that

- **(i)** is of class $C^{1,1}$ with respect to $\chi : A_{ijkl}(\cdot) \in C^1(\mathbb{R})$ with $A'_{ijkl}(\cdot)$ Lipschitz continuous,
- **(ii)** satisfies the symmetry conditions $A_{ijkl}(\cdot) = A_{jikl}(\cdot) = A_{klij}(\cdot)$,
- **(iii)** is positive definite and bounded uniformly with respect to χ : there exist constants $0 < \underline{c}_A < \overline{c}_A$ such that

$$\underline{c}_A|\boldsymbol{\varepsilon}|^2 \leq \boldsymbol{\varepsilon}\cdot\mathbf{A}(\chi)\boldsymbol{\varepsilon} \leq \overline{c}_A|\boldsymbol{\varepsilon}|^2 \quad \forall\, \boldsymbol{\varepsilon} \in \mathcal{S}^2 \text{ and } \chi \in \mathbb{R},$$

- **(iv)** the mapping $\mathbf{A}'(\chi) = (A'_{ijkl}(\chi)) : \mathcal{S}^2 \to \mathcal{S}^2$ is uniformly bounded with respect to χ: there exists a constant $c_{A'} > 0$ such that

$$|\mathbf{A}'(\chi)\boldsymbol{\varepsilon}| \leq c_{A'}|\boldsymbol{\varepsilon}| \quad \forall\, \boldsymbol{\varepsilon} \in \mathcal{S}^2 \text{ and } \chi \in \mathbb{R}.$$

We mention that we do not require that $\mathbf{A}(\chi)$ be isotropic.

(A3) The eigenstrain $\overline{\boldsymbol{\varepsilon}}(\chi) = (\overline{\varepsilon}_{ij}(\chi)) \in \mathcal{S}^2$ is

(i) of class $C^{1,1}$ with respect to $\chi : \overline{\varepsilon}_{ij}(\cdot) \in C^1(\mathbb{R})$ with $\overline{\varepsilon}'_{ij}(\cdot)$ Lipschitz continuous,

(ii) satisfies growth conditions: there exists a constant $c > 0$ such that

$$|\overline{\boldsymbol{\varepsilon}}(\chi)| \le c(|\chi| + 1), \quad |\overline{\boldsymbol{\varepsilon}}'(\chi)| \le c \quad \forall\, \chi \in \mathbb{R}.$$

In view of expressions (6), (7), assumptions (A2), (A3) imply that $W(\boldsymbol{\varepsilon}, \chi)$, $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi)$ and $W_{,\chi}(\boldsymbol{\varepsilon}, \chi)$ are Lipschitz continuous functions with respect to $\boldsymbol{\varepsilon}$, χ, satisfying the growth conditions

$$|W(\boldsymbol{\varepsilon}, \chi)| \le c(|\boldsymbol{\varepsilon}|^2 + \chi^2 + 1), \tag{30}$$

$$\begin{aligned} &|W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi)| \le c(|\boldsymbol{\varepsilon}| + |\chi| + 1), \\ &|W_{,\chi}(\boldsymbol{\varepsilon}, \chi)| \le c(|\boldsymbol{\varepsilon}|^2 + \chi^2 + 1), \quad \forall\, (\boldsymbol{\varepsilon}, \chi) \in \mathcal{S}^2 \times \mathbb{R}. \end{aligned} \tag{31}$$

(A4) The double-well potential $\Psi(\cdot) : \mathbb{R} \to \mathbb{R}$ satisfies

(i) is of class $C^{1,1} : \Psi(\cdot) \in C^1(\mathbb{R})$ with $\Psi'(\cdot)$ Lipschitz continuous,

(ii) the bound from below: there exist constants $c_1 > 0$, $c_2 \ge 0$ and a number $r > 2$ such that

$$\Psi(\chi) \ge c_1 |\chi|^r - c_2 \quad \forall\, \chi \in \mathbb{R},$$

(iii) the growth conditions: there exists a constant $c > 0$ such that

$$\begin{aligned} &\Psi(\chi) \le c(|\chi|^{\frac{q_n}{2}+1} + 1), \\ &\Psi'(\chi) \le c(|\chi|^{\frac{q_n}{2}} + 1), \quad \forall\, \chi \in \mathbb{R}, \end{aligned}$$

where q_n is the Sobolev exponent for which the imbedding of $H^1(\Omega)$ into $L_{q_n}(\Omega)$ is continuous, i.e., $q_n = 2n/(n-2)$ for $n \ge 3$ and q_n is any finite number for $n = 2$. We note that $\Psi(\chi)$ defined by (10) satisfies (A4)(ii):

$$\Psi(\chi) \ge \frac{1}{8}\chi^4 - \frac{1}{2},$$

and obviously (A4)(iii). We remark that the growth condition (A4)(iii) on $\Psi'(\chi)$ is needed in the proof of the convergence of the Faedo-Galerkin approximations (see Lemma 6.1).

(A5) The gradient energy tensor $\boldsymbol{\Gamma}(\chi) = (\Gamma_{ij}(\chi)) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping such that

(i) is of class $C^{1,1}$ with respect to $\chi : \Gamma_{ij}(\cdot) \in C^1(\mathbb{R})$ with $\Gamma'_{ij}(\cdot)$ Lipschitz continuous,

(ii) is symmetric $\Gamma_{ij}(\cdot) = \Gamma_{ji}(\cdot)$,

(iii) is positive definite and bounded uniformly with respect to χ : there exist constants $0 < \underline{c}_\Gamma < \overline{c}_\Gamma$ such that

$$\underline{c}_\Gamma |\boldsymbol{\xi}|^2 \le \boldsymbol{\xi} \cdot \boldsymbol{\Gamma}(\chi)\boldsymbol{\xi} \le \overline{c}_\Gamma |\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^n \text{ and } \chi \in \mathbb{R},$$

(iv) the mapping $\mathbf{\Gamma}'(\chi) = (\Gamma'_{ij}(\chi)) : \mathbb{R}^n \to \mathbb{R}^n$ is uniformly bounded with respect to χ: there exists a constant $c_{\Gamma'} > 0$ such that

$$|\mathbf{\Gamma}'(\chi)\boldsymbol{\xi}| \leq c_{\Gamma'}|\boldsymbol{\xi}| \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^n \text{ and } \chi \in \mathbb{R}.$$

We note that in view of (A2)(iii), (A4)(ii) and (A3)(ii), using Young's inequality and the fact that $r > 2$,

$$\begin{aligned} W(\boldsymbol{\varepsilon},\chi) + \Psi(\chi) &\geq \frac{1}{2}\underline{c}_A|\boldsymbol{\varepsilon} - \overline{\boldsymbol{\varepsilon}}(\chi)|^2 + c_1|\chi|^r - c_2 \\ &\geq \frac{1}{2}\underline{c}_A|\boldsymbol{\varepsilon}|^2 - \underline{c}_A\boldsymbol{\varepsilon}\cdot\overline{\boldsymbol{\varepsilon}}(\chi) + c_1|\chi|^r - c_2 \\ &\geq \frac{1}{4}\underline{c}_A|\boldsymbol{\varepsilon}|^2 - \underline{c}_A|\overline{\boldsymbol{\varepsilon}}(\chi)|^2 + c_1|\chi|^r - c_2 \\ &\geq \frac{1}{4}\underline{c}_A|\boldsymbol{\varepsilon}|^2 - c|\chi|^2 + c_1|\chi|^r - c \\ &\geq c(|\boldsymbol{\varepsilon}|^2 + |\chi|^r) - c \quad \forall\, (\boldsymbol{\varepsilon},\chi) \in \mathcal{S}^2 \times \mathbb{R} \end{aligned} \tag{32}$$

with some constant $c > 0$. Consequently, taking into account (A5)(iii) we can see that the free energy satisfies the following bound from below

$$f(\boldsymbol{\varepsilon},\chi,\nabla\chi) \geq c(|\boldsymbol{\varepsilon}|^2 + |\chi|^r + |\nabla\chi|^2) - c \quad \forall\, (\boldsymbol{\varepsilon},\chi,\nabla\chi) \in \mathcal{S}^2 \times \mathbb{R} \times \mathbb{R}^n \tag{33}$$

with some constant $c > 0$. This is the first main structural postulate that we use in deriving energy estimates (see Section 4).

The next two assumptions concern the mobility matrix and the coupling terms.

(A6) The mobility matrix $\mathbf{M}(\chi) = (M_{ij}(\chi)) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping which

- **(i)** is of class $C^{0,1}$ with respect to $\chi : M_{ij}(\cdot) \in C^0(\mathbb{R})$ are Lipschitz continuous,
- **(ii)** is symmetric $M_{ij} = M_{ji}$,
- **(iii)** is positive definite and bounded uniformly with respect to χ : there exist constants $0 < \underline{c}_M < \overline{c}_M$ such that

$$\underline{c}_M|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi}\cdot\mathbf{M}(\chi)\boldsymbol{\xi} \leq \overline{c}_M|\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^n \text{ and } \chi \in \mathbb{R}.$$

(A7) The coupling vectors $\mathbf{g} = (g_i)$, $\mathbf{h} = (h_i)$ are constant, the viscosity coefficient is a positive constant $\beta > 0$, and the coefficients matrix

$$\mathbf{B}(\chi) = \begin{bmatrix} \mathbf{M} & \mathbf{h} \\ \mathbf{g}^T & \beta \end{bmatrix}$$

is positive definite in the sense that there exist constants $\underline{c}^*_M > 0$ and $\underline{c}^*_\beta > 0$ such that

$$\begin{aligned} \mathbf{X}\cdot\mathbf{B}(\chi)\mathbf{X} &= \nabla\mu\cdot\mathbf{M}(\chi)\nabla\mu + \chi_t(\mathbf{g}+\mathbf{h})\cdot\nabla\mu + \beta\chi_t^2 \\ &\geq \underline{c}^*_M|\nabla\mu|^2 + \underline{c}^*_\beta\chi_t^2 \quad \forall\, \mathbf{X} = (\nabla\mu,\chi_t) \in \mathbb{R}^n \times \mathbb{R}. \end{aligned} \tag{34}$$

The condition (34) is the second main structural postulate which is used in derivation of energy estimates.

In the standard Cahn-Hilliard case the assumption (A7) is replaced by

(A7)′ The vectors $\mathbf{g} = \mathbf{h} = \mathbf{0}$, the coefficient $\beta = 0$, and the matrix $\mathbf{M}(\chi)$ is positive definite uniformly with respect to χ, i.e., satisfies (A6)(iii).

The last assumption concerns the data of the problem.

(A8) The initial data $\mathbf{u}_0$, $\mathbf{u}_1$, χ_0, and μ_0 in case of problem $(P)^\nu$, and the force term $\mathbf{b}$ satisfy

$$\mathbf{u}_0 \in \mathbf{V}_0, \mathbf{u}_1 \in \mathbf{L}_2(\Omega), \chi_0 \in H^1(\Omega), \mu_0 \in L_2(\Omega), \mathbf{b} \in L_2(0,T;\mathbf{L}_2(\Omega)).$$

We note that in view of growth conditions (30) on $W(\boldsymbol{\varepsilon},\chi)$, (A4)(iii) on $\Psi(\chi)$, and the uniform boundedness (A5)(iii) on $\boldsymbol{\Gamma}(\chi)$, it follows that the total free energy corresponding to the initial data is bounded

$$\left|\int_\Omega f(\boldsymbol{\varepsilon}(\mathbf{u}_0),\chi_0,\nabla\chi_0)dx\right| \le c(\|\boldsymbol{\varepsilon}(\mathbf{u}_0)\|^2_{\mathbf{L}_2(\Omega)} + \|\chi_0\|^2_{H^1(\Omega)} + 1) \le c. \tag{35}$$

We formulate now the main results of the paper which are restricted to the homogeneous problem with constant tensors $\mathbf{A}$ and $\boldsymbol{\Gamma}$. The first result states the existence of weak solutions to problem $(P)^\nu$.

Theorem 3.1 *Let the assumptions (A1)÷(A6), (A7), (A8) be satisfied, and in addition tensors* $\boldsymbol{\Gamma}$ *and* $\mathbf{A}$ *are constant. Then there exist functions* $(\mathbf{u}^\nu, \chi^\nu, \mu^\nu)$ *such that*

(i) $\mathbf{u}^\nu \in L_\infty(0,T;\mathbf{V}_0)$, $\mathbf{u}^\nu_t \in L_\infty(0,T;\mathbf{L}_2(\Omega))$, $\mathbf{u}_{tt} \in L_2(0,T;\mathbf{V}'_0)$, $\mathbf{u}^\nu(0) = \mathbf{u}_0$, $\mathbf{u}^\nu_t(0) = \mathbf{u}_1$,

(ii) $\chi^\nu \in L_\infty(0,T;H^1(\Omega))$, $\chi^\nu_t \in L_2(\Omega^T)$, $\chi^\nu(0) = \chi_0$,

(iii) $\nu^{\frac{1}{2}}\mu^\nu \in L_\infty(0,T;L_2(\Omega))$, $\mu^\nu \in L_2(0,T;H^1(\Omega))$,

which satisfy $(P)^\nu$ *in the following weak sense:*

$$\int_0^T < \mathbf{u}^\nu_{tt}, \boldsymbol{\eta} >_{\mathbf{V'}_0,\mathbf{V}_0} dt + \int_{\Omega^T} \mathbf{A}(\underline{\boldsymbol{\varepsilon}}(\mathbf{u}^\nu) - \overline{\boldsymbol{\varepsilon}}(\mathbf{u}^\nu))\cdot\boldsymbol{\varepsilon}(\boldsymbol{\eta})\ dxdt \tag{36}$$
$$= \int_{\Omega^T} \mathbf{b}\cdot\boldsymbol{\eta}\ dxdt \quad \forall\ \boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_0),$$

$$-\int_{\Omega^T} \nu\mu^\nu\xi_t\ dxdt + \int_{\Omega^T} [\chi^\nu_t\xi + (\mathbf{M}(\chi^\nu)\nabla\mu^\nu + \mathbf{h}\chi^\nu_t)\cdot\nabla\xi]\ dxdt \tag{37}$$
$$= \nu\int \mu_0\xi(0)\ dx \quad \forall\ \xi \in C^1([0,T];H^1(\Omega)) \text{ with } \xi(T) = 0,$$

$$\int_{\Omega^T} (\mu^\nu - \mathbf{g}\cdot\nabla\mu^\nu)\zeta\ dxdt - \int_{\Omega^T} \boldsymbol{\Gamma}\nabla\chi^\nu\cdot\nabla\zeta\ dxdt \tag{38}$$
$$-\int_{\Omega^T} [\Psi'(\chi^\nu) - \overline{\boldsymbol{\varepsilon}}'(\chi^\nu)\cdot\mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^\nu) - \overline{\boldsymbol{\varepsilon}}(\chi^\nu)) + \beta\chi^\nu_t]\zeta\ dxdt = 0$$
$$\forall\ \zeta \in L_2(0,T;H^1(\Omega)).$$

Moreover, $(\mathbf{u}^\nu, \chi^\nu, \mu^\nu)$ *satisfy a priori estimates*

$$\begin{aligned} &\|\mathbf{u}^\nu\|_{L_\infty(0,T;\mathbf{V}_0)} + \|\mathbf{u}_t^\nu\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \\ &+\|\mathbf{u}_{tt}^\nu\|_{L_2(0,T;\mathbf{V}_0')} + \|\chi^\nu\|_{L_\infty(0,T;H^1(\Omega))} + \|\chi_t^\nu\|_{L_2(\Omega^T)} \\ &+\nu^{\frac{1}{2}}\|\mu^\nu\|_{L_\infty(0,T;L_2(\Omega))} + \|\mu^\nu\|_{L_2(0,T;H^1(\Omega))} \le c \ne c(\nu) \end{aligned} \tag{39}$$

with constant c depending only on the data.

The solutions to problem (P) arise as limits of solutions to problems $(P)^\nu$.

Theorem 3.2 *Let the assumptions of Theorem 3.1 be satisfied. Then there exists a triple* $(\mathbf{u}, \chi, \mu)$ *with*

(i) $\mathbf{u} \in L_\infty(0,T;\mathbf{V}_0)$, $\mathbf{u}_t \in L_\infty(0,T;\mathbf{L}_2(\Omega))$, $\mathbf{u}_{tt} \in L_2(0,T;\mathbf{V}_0')$, $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}_t(0) = \mathbf{u}_1$,
(ii) $\chi \in L_\infty(0,T;H^1(\Omega))$, $\chi_t \in L_2(\Omega^T)$, $\chi(0) = \chi_0$,
(iii) $\mu \in L_2(0,T;H^1(\Omega))$,

which for a subsequence $\nu \to 0$ *is a limit of solutions* $(\mathbf{u}^\nu, \chi^\nu, \mu^\nu)$ *to problem* $(P)^\nu$*, and* $(\mathbf{u}, \chi, \mu)$ *satisfy* (P) *in the following weak sense:*

$$\begin{aligned} &\int_0^T < \mathbf{u}_{tt}, \boldsymbol{\eta} >_{\mathbf{V}_0',\mathbf{V}_0} dt + \int_{\Omega^T} \mathbf{A}(\underline{\boldsymbol{\varepsilon}}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, dxdt \\ &= \int_{\Omega^T} \mathbf{b} \cdot \boldsymbol{\eta}\, dxdt \quad \forall\, \boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_0), \end{aligned} \tag{40}$$

$$\begin{aligned} &\int_{\Omega^T} [\chi_t \xi + (\mathbf{M}(\chi)\nabla\mu + \mathbf{h}\chi_t) \cdot \nabla\xi]\, dxdt = 0 \\ &\forall\, \xi \in L_2(0,T;H^1(\Omega)), \end{aligned} \tag{41}$$

$$\begin{aligned} &\int_{\Omega^T} (\mu - \mathbf{g} \cdot \nabla\mu)\zeta\, dxdt - \int_{\Omega^T} \boldsymbol{\Gamma}\nabla\chi \cdot \nabla\zeta\, dxdt \\ &- \int_{\Omega^T} [\Psi'(\chi) - \overline{\boldsymbol{\varepsilon}}'(\chi) \cdot \mathbf{A}(\boldsymbol{\varepsilon}(u) - \overline{\boldsymbol{\varepsilon}}(\chi)) + \beta\chi_t]\zeta\, dxdt = 0 \\ &\forall\, \zeta \in L_2(0,T;H^1(\Omega)). \end{aligned} \tag{42}$$

Moreover, the following a priori estimates hold

$$\begin{aligned} &\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{V}_0)} + \|\mathbf{u}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \\ &+\|\mathbf{u}_{tt}\|_{L_2(0,T;\mathbf{V}_0')} + \|\chi\|_{L_\infty(0,T;H^1(\Omega))} + \|\chi_t\|_{L_2(\Omega^T)} \\ &+\|\mu\|_{L_2(0,T;H^1(\Omega))} \le c \end{aligned} \tag{43}$$

with constant c depending only on the data.

The next result concerns the special case of problem (P) with $\mathbf{g} = \mathbf{h} = \mathbf{0}$ and $\beta = 0$ which corresponds to the standard Cahn-Hilliard system coupled with elasticity.

THEOREM 3.3 *Let the assumptions (A1)÷(A6), (A7)′, (A8) be satisfied, and in addition tensors* $\mathbf{\Gamma}$ *and* $\mathbf{A}$ *be constant. Then there exist functions* $(\mathbf{u}, \chi, \mu)$ *such that*

(i) $\mathbf{u} \in L_\infty(0,T;\mathbf{V}_0)$, $\mathbf{u}_t \in L_\infty(0,T;\mathbf{L}_2(\Omega))$, $\mathbf{u}_{tt} \in L_2(0,T;\mathbf{V}_0')$, $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}_t(0) = \mathbf{u}_1$,

(ii) $\chi \in L_\infty(0,T;H^1(\Omega))$, $\chi_t \in L_2(0,T;V')$, $\chi(0) = \chi_0$,

(iii) $\mu \in L_2(0,T;H^1(\Omega))$,

which satisfy (P) *in the sense of identities (40), (42) (with* $\mathbf{g} = \mathbf{0}$*) and (41) replaced by*

$$\int_0^T < \chi_t, \xi >_{V',V} \, dt + \int_{\Omega^T} \mathbf{M}(\chi)\nabla\mu \cdot \nabla\xi \, dxdt = 0 \quad \forall\, \xi \in L_2(0,T;H^1(\Omega)). \tag{44}$$

Moreover, $(\mathbf{u}, \chi, \mu)$ *satisfy a priori estimates*

$$\begin{aligned} &\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{V}_0)} + \|\mathbf{u}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{u}_{tt}\|_{L_2(0,T;\mathbf{V}_0')} \\ &+ \|\chi\|_{L_\infty(0,T;H^1(\Omega))} + \|\chi_t\|_{L_2(0,T;V')} + \|\mu\|_{L_2(0,T;H^1(\Omega))} \le c \end{aligned} \tag{45}$$

with constant c depending only on the data.

REMARK 3.1 *As is common, we can also introduce modified weak formulations of problems* $(P)^\nu$ *and* (P) *with the identity (36) corresponding to the elasticity system replaced by*

$$\begin{aligned} &-\int_{\Omega^T} \mathbf{u}_t^\nu \cdot \boldsymbol{\eta}_t \, dxdt + \int_{\Omega^T} \mathbf{A}(\underline{\boldsymbol{\varepsilon}}(\mathbf{u}^\nu) - \overline{\boldsymbol{\varepsilon}}(\chi^\nu)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, dxdt \\ &= \int_{\Omega^T} \mathbf{b} \cdot \boldsymbol{\eta} \, dxdt + \int_\Omega \mathbf{u}_1 \cdot \boldsymbol{\eta}(0) \, dx \quad \forall\, \boldsymbol{\eta} \in C^1([0,T];\mathbf{V}_0) \text{ with } \boldsymbol{\eta}(T) = 0, \end{aligned} \tag{46}$$

and the analogous modification of (40). By virtue of the identity

$$\begin{aligned} &\int_0^T < \phi_t, \eta >_{V',V} \, dt = -\int_0^T (\phi, \eta_t)_{L_2(\Omega)} \, dt - (\phi_0, \eta(0))_{L_2(\Omega)} \\ &\forall\, \phi \in L_2(0,T;V) \cap H^1(0,T;V') \text{ with } \phi(0) = \phi_0, \text{ and} \\ &\eta \in L_2(0,T;V) \cap H^1(0,T;L_2(\Omega)) \text{ with } \eta(T) = 0, \end{aligned}$$

the existence results of Theorems 3.1-3.3 hold true for the above mentioned modified formulations.

The last result concerns the existence of weak solutions to the quasi-stationary version of problem (P). We consider a time re-scaled problem $(P)^\alpha$, $\alpha \in (0,1]$ with the term $\mathbf{u}_{tt}$ replaced by $\alpha\mathbf{u}_{tt}$. For simplicity we confine ourselves to the situation of Theorem 3.3, i.e., $\mathbf{g} = \mathbf{h} = \mathbf{0}, \beta = 0$ and constant tensors $\mathbf{\Gamma}, \mathbf{A}$.

Let $(\mathbf{u}^\alpha, \chi^\alpha, \mu^\alpha)$ denote a weak solution to $(P)^\alpha$ in the sense of Theorem 3.3 with the modification in Remark 3.1, satisfying the identities

$$-\alpha \int_{\Omega^T} \mathbf{u}_t^\alpha \cdot \boldsymbol{\eta}_t \, dxdt + \int_{\Omega^T} \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^\alpha) - \overline{\boldsymbol{\varepsilon}}(\chi^\alpha)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, dxdt \tag{47}$$
$$= \int_{\Omega^T} \mathbf{b} \cdot \boldsymbol{\eta} \, dxdt + \alpha \int_\Omega \mathbf{u}_1 \cdot \boldsymbol{\eta}(0) \, dx \quad \forall \, \boldsymbol{\eta} \in C^1([0,T]; \mathbf{V}_0) \text{ with } \boldsymbol{\eta}(T) = 0,$$

$$\int_0^T < \chi_t^\alpha, \xi >_{V',V} \, dt + \int_{\Omega^T} \mathbf{M}(\chi^\alpha) \nabla \mu^\alpha \cdot \nabla \xi \, dxdt = 0 \tag{48}$$
$$\forall \, \xi \in L_2(0,T; H^1(\Omega)),$$

$$\int_{\Omega^T} \mu^\alpha \zeta \, dxdt - \int_{\Omega^T} \boldsymbol{\Gamma} \nabla \chi^\alpha \cdot \nabla \zeta \, dxdt \tag{49}$$
$$- \int_{\Omega^T} [\Psi'(\chi^\alpha) - \overline{\boldsymbol{\varepsilon}}'(\chi^\alpha) \cdot \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^\alpha) - \overline{\boldsymbol{\varepsilon}}(\chi^\alpha))] \zeta \, dxdt = 0$$
$$\forall \, \zeta \in L_2(0,T; H^1(\Omega)).$$

An inspection of the proof of Theorem 3.3 shows that the following estimate uniform in α holds true

$$\|\mathbf{u}^\alpha\|_{L_\infty(0,T;\mathbf{V}_0)} + \alpha^{\frac{1}{2}} \|\mathbf{u}_t^\alpha\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \tag{50}$$
$$+ \alpha \|\mathbf{u}_{tt}^\alpha\|_{L_2(0,T;\mathbf{V}_0')} + \|\chi^\alpha\|_{L_\infty(0,T;H^1(\Omega))} + \|\chi_t^\alpha\|_{L_2(0,T;V')}$$
$$+ \|\mu^\alpha\|_{L_2(0,T;H^1(\Omega))} \le c \neq c(\alpha)$$

with constant c depending only on the data. Due to this estimate we can pass to the limit $\alpha \to 0$ in (47)-(49) to obtain

THEOREM 3.4 *Let the assumptions of Theorem 3.3 be satisfied. Then there exists a triple* $(\mathbf{u}, \chi, \mu)$ *with*
(i) $\mathbf{u} \in L_\infty(0,T; \mathbf{V}_0)$,
(ii) $\chi \in L_\infty(0,T; H^1(\Omega))$, $\chi_t \in L_2(0,T; V')$, $\chi(0) = \chi_0$,
(iii) $\mu \in L_2(0,T; H^1(\Omega))$,
which for a subsequence $\alpha \to 0$ *is a limit of solutions* $(\mathbf{u}^\alpha, \chi^\alpha, \mu^\alpha)$ *to problem* $(P)^\alpha$*, and* $(\mathbf{u}, \chi, \mu)$ *satisfy the quasi-stationary version of* (P) *in the sense of identities*

$$\int_{\Omega^T} \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, dxdt = \int_{\Omega^T} \mathbf{b} \cdot \boldsymbol{\eta} \, dxdt \quad \forall \, \boldsymbol{\eta} \in L_2(0,T; \mathbf{V}_0), \tag{51}$$

together with (44) and (42) (with $\mathbf{g} = \mathbf{0}$*,* $\beta = 0$*). Moreover,* $(\mathbf{u}, \chi, \mu)$ *satisfy estimates*

$$\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{V}_0)} + \|\chi\|_{L_\infty(0,T;H^1(\Omega))} + \|\chi_t\|_{L_2(0,T;V')} + \|\mu\|_{L_2(0,T;H^1(\Omega))} \le c \tag{52}$$

with constant c *depending only on the data.*

The above result coincides with that obtained in Garcke (2000, 2003) where more general problem with multicomponent order parameter has been considered.

REMARK 3.2 *As mentioned in the introduction, problem (P) can be considered with other boundary conditions for the displacement. In particular, the presented above existence results extend directly to the following boundary conditions*

$$\begin{aligned} &\mathbf{u} = \mathbf{0} \ \textit{on} \ S_D^T = S_D \times (0,T), \\ &W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi) = \mathbf{0} \ \textit{on} \ S_N^T = S_N \times (0,T), \end{aligned} \tag{53}$$

where $S = \overline{S}_D \cup \overline{S}_N$, $S_D \cap S_N = \emptyset$ *and meas* $S_D > 0$, *i.e.* S_D *and* S_N *are disjoint parts of the boundary* S *on which zero displacement and zero traction boundary conditions are prescribed.*
In such a case the procedure of getting energy estimates is the same as presented in Section 2 (see estimate (26). In fact, the boundary conditions (53), (4)$_3$ *and (5)*$_3$ *are consistent with the requirement (27) so that as in case (3)*$_3$ *the boundary integrals do not enter the weak formulations.*
The condition meas $S_D > 0$ *assures the validity of Korn's inequality which is used in Section 4.3. All other arguments remain unchanged.*

4. The Faedo-Galerkin approximation $(P)^{\nu,m}$

4.1. Approximate problems

Let $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathbf{V}_0$ and $\{z_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $H^1(\Omega)$. Without loss of generality we assume that $z_1 = 1$. Further, for $m \in \mathbb{N}$, we set

$$\begin{aligned} &\mathbf{V}_m = span\{\mathbf{v}_1, ..., \mathbf{v}_m\}, \ V_m = span\{z_1, ..., z_m\}, \\ &\mathbf{V}_\infty = \bigcup_{m=1}^{\infty} \mathbf{V}_m, \ V_\infty = \bigcup_{m=1}^{\infty} V_m. \end{aligned}$$

First we introduce the Faedo-Galerkin approximation of (P):

Problem $(P)^m$. For any $m \in \mathbb{N}$ find a triple of functions $(\mathbf{u}^m, \chi^m, \mu^m)$ of the form

$$\begin{aligned} &\mathbf{u}^m(\mathbf{x},t) = \sum_{i=1}^{m} e_i^m(t)\mathbf{v}_i(\mathbf{x}), \\ &\chi^m(\mathbf{x},t) = \sum_{i=1}^{m} c_i^m(t) z_i(\mathbf{x}), \\ &\mu^m(\mathbf{x},t) = \sum_{i=1}^{m} d_i^m(t) z_i(\mathbf{x}) \end{aligned} \tag{54}$$

satisfying for a.e. $t \in [0,T]$:

$$(\mathbf{u}_{tt}^m, \boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)} + (W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}^m), \chi^m), \boldsymbol{\varepsilon}(\boldsymbol{\eta}^m))_{\mathbf{L}_2(\Omega)} \\ = (\mathbf{b}, \boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)} \quad \forall\, \boldsymbol{\eta}^m \in \mathbf{V}_m, \tag{55}$$

$$(\chi_t^m, \xi^m)_{L_2(\Omega)} + (\mathbf{M}(\chi^m)\nabla\mu^m + \mathbf{h}\chi_t^m, \nabla\xi^m)_{\mathbf{L}_2(\Omega)} = 0 \quad \forall\, \xi^m \in V_m, \tag{56}$$

$$(\mu^m - \mathbf{g}\cdot\nabla\mu^m, \zeta^m)_{L_2(\Omega)} - (\boldsymbol{\Gamma}(\chi^m)\nabla\chi^m, \nabla\zeta^m)_{\mathbf{L}_2(\Omega)} \\ -(\frac{1}{2}\nabla\chi^m\cdot\boldsymbol{\Gamma}'(\chi^m)\nabla\chi^m + \Psi'(\chi^m) \\ +W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}^m), \chi^m) + \beta\chi_t^m, \zeta^m)_{L_2(\Omega)} = 0 \quad \forall\, \zeta^m \in V_m, \tag{57}$$

$$\mathbf{u}^m(0) = \mathbf{u}_0^m,\ \mathbf{u}_t^m(0) = \mathbf{u}_1^m,\ \chi^m(0) = \chi_0^m \tag{58}$$

where $\mathbf{u}_0^m$, $\mathbf{u}_1^m \in \mathbf{V}_m$, $\chi_0^m \in V_m$ satisfy for $m \to \infty$

$$\mathbf{u}_0^m \to \mathbf{u}_0 \text{ strongly in } \mathbf{V}_0, \\ \mathbf{u}_1^m \to \mathbf{u}_1 \text{ strongly in } \mathbf{L}_2(\Omega), \\ \chi_0^m \to \chi_0 \text{ strongly in } H^1(\Omega). \tag{59}$$

Similarly, we introduce the Faedo-Galerkin approximation of $(P)^\nu$.

Problem $(P)^{\nu,m}$. For any $\nu \in (0,1]$, $m \in \mathbb{N}$, find a triple of functions $(\mathbf{u}^{\nu,m}, \chi^{\nu,m}, \mu^{\nu,m})$ of the form

$$\mathbf{u}^{\nu,m}(\mathbf{x},t) = \sum_{i=1}^m e_i^{\nu,m}(t)\mathbf{v}_i(\mathbf{x}), \\ \chi^{\nu,m}(\mathbf{x},t) = \sum_{i=1}^m c_i^{\nu,m}(t) z_i(\mathbf{x}), \\ \mu^{\nu,m}(\mathbf{x},t) = \sum_{i=1}^m d_i^{v,m}(t) z_i(\mathbf{x}) \tag{60}$$

satisfying for a.e. $t \in [0,T]$:

$$(\mathbf{u}_{tt}^{\nu,m}, \boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)} + (W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}), \chi^{\nu,m}), \boldsymbol{\varepsilon}(\boldsymbol{\eta}^m))_{\mathbf{L}_2(\Omega)} \\ = (\mathbf{b}, \boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)} \quad \forall\, \boldsymbol{\eta}^m \in \mathbf{V}_m, \tag{61}$$

$$\nu(\mu_t^{\nu,m}, \xi^m)_{L_2(\Omega)} + (\chi_t^{\nu,m}, \xi^m)_{L_2(\Omega)} \\ +(\mathbf{M}(\chi^{\nu,m})\nabla\mu^{\nu,m} + \mathbf{h}\chi_t^{\nu,m}, \nabla\xi^m)_{\mathbf{L}_2(\Omega)} = 0 \quad \forall\, \xi^m \in V_m, \tag{62}$$

$$(\mu^{\nu,m} - \mathbf{g}\cdot\nabla\mu^{\nu,m}, \zeta^m)_{L_2(\Omega)} - (\boldsymbol{\Gamma}(\chi^{\nu,m})\nabla\chi^{\nu,m}, \nabla\zeta^m)_{\mathbf{L}_2(\Omega)} \\ -(\frac{1}{2}\nabla\chi^{\nu,m}\cdot\boldsymbol{\Gamma}'(\chi^{\nu,m})\nabla\chi^{\nu,m} + \Psi'(\chi^{\nu,m}) \\ +W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}), \chi^{\nu,m}) + \beta\chi_t^{\nu,m}, \zeta^m)_{L_2(\Omega)} = 0 \quad \forall\, \zeta^m \in V_m, \tag{63}$$

$$\mathbf{u}^{\nu,m}(0) = \mathbf{u}_0^m, \quad \mathbf{u}_t^{\nu,m}(0) = \mathbf{u}_1^m, \quad \chi^{\nu,m} = \chi_0^m, \quad \mu^{\nu,m} = \mu_0^m \tag{64}$$

where $\mathbf{u}_0^m, \mathbf{u}_1^m \in \mathbf{V}_m$, $\chi_0^m \in V_m$ satisfy (59), and $\mu_0^m \in V_m$ are such that

$$\mu_0^m \to \mu_0 \text{ strongly in } L_2(\Omega) \textit{ as } m \to \infty. \tag{65}$$

4.2. Existence of solutions to $(P)^{\nu,m}$

We prove first the local in time existence.

LEMMA 4.1 *Assume that $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi)$, $W_{,\chi}(\boldsymbol{\varepsilon},\chi)$, $\mathbf{M}(\chi)$, $\boldsymbol{\Gamma}(\chi)$, $\boldsymbol{\Gamma}'(\chi)$, $\Psi'(\chi)$ are Lipschitz continuous functions of their arguments, and $\beta = const > 0$. Then problem $(P)^{\nu,m}$ has a unique solution $(\mathbf{u}^{\nu,m}, \chi^{\nu,m}, \mu^{\nu,m})$ on a time interval $[0, T^{\nu,m}]$ with $T^{\nu,m} > 0$ depending on ν, m.*

Proof. For simplicity we omit the upper indices ν, m, writing $(\mathbf{u}, \chi, \mu) = (\mathbf{u}^{\nu,m}, \chi^{\nu,m}, \mu^{\nu,m})$. Moreover, we denote by $\mathbf{e}$, $\mathbf{c}$, $\mathbf{d}$ the vectors $\mathbf{e} = (e_1^{\nu,m}, ..., e_m^{\nu,m})$, $\mathbf{c} = (c_1^{\nu,m}, ..., c_m^{\nu,m})$, $\mathbf{d} = (d_1^{\nu,m}, ..., d_m^{\nu,m})$.

From (61)-(64) we obtain an initial value problem for the system of ODE's for $\mathbf{e}$, $\mathbf{c}$, $\mathbf{d}$:

$$\begin{aligned}
&\partial_t^2 e_j = -\int_\Omega W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\sum_{i=1}^m e_i \mathbf{v}_i), \sum_{i=1}^m c_i z_i) \cdot \boldsymbol{\varepsilon}(\mathbf{v}_j)\, dx \qquad (66)\\
&\quad + \int_\Omega \mathbf{b} \cdot \mathbf{v}_j\, dx \equiv F_j^1(\mathbf{e}, \mathbf{c}),\\
&\nu \partial_t d_j = -\partial_t c_j - \sum_{i=1}^m \partial_t c_i \int_\Omega z_i \mathbf{h} \cdot \nabla z_j\, dx \qquad (67)\\
&\quad - \sum_{i=1}^m d_i \int_\Omega (\mathbf{M}(\sum_{k=1}^m c_k z_k) \nabla z_i) \cdot \nabla z_j\, dx \equiv F_j^2(\mathbf{c}, \partial_t \mathbf{c}, \mathbf{d}),\\
&\beta \partial_t c_j = d_j - \sum_{i=1}^m d_i \int_\Omega \mathbf{g} \cdot \nabla z_i z_j\, dx \qquad (68)\\
&\quad - \sum_{i=1}^m c_i \int_\Omega (\boldsymbol{\Gamma}(\sum_{k=1}^m c_k z_k) \nabla z_i) \cdot \nabla z_j\, dx\\
&\quad - \frac{1}{2} \int_\Omega (\sum_{i=1}^m c_i \nabla z_i) \cdot \boldsymbol{\Gamma}'(\sum_{k=1}^m c_k z_k)(\sum_{i=1}^m c_i \nabla z_i) z_j\, dx - \int_\Omega \Psi'(\sum_{i=1}^m c_i z_i) z_j\, dx\\
&\quad - \int_\Omega W_{,\chi}(\boldsymbol{\varepsilon}(\sum_{i=1}^m e_i \mathbf{v}_i), \sum_{i=1}^m c_i z_i) z_j\, dx \equiv F_j^3(\mathbf{e}, \mathbf{c}, \mathbf{d}),\\
&e_j(0) = (\mathbf{u}_0^m, \mathbf{v}_j)_{\mathbf{L}_2(\Omega)},\ \partial_t e_j(0) = (\mathbf{u}_1^m, \mathbf{v}_j)_{\mathbf{L}_2(\Omega)}, \qquad (69)\\
&c_j(0) = (\chi_0^m, z_j)_{L_2(\Omega)},\ d_j(0) = (\mu_0^m, z_j)_{L_2(\Omega)},
\end{aligned}$$

which has to hold for $j = 1, ..., m$.

Substituting $\partial_t c_j$ from (68) into (67) ($\beta > 0$) and introducing vector $\mathbf{f} = (\partial_t e_1, ..., \partial_t e_m)$, we rewrite (66)-(69) in the form of the Cauchy problem for the first order system:

$$\begin{aligned}
&\partial_t e_j = f_j, \qquad (70)\\
&\partial_t f_j = F_j^1(\mathbf{e}, \mathbf{c}),\\
&\nu \partial_t d_j = \frac{1}{\beta} F_j^3(\mathbf{e}, \mathbf{c}, \mathbf{d}) - \frac{1}{\beta} \sum_{i=1}^m F_i^3(\mathbf{e}, \mathbf{c}, \mathbf{d}) \int_\Omega z_i \mathbf{h} \cdot \nabla z_j \, dx\\
&\quad - \sum_{i=1}^m d_i \int_\Omega (\mathbf{M}(\sum_{k=1}^m c_k z_k) \nabla z_i) \cdot \nabla z_j \, dx \equiv F_j^4(\mathbf{e}, \mathbf{c}, \mathbf{d}),\\
&\beta \partial_t c_j = F_j^3(\mathbf{e}, \mathbf{c}, \mathbf{d}),\\
&e_j(0) = (\mathbf{u}_0^m, \mathbf{v}_j)_{\mathbf{L}_2(\Omega)}, \ f_j(0) = (\mathbf{u}_1^m, \mathbf{v}_j)_{\mathbf{L}_2(\Omega)},\\
&d_j(0) = (\mu_0^m, z_j)_{L_2(\Omega)}, \ c_j(0) = (\chi_0^m, z_j)_{L_2(\Omega)},
\end{aligned}$$

for $j = 1, ..., m$.

By assumption the right-hand sides F_j^1, F_j^4, F_j^3 are Lipschitz continuous functions with respect to $\mathbf{e}$, $\mathbf{c}$, $\mathbf{d}$. Hence, by virtue of the Cauchy theorem for ODE's it follows that the initial value problem (70) has a unique solution $(\mathbf{e}, \mathbf{f}, \mathbf{c}, \mathbf{d})$ on an interval $[0, T^{\nu,m}]$. In view of the representation (60) this gives a solution to problem $(P)^{\nu,m}$. ■

LEMMA 4.2 *Assume that the free energy $f(\boldsymbol{\varepsilon}, \chi, \nabla\chi)$ satisfies structure condition (33), and the coefficients matrix $\mathbf{B}(\chi)$ satisfies structure condition (34). Moreover, assume that the data comply with (A8) so that the bound (35) holds true. Then there exists a solution $(\mathbf{u}^{\nu,m}, \chi^{\nu.m}, \mu^{\nu,m})$ to problem $(P)^{\nu,m}$ on the interval $[0, T]$, satisfying the following energy estimates:*

$$\begin{aligned}
&\|\mathbf{u}_t^{\nu,m}\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \nu^{\frac{1}{2}} \|\mu^{\nu,m}\|_{L_\infty(0,T;L_2(\Omega))} \qquad (71)\\
&+\|\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}\\
&+\|\chi^{\nu,m}\|_{L_\infty(0,T;L_r(\Omega))} + \|\nabla\chi^{\nu,m}\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}\\
&+\|\nabla\mu^{\nu,m}\|_{\mathbf{L}_2(\Omega^T)} + \|\chi_t^{\nu,m}\|_{L_2(\Omega^T)} \le c \ne c(\nu, m),
\end{aligned}$$

where constant c depends only on the data $\mathbf{u}_0$, $\mathbf{u}_1$, χ_0, μ_0, b, and on the constants in the structure conditions.

Proof. We derive the energy identity for solutions of $(P)^{\nu,m}$. To this purpose we follow the idea presented in Section 2 (see estimates (24) and (26)).
For simplicity we write $(\mathbf{u}, \chi, \mu)$=$(\mathbf{u}^{\nu,m}, \chi^{\nu,m}, \mu^{\nu,m})$. For $t \in (0, T^{\nu,m}]$, setting in (61)-(63) $\boldsymbol{\eta}^m = \mathbf{u}_t$, $\xi^m = \mu$, $\zeta^m = -\chi_t$ as test functions, we get

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_t\|^2_{\mathbf{L}_2(\Omega)} + (W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}), \chi), \boldsymbol{\varepsilon}(\mathbf{u}_t))_{\mathbf{L}_2(\Omega)} = (\mathbf{b}, \mathbf{u}_t)_{\mathbf{L}_2(\Omega)},$$

$$\frac{\nu}{2}\frac{d}{dt}\|\mu\|^2_{L_2(\Omega)} + (\chi_t,\mu)_{L_2(\Omega)} + (\mathbf{M}(\chi)\nabla\mu + \mathbf{h}\chi_t, \nabla\mu)_{\mathbf{L}_2(\Omega)} = 0,$$

$$\beta\|\chi_t\|^2_{L_2(\Omega)} + \frac{d}{dt}\int_\Omega [\frac{1}{2}\nabla\chi\cdot\mathbf{\Gamma}(\chi)\nabla\chi + \Psi(\chi)]\ dx + (W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi),\chi_t)_{L_2(\Omega)}$$
$$-(\mu - \mathbf{g}\cdot\nabla\mu, \chi_t)_{L_2(\Omega)} = 0.$$

Summing up the above equations, noting that the terms $(\chi_t,\mu)_{L_2(\Omega)}$ cancel out, we arrive at the energy identity

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2_{\mathbf{L}_2(\Omega)} + \frac{\nu}{2}\frac{d}{dt}\|\mu\|^2_{L_2(\Omega)} \tag{72}$$
$$+\frac{d}{dt}\int_\Omega [W(\boldsymbol{\varepsilon}(\mathbf{u},\chi) + \Psi(\chi) + \frac{1}{2}\nabla\chi\cdot\mathbf{\Gamma}(\chi)\nabla(\chi)]\ dx$$
$$+\int_\Omega \begin{bmatrix}\nabla\mu\\ \chi_t\end{bmatrix}\cdot\begin{bmatrix}\mathbf{M} & \mathbf{h}\\ \mathbf{g}^T & \beta\end{bmatrix}\begin{bmatrix}\nabla\mu\\ \chi_t\end{bmatrix} dx = \int_\Omega \mathbf{b}\cdot\mathbf{u}_t\ dx.$$

Integration of (72) over $(0,t)$ for $t\in(0,T^{\nu,m}]$ gives

$$\frac{1}{2}\|\mathbf{u}_t(t)\|^2_{\mathbf{L}_2(\Omega)} + \frac{\nu}{2}\|\mu(t)\|^2_{L_2(\Omega)} + \int_\Omega f(\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{t})),\chi(t),\nabla\chi(t))\ dx \tag{73}$$
$$+\int_{\Omega^t}\mathbf{X}\cdot\mathbf{B}(\chi)\mathbf{X}\ dxdt'$$
$$= \frac{1}{2}\|\mathbf{u}_1^m\|^2_{\mathbf{L}_2(\Omega)} + \frac{\nu}{2}\|\mu_0^m\|^2_{L_2(\Omega)} + \int_\Omega (f(\boldsymbol{\varepsilon}(\mathbf{u_0^m}),\chi_0^m,\nabla\chi_0^m))\ dx$$
$$+\int_{\Omega^t}\mathbf{b}\cdot\mathbf{u}_t\ dxdt'.$$

The left-hand side of (73) is bounded from below in view of structure conditions (33) and (34). By convergences (59), (65) and estimate (35), the sum of the first three terms on the right-hand side of (73) is bounded from above by a constant depending on $\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}$, $\|\mathbf{u}_1\|_{\mathbf{L}_2(\Omega)}$, $\|\chi_0\|_{H^1(\Omega)}$, $\|\mu_0\|_{L_2(\Omega)}$. The last term is estimated with the help of Young's inequality by

$$\left|\int_{\Omega^t}\mathbf{b}\cdot\mathbf{u}_{t'}\ dxdt'\right| \le \frac{1}{4}\|\mathbf{u}_{t'}\|^2_{L_\infty(0,t;\mathbf{L}_2(\Omega))} + \|\mathbf{b}\|^2_{L_1(0,t;\mathbf{L}_2(\Omega))},$$

so that the term with $\mathbf{u}_t$ is absorbed by the left-hand side of (73). Consequently, we arrive at the estimate

$$\frac{1}{4}\|\mathbf{u}_t(t)\|^2_{\mathbf{L}_2(\Omega)} + \frac{\nu}{2}\|\mu(t)\|^2_{L_2(\Omega)} \tag{74}$$
$$+c(\|\boldsymbol{\varepsilon}(t)\|^2_{\mathbf{L}_2(\Omega)} + \|\chi(t)\|^r_{L_r(\Omega)} + \|\nabla\chi\|^2_{\mathbf{L}_2(\Omega)}) - c$$
$$+\underline{c}^*_M\|\nabla\mu\|^2_{\mathbf{L}_2(\Omega^t)} + \underline{c}^*_\beta\|\chi_{t'}\|^2_{L_2(\Omega^t)} \le c \ne c(\nu,m) \quad \text{for } t\in(0,T^{\nu,m}]$$

with constant c depending only on the data $\mathbf{u}_0$, $\mathbf{u}_1$, χ_0, μ_0 and $\mathbf{b}$.

From (74) it follows that for fixed $\nu \in (0,1]$, the functions $\mathbf{e}$, $\partial_t\mathbf{e}$, $\mathbf{c}$ and $\mathbf{d}$ are uniformly bounded with respect to m over the interval $[0, T^{\nu,m}]$. This means that $\mathbf{u}^{\nu,m}(T^{\nu,m})$, $\mathbf{u}_t^{\nu,m}(T^{\nu,m})$, $\chi^{\nu,m}(T^{\nu,m})$, $\mu^{\nu,m}(T^{\nu,m})$ can be taken as the initial conditions for the next time interval. In such a way the solution of problem $(P)^{\nu,m}$ can be extended onto the time interval $[0,T]$ in a finite number of steps. Hence, estimate (74) holds true on $[0,T]$. Thereby the assertion is proved. ■

4.3. Additional estimates for $(P)^{\nu,m}$

First, we shall note that by virtue of Korn's inequality (see e.g. Duvaut and Lions, 1972, Chapter III, Theorem 3.3) it follows from (71) that

$$\|\mathbf{u}^{\nu,m}\|_{L_\infty(0,T;\mathbf{V}_0)} \leq c\|\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \leq c \neq c(\nu,m). \tag{75}$$

Further, (71) implies that

$$\|\chi^{\nu,m}\|_{L_\infty(0,T;H^1(\Omega))} \leq c \neq c(\nu,m). \tag{76}$$

Hence, by Sobolev's imbedding,

$$\|\chi^{\nu,m}\|_{L_\infty(0,T;L_{q_n}(\Omega))} \leq c \neq c(\nu,m). \tag{77}$$

We shall prove now an additional estimate on $\mu^{\nu,m}$.

LEMMA 4.3 *Let the assumptions of Lemma 4.2 be satisfied. Moreover assume that $W_{,\chi}(\boldsymbol{\varepsilon},\chi)$ satisfies growth condition (31), $\Psi'(\chi)$ satisfies*

$$|\Psi'(\chi)| \leq c(|\chi|^{q_n} + 1)$$

with some constant $c > 0$, tensor $\mathbf{\Gamma}'(\chi)$ satisfies the uniform in χ bound (A5)(iv), and vector $\mathbf{g}$ is constant. Then

$$\|\mu^{\nu,m}\|_{L_2(0,T;H^1(\Omega))} \leq c \neq c(\nu,m), \tag{78}$$

with constant c depending only on the data, Ω and time T.

Proof. For simplicity we write $(\mathbf{u},\chi,\mu) = (\mathbf{u}^{\nu,m},\chi^{\nu,m},\mu^{\nu,m})$. Firstly we note that by virtue of the Poincaré inequality,

$$\int_\Omega |\mu|^2\, dx \leq c\int_\Omega |\nabla\mu|^2\, dx + |\int_\Omega \mu\, dx|^2, \tag{79}$$

with constant c depending only on Ω. Hence,

$$\int_{\Omega^T} |\mu|^2\, dxdt \leq c\int_{\Omega^T} |\nabla\mu|^2\, dxdt + \int_0^T \left|\int_\Omega \mu\, dx\right|^2 dt. \tag{80}$$

The first term on the right-hand side of (80) is bounded due to energy estimate (71). We estimate the second term in (80). Setting $\zeta^m = 1$ in (63) (admissible by assumption) it follows that

$$\int_\Omega \mu\ dx = \int_\Omega \mathbf{g}\cdot\nabla\mu\ dx + \frac{1}{2}\int_\Omega \nabla\chi\cdot\mathbf{\Gamma}'(\chi)\nabla\chi\ dx + \int_\Omega \Psi'(\chi)\ dx + \int_\Omega W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)\ dx + \beta\int_\Omega \chi_t\ dx.$$

Hence,

$$\begin{aligned}\int_0^T |\int_\Omega \mu\ dx|^2\ dt \le c\Bigg[&\int_0^T |\int_\Omega \mathbf{g}\cdot\nabla\mu|^2\ dt\\ &+\int_0^T |\int_\Omega \nabla\chi\cdot\mathbf{\Gamma}'(\chi)\nabla\chi\ dx|^2\ dt\\ &+\int_0^T |\int_\Omega \Psi'(\chi)\ dx|^2\ dt \quad +\int_0^T |\int_\Omega W_{,\chi}(\boldsymbol{\varepsilon}(u),\chi)\ dx|^2\ dt\\ &+\beta^2\int_0^T |\int_\Omega \chi_t\ dx|^2\ dt\Bigg] \equiv R_1+R_2+R_3+R_4+R_5.\end{aligned}$$

In view of the assumptions, recalling estimates (71) and (77), we have

$$\begin{aligned}&R_1 \le c\|\nabla\mu\|^2_{\mathbf{L}_2(\Omega^T)} \le c,\\ &R_2 \le c_{\Gamma'}\int_0^T \|\nabla\chi\|^4_{\mathbf{L}_2(\Omega)}\ dt \le cT\|\nabla\chi\|^4_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \le c,\\ &R_3 \le cT\ ess\ sup_{t\in[0,T]}(\int_\Omega |\Psi'(\chi)|\ dx)^2\\ &\quad \le cT\ ess\ sup_{t\in[0,T]}(\|\chi(t)\|^{q_n}_{L_{q_n}(\Omega)}+1)^2 \le c,\\ &R_4 \le cT\ ess\ sup_{t\in[0,T]}(\int_\Omega |W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)|\ dx)^2\\ &\quad \le cT\ ess\ sup_{t\in[0,T]}(\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|^2_{\mathbf{L}_2(\Omega)}+\|\chi(t)\|^2_{L_2(\Omega)}+1)^2 \le c,\\ &R_5 \le \beta^2\|\chi_t\|^2_{L_2(\Omega^T)} \le c.\end{aligned}$$

Consequently,

$$\int_0^T |\int_\Omega \mu\ dx|^2\ dt \le c, \tag{81}$$

and, in view of (80),

$$\int_{\Omega^T} |\mu|^2\ dxdt \le c. \tag{82}$$

This yields estimate (78). ∎

Using standard duality arguments we shall estimate time derivative $\mathbf{u}_{tt}^{\nu,m}$.

LEMMA 4.4 *Let the assumptions of Lemma 4.2 hold true and $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi)$ satisfy growth condition (31). Then*

$$\|\mathbf{u}_{tt}^{\nu,m}\|_{L_2(0,T;\mathbf{V}_0')} \leq c \neq c(\nu,m). \tag{83}$$

Proof. We use in (61) the test function $\boldsymbol{\eta}^m = \mathbf{P}_m\boldsymbol{\eta}$ for $\boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_0)$, where $\mathbf{P}_m$ denotes the projection of $\mathbf{L}_2(\Omega)$ onto $\mathbf{V}_m$. Taking into account that

$$(\mathbf{v},\boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)} = (\mathbf{P}_m\mathbf{v},\boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)} \quad \forall\, \boldsymbol{\eta}^m \in \mathbf{V}_m,$$

writing for simplicity $(\mathbf{u},\chi,\mu) = (\mathbf{u}^{\nu,m},\chi^{\nu,m},\mu^{\nu,m})$, we get

$$\begin{aligned}
&|\int_{\Omega^T} \mathbf{u}_{tt}\cdot\boldsymbol{\eta}\, dxdt| = |\int_{\Omega^T} \mathbf{u}_{tt}\cdot\mathbf{P}_m\boldsymbol{\eta}\, dxdt| \qquad (84)\\
&= |-\int_{\Omega^T} W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)\cdot\boldsymbol{\varepsilon}(\mathbf{P}_m\boldsymbol{\eta})\, dxdt + \int_{\Omega^T} \mathbf{b}\cdot\mathbf{P}_m\boldsymbol{\eta}\, dxdt|\\
&\leq \|W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)\|_{\mathbf{L}_2(\Omega^T)}\|\nabla\mathbf{P}_m\boldsymbol{\eta}\|_{\mathbf{L}_2(\Omega^T)} + \|\mathbf{b}\|_{\mathbf{L}_2(\Omega^T)}\|\mathbf{P}_m\boldsymbol{\eta}\|_{\mathbf{L}_2(\Omega^T)}\\
&\leq c(\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{\mathbf{L}_2(\Omega^T)} + \|\chi\|_{L_2(\Omega^T)} + \|\mathbf{b}\|_{\mathbf{L}_2(\Omega^T)} + 1)\|\mathbf{P}_m\boldsymbol{\eta}\|_{L_2(0,T;\mathbf{V}_0)}\\
&\leq c\|\boldsymbol{\eta}\|_{L_2(0,T;\mathbf{V}_0)} \quad \forall\, \boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_0),
\end{aligned}$$

where in the last inequality we have used (75) and (77). This shows the assertion. ■

5. The Faedo-Galerkin approximation $(P)^m$ in the case of $\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = 0$

In this section we study the approximate problem $(P)^m$ in the special case $\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = 0$.

LEMMA 5.1 *Assume that $\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = 0$, and $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi)$, $W_{,\chi}(\boldsymbol{\varepsilon},\chi)$, $\mathbf{M}(\chi)$, $\boldsymbol{\Gamma}'(\chi)$, $\Psi'(\chi)$ are Lipschitz continuous functions of their arguments. Then, problem $(P)^m$ has a unique solution $(\mathbf{u}^m,\chi^m,\mu^m)$ on a time interval $[0,T^m]$ with $T^m > 0$ depending on m.*

Proof. We proceed as in Lemma 4.1. Let $\mathbf{e}$, $\mathbf{c}$, $\mathbf{d}$ denote the vectors $\mathbf{e} = (e_1^m,...,e_m^m)$, $\mathbf{c} = (c_1^m,...,c_m^m)$, $\mathbf{d} = (d_1^m,...,d_m^m)$. From (55)-(58) (with $\mathbf{g} = \mathbf{h} = \mathbf{0}$, $\beta = 0$) we obtain an initial value problem for the system of ODE's for $\mathbf{e}$, $\mathbf{c}$, $\mathbf{d}$ which includes (66) and

$$\partial_t c_j = -\sum_{i=1}^m d_i \int_\Omega (\mathbf{M}(\sum_{k=1}^m c_k z_k)\nabla z_i)\cdot\nabla z_j\, dx \equiv G_j^1(\mathbf{c},\mathbf{d}), \tag{85}$$

$$\begin{aligned} d_j = & \sum_{i=1}^m c_i \int_\Omega (\mathbf{\Gamma}(\sum_{k=1}^m c_k z_k)\nabla z_i) \cdot \nabla z_j \; dx \\ & + \frac{1}{2} \int_\Omega (\sum_{i=1}^m c_i \nabla z_i) \cdot \mathbf{\Gamma}'(\sum_{k=1}^m c_k z_k)(\sum_{i=1}^m c_i \nabla z_i) z_j \; dx \\ & + \int_\Omega \Psi'(\sum_{i=1}^m c_i z_i) z_j \; dx \\ & + \int_\Omega W_{,\chi}(\boldsymbol{\varepsilon}(\sum_{i=1}^m e_i \mathbf{v}_i), \sum_{i=1}^m c_i z_i) z_j \; dx \equiv G_j^2(\mathbf{e}, \mathbf{c}), \end{aligned} \tag{86}$$

$$e_j(0) = (\mathbf{u}_0^m, \mathbf{v}_j)_{\mathbf{L}_2(\Omega)}, \;\; \partial_t e_j(0) = (\mathbf{u}_1^m, \mathbf{v}_j)_{\mathbf{L}_2(\Omega)}, \;\; c_j(0) = (\chi_0^m, z_j)_{L_2(\Omega)}. \tag{87}$$

Substituting d_j from (86) into (85) and introducing vector $\mathbf{f} = (\partial_t e_1^m, ..., \partial_t e_m^m)$, we rewrite (66), (85)-(87) in the form of the Cauchy problem for the first order system:

$$\begin{aligned} & \partial_t e_j = f_j, \\ & \partial_t f_j = F_j^1(\mathbf{e}, \mathbf{c}), \\ & \partial_t c_j = -\sum_{i=1}^m G_i^2(\mathbf{e}, \mathbf{c}) \int_\Omega (\mathbf{M}(\sum_{k=1}^m c_k z_k)\nabla z_i) \cdot \nabla z_j \; dx \equiv G_j^3(\mathbf{e}, \mathbf{c}), \\ & e_j(0) = (\mathbf{u}_0^m, \mathbf{v}_j)_{\mathbf{L}_2(\Omega)}, f_j(0) = (\mathbf{u}_1^m, \mathbf{v}_j)_{\mathbf{L}_2(\Omega)}, c_j(0) = (\chi_0^m, z_j)_{L_2(\Omega)}, \end{aligned} \tag{88}$$

for $j = 1, ..., m$.

By assumption, the right-hand sides are Lipschitz continuous functions with respect to $\mathbf{e}$, $\mathbf{c}$. Hence, the initial value problem (88) has a unique solution on an interval $[0, T^m]$. This shows the assertion. ∎

By repeating the proof of Lemma 4.2 we can extend the local solution onto the whole interval $[0, T]$. As a result we have

LEMMA 5.2 *Let the assumptions of Lemma 5.1 be satisfied. Further, assume that the free energy satisfies structure condition (33), the matrix $\mathbf{M}(\chi)$ satisfies the uniform in χ positive definitness condition (A6)(iii), and the data comply with (A8) so that the bound (35) holds true. Then there exists a solution $(\mathbf{u}^m, \chi^m, \mu^m)$ to problem $(P)^m$ on the interval $[0, T]$, satisfying energy estimates*

$$\begin{aligned} & \|\mathbf{u}_t^m\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\boldsymbol{\varepsilon}(\mathbf{u}^m)\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\chi^m\|_{L_\infty(0,T;L_r(\Omega))} \\ & + \|\nabla \chi^m\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\nabla \mu^m\|_{\mathbf{L}_2(\Omega^T)} \le c \ne c(m) \end{aligned} \tag{89}$$

with constant c depending only on the data.

As in (75)-(77), we conclude from (89) the estimates

$$\|\mathbf{u}^m\|_{L_\infty(0,T;\mathbf{V}_0)} \leq c \neq c(m), \tag{90}$$

$$\|\chi^m\|_{L_\infty(0,T;H^1(\Omega))} + \|\chi^m\|_{L_\infty(0,T;L_{q_n}(\Omega))} \leq c \neq c(m). \tag{91}$$

The next lemma provides an additional estimate in $L_2(0,T;H^1(\Omega))$ - norm for μ^m.

LEMMA 5.3 *Let the assumptions of Lemma 5.2 be satisfied. Moreover, assume that $W_{,\chi}(\boldsymbol{\varepsilon},\chi)$ satisfies growth condition (31), $\Psi'(\chi)$ satisfies*

$$|\Psi'(\chi)| \leq c(|\chi|^{q_n} + 1)$$

with some constant $c > 0$, and tensor $\mathbf{\Gamma}'(\chi)$ satisfies the uniform in χ bound (A5)(iv). Then

$$\|\mu^m\|_{L_2(0,T;H^1(\Omega))} \leq c \neq c(m) \tag{92}$$

with constant c depending only on the data.

Proof. Setting $\zeta^m = 1$ in (57) (with $\mathbf{g} = \mathbf{0}, \beta = 0$) gives

$$\int_\Omega \mu^m \, dx = \frac{1}{2}\int_\Omega \nabla\chi^m \cdot \mathbf{\Gamma}'(\chi^m)\nabla\chi^m \, dx + \int_\Omega \Psi'(\chi^m)\, dx + \int_\Omega W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}^m),\chi^m)\, dx.$$

Recalling the assumptions we have

$$\begin{aligned} |\int_\Omega \mu^m \, dx| &\leq \frac{1}{2}c_{\Gamma'}\int_\Omega |\nabla\chi^m|^2 \, dx + c\int_\Omega (|\chi^m|^{q_n} + 1)\, dx \\ &\quad + c\int_\Omega (|\boldsymbol{\varepsilon}(\mathbf{u}^m)|^2 + |\chi^m|^2 + 1)\, dx \\ &\leq c(\|\nabla\chi^m\|^2_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\chi^m\|^{q_n}_{L_\infty(0,T;L_{q_n}(\Omega))}) \\ &\quad + \|\boldsymbol{\varepsilon}(\mathbf{u}^m)\|^2_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + 1) \leq c \neq c(m), \end{aligned} \tag{93}$$

where in the last inequality we have used estimates (89) and (91). By the Poincaré inequality, estimate (89) on $\nabla\mu^m$ and (93) yield (92). ∎

Using duality arguments, as in Lemma 4.4, we estimate time derivatives $\mathbf{u}^m_{tt}$ and χ^m_t.

LEMMA 5.4 *Let the assumptions of Lemma 5.2 hold true, $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi)$ satisfy growth condition (31), and the matrix $\mathbf{M}(\chi)$ satisfy the uniform in χ bound (A6)(iii). Then*

$$\|\mathbf{u}^m_{tt}\|_{L_2(0,T;\mathbf{V}_0')} \leq c \neq c(m), \tag{94}$$

$$\|\chi^m_t\|_{L_2(0,T;V')} \leq c \neq c(m). \tag{95}$$

Proof. Estimate (94) follows by the same arguments as in Lemma 4.4. To show (95) we use $\xi^m = P_m\xi$ for $\xi \in L_2(0,T;H^1(\Omega))$ as test function in (56) (with $\mathbf{h}=\mathbf{0}$), where P_m denotes the projection of $L_2(\Omega)$ onto V_m. Then

$$\begin{aligned}
&|\int_{\Omega^T} \chi_t^m \xi \, dxdt| = |\int_{\Omega^T} \chi_t^m P_m\xi \, dxdt| \\
&= |\int_{\Omega^T} (\mathbf{M}(\chi^m)\nabla\mu^m)\cdot\nabla(P_m\xi)\, dxdt| \\
&\le \left(\int_{\Omega^T} |\mathbf{M}(\chi^m)\nabla\mu^m|^2 \, dxdt\right)^{\frac{1}{2}} \left(\int_{\Omega^T} |\nabla(P_m\xi)|^2 \, dxdt\right)^{\frac{1}{2}} \\
&\le \overline{c}_M \|\nabla\mu^m\|_{\mathbf{L}_2(\Omega^T)} \|\nabla(P_m\xi)\|_{\mathbf{L}_2(\Omega^T)} \\
&\le c\|\nabla\xi\|_{\mathbf{L}_2(\Omega^T)} \quad \forall\, \xi \in L_2(0,T;H^1(\Omega)),
\end{aligned}$$

where in the last inequality we have used estimate (89). This shows (95). ■

6. Proof of Theorem 3.1: Passage to the limit $m \to \infty$ in $(P)^{\nu,m}$

From the estimates (71), (75)-(78), (83) it follows that there exists a triple $(\mathbf{u}^\nu, \chi^\nu, \mu^\nu)$ with

$$\begin{aligned}
&\mathbf{u}^\nu \in L_\infty(0,T;\mathbf{V}_0),\ \mathbf{u}_t^\nu \in L_\infty(0,T;\mathbf{L}_2(\Omega)),\ \mathbf{u}_{tt}^\nu \in L_2(0,T;\mathbf{V}_0'), \qquad (96)\\
&\chi^\nu \in L_\infty(0,T;H^1(\Omega)),\ \chi_t^\nu \in L_2(\Omega^T), \\
&\mu^\nu \in L_2(0,T;H^1(\Omega)),\ \nu^{\frac{1}{2}}\mu^\nu \in L_\infty(0,T;L_2(\Omega)),
\end{aligned}$$

and a subsequence of solutions $(\mathbf{u}^{\nu,m}, \chi^{\nu,m}, \mu^{\nu,m})$ to $(P)^{\nu,m}$ (which we still denote by the same indices) such that as $m \to \infty$:

$$\begin{aligned}
&\mathbf{u}^{\nu,m} \to \mathbf{u}^\nu \text{ weakly-* in } L_\infty(0,T;\mathbf{V}_0), \qquad (97)\\
&\mathbf{u}_t^{\nu,m} \to \mathbf{u}_t^\nu \text{ weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)), \\
&\mathbf{u}_{tt}^{\nu,m} \to \mathbf{u}_{tt}^\nu \text{ weakly in } L_2(0,T;\mathbf{V}_0'),
\end{aligned}$$

$$\begin{aligned}
&\chi^{\nu,m} \to \chi^\nu \text{ weakly-* in } L_\infty(0,T;H^1(\Omega)), \qquad (98)\\
&\chi_t^{\nu,m} \to \chi_t^\nu \text{ weakly in } L_2(\Omega^T),
\end{aligned}$$

$$\begin{aligned}
&\mu^{\nu,m} \to \mu^\nu \text{ weakly in } L_2(0,T;H^1(\Omega)), \qquad (99)\\
&\nu^{\frac{1}{2}}\mu^{\nu,m} \to \nu^{\frac{1}{2}}\mu^\nu \text{ weakly-* in } L_\infty(0,T;L_2(\Omega)).
\end{aligned}$$

Then by the standard compactness results (see Simon, 1987, Corollary 4) it follows in particular that

$$\mathbf{u}^{\nu,m} \to \mathbf{u}^{\nu} \text{ strongly in } L_2(0,T;\mathbf{L}_q(\Omega)) \cap C([0,T];\mathbf{L}_q(\Omega)) \tag{100}$$
$$\text{and a.e. in } \Omega^T,$$
$$\mathbf{u}_t^{\nu,m} \to \mathbf{u}_t^{\nu} \text{ strongly in } C([0,T];\mathbf{V}_0'),$$

$$\chi^{\nu,m} \to \chi^{\nu} \text{ strongly in } L_2(0,T;L_q(\Omega)) \cap C([0,T];L_q(\Omega)) \tag{101}$$
$$\text{and a.e. in } \Omega^T,$$

where $q < q_n$. Hence,

$$\begin{aligned} &\mathbf{u}^{\nu,m}(0) = \mathbf{u}_0^m \to \mathbf{u}^{\nu}(0) \text{ strongly in } L_q(\Omega),\\ &\mathbf{u}_t^{\nu,m}(0) = \mathbf{u}_1^m \to \mathbf{u}_t^{\nu}(0) \text{ strongly in } \mathbf{V}_0',\\ &\chi^{\nu,m}(0) = \chi_0^m \to \chi^{\nu}(0) \text{ strongly in } L_q(\Omega), \end{aligned} \tag{102}$$

what together with convergences (59) implies that

$$\mathbf{u}^{\nu}(0) = \mathbf{u}_0, \quad \mathbf{u}_t^{\nu}(0) = \mathbf{u}_1, \quad \chi^{\nu}(0) = \chi_0. \tag{103}$$

The relations (96) and (103) imply assertions (i)-(iii) of Theorem 3.1.

Now we introduce the following weak formulation of problem $(P)^{\nu,m}$:

$$\begin{aligned} &\int_0^T < \mathbf{u}_{tt}^{\nu,m}, \boldsymbol{\eta} >_{\mathbf{V}_0',\mathbf{V}_0} dt\\ &\quad + \int_0^T (\mathbf{A}(\chi^{\nu,m})(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m})), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))_{\mathbf{L}_2(\Omega)}\, dt\\ &= \int_0^T (\mathbf{b}, \boldsymbol{\eta})_{\mathbf{L}_2(\Omega)}\, dt \quad \forall\, \boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_m), \end{aligned} \tag{104}$$

$$\begin{aligned} &\int_0^T [-(\nu\mu^{\nu,m}, \xi_t)_{L_2(\Omega)} + (\chi_t^{\nu,m}, \xi)_{L_2(\Omega)}\\ &\quad +(\mathbf{M}(\chi^{\nu,m})\nabla\mu^{\nu,m} + \mathbf{h}\chi_t^{\nu,m}, \nabla\xi)_{\mathbf{L}_2(\Omega)}]\, dt\\ &= (\nu\mu_0^m, \xi(0))_{L_2(\Omega)} \quad \forall\, \xi \in C^1([0,T], V_m) \textit{ with } \xi(T) = 0, \end{aligned} \tag{105}$$

$$\begin{aligned} &\int_0^T [(\mu^{\nu,m} - \mathbf{g}\cdot\nabla\mu^{\nu,m}, \zeta)_{L_2(\Omega)} - (\boldsymbol{\Gamma}(\chi^{\nu,m})\nabla\chi^{\nu,m}, \nabla\zeta)_{\mathbf{L}_2(\Omega)}\\ &\quad -(\frac{1}{2}\nabla\chi^{\nu,m}\cdot\boldsymbol{\Gamma}'(\chi^{\nu,m})\nabla\chi^{\nu,m}, \zeta)_{L_2(\Omega)} - (\Psi'(\chi^{\nu,m}), \zeta)_{L_2(\Omega)}\\ &\quad +(\overline{\boldsymbol{\varepsilon}}'(\chi^{\nu,m})\cdot\mathbf{A}(\chi^{\nu,m})(\overline{\boldsymbol{\varepsilon}}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m})), \zeta)_{L_2(\Omega)}\\ &\quad -\frac{1}{2}((\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m}))\cdot\mathbf{A}'(\chi^{\nu,m})(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m})), \zeta)_{L_2(\Omega)}\\ &\quad -(\beta\chi_t^{\nu,m}, \zeta)_{L_2(\Omega)}]\, dt = 0 \quad \forall\, \zeta \in L_2(0,T;V_m). \end{aligned} \tag{106}$$

The goal is to pass to the limit in (104)-(106) with $m \to \infty$. Before doing this we point out the difficulties arising from the nonlinear terms in (106) with the highest space derivatives of $\chi^{\nu,m}$ and $\mathbf{u}^{\nu,m}$, that is the terms:

$$\frac{1}{2}\nabla\chi^{\nu,m} \cdot \mathbf{\Gamma}'(\chi^{\nu,m})\nabla\chi^{\nu,m} \text{ and}$$
$$\frac{1}{2}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m})) \cdot \mathbf{A}'(\chi^{\nu,m})(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m})).$$

From (97), (98) it follows that as $m \to \infty$,

$$\begin{aligned} &\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) \to \boldsymbol{\varepsilon}(\mathbf{u}^{\nu}) \text{ weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)), \\ &\nabla\chi^{\nu,m} \to \nabla\chi^{\nu} \text{ weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)). \end{aligned} \tag{107}$$

Clearly, the convergences (107) are not sufficient to pass to the limit $m \to \infty$ in the above mentioned terms. Because we are unable to establish any strong convergences of $\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m})$ and $\nabla\chi^{\nu,m}$, we shall restrict ourselves to the study of the homogeneous problem with constant tensors $\mathbf{\Gamma}$ and $\mathbf{A}$. In such a case (105) remains unchanged, and (104), (106) reduce to

$$\begin{aligned} &\int_0^T < \mathbf{u}_{tt}^{\nu,m}, \boldsymbol{\eta} >_{\mathbf{V}_0',\mathbf{V}_0} dt \\ &\quad + \int_0^T (\mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m}), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))_{\mathbf{L}_2(\Omega)}\, dt \\ &= \int_0^T (\mathbf{b}, \boldsymbol{\eta})_{\mathbf{L}_2(\Omega)}\, dt \quad \forall\, \boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_m), \end{aligned} \tag{108}$$

$$\begin{aligned} &\int_0^T [(\mu^{\nu,m} - \mathbf{g}\cdot\nabla\mu^{\nu,m}, \zeta)_{L_2(\Omega)} - (\mathbf{\Gamma}\nabla\chi^{\nu,m}, \nabla\zeta)_{\mathbf{L}_2(\Omega)} \\ &\quad -(\Psi'(\chi^{\nu,m}),\zeta)_{L_2(\Omega)} + (\overline{\boldsymbol{\varepsilon}}'(\chi^{\nu,m})\cdot\mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m}),\zeta)_{L_2(\Omega)} \\ &\quad -(\beta\chi_t^{\nu,m},\zeta)_{L_2(\Omega)}]\, dt = 0 \quad \forall\zeta \in L_2(0,T;V_m). \end{aligned} \tag{109}$$

To pass to the limit $m \to \infty$ in (105), (108), (109) we follow the standard procedure (see e.g. Duvaut and Lions, 1972). Namely, we fix $m = m_0 \in \mathbb{N}$ in the spaces of test functions $\boldsymbol{\eta}, \xi, \zeta$, and take subsequences (97)-(99) with $m \geq m_0$.

Clearly, by virtue of the weak convergences (97)-(99), the linear terms in (105), (108), (109) converge to the corresponding limits. The convergence of the remaining nonlinear terms follows from the following

LEMMA 6.1 *Assume that tensors* $\mathbf{\Gamma}$ *and* $\mathbf{A}$ *are constant,* $\overline{\boldsymbol{\varepsilon}}(\chi)$ *satisfies (A3)(i)(ii),* $\Psi(\chi)$ *satisfies (A4)(i)(iii), the matrix* $\mathbf{M}(\chi)$ *satisfies (A6)(i) and the uniform in* χ *bound (A6)(iii). Then for a subsequence* $m \to \infty$,

$$\begin{aligned}
&\Psi'(\chi^{\nu,m}) \to \Psi'(\chi^{\nu}) \text{ weakly-* in } L_\infty(0,T;L_2(\Omega)), \qquad (110)\\
&\mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m})) \to \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu}))\\
&\qquad \text{weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)),\\
&\overline{\boldsymbol{\varepsilon}}'(\chi^{\nu,m}) \cdot \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu,m}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m})) \to \overline{\boldsymbol{\varepsilon}}'(\chi^{\nu}) \cdot \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu}))\\
&\qquad \text{weakly-* in } L_\infty(0,T;L_2(\Omega)),\\
&\mathbf{M}(\chi^{\nu,m})\nabla\mu^{\nu,m} \to \mathbf{M}(\chi^{\nu})\nabla\mu^{\nu} \text{ weakly in } \mathbf{L}_2(\Omega^T).
\end{aligned}$$

Proof. We use convergences (97)-(101). In view of the bounds

$$\begin{aligned}
&\|\overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \le c(\|\chi^{\nu,m}\|_{L_\infty(0,T;L_2(\Omega))} + 1) \le c \ne c(\nu, m),\\
&\|\Psi'(\chi^{\nu,m})\|_{L_\infty(0,T;L_2(\Omega))} \le c(\|\chi^{\nu,m}\|^{\frac{q_n}{2}}_{L_\infty(0,T;L_{q_n}(\Omega))} + 1) \le c \ne c(\nu, m),
\end{aligned}$$

and the convergence $\chi^{\nu,m} \to \chi^{\nu}$ a.e. in Ω^T, we can apply the classical convergence result (see e.g. Lions, 1969, Chapter 1, Lemma 1.3) to conclude that $(110)_1$ holds, and

$$\overline{\boldsymbol{\varepsilon}}(\chi^{\nu,m}) \to \overline{\boldsymbol{\varepsilon}}(\chi^{\nu}) \text{ weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)). \qquad (111)$$

Hence, recalling $(107)_1$, convergence $(110)_2$ follows. Further, due to assumption on $\overline{\boldsymbol{\varepsilon}}'(\cdot)$, we have

$$\overline{\boldsymbol{\varepsilon}}'(\chi^{\nu,m}) \to \overline{\boldsymbol{\varepsilon}}'(\chi^{\nu}) \text{ strongly in } \mathbf{L}_2(\Omega^T) \text{ and a.e. in } \Omega^T.$$

Therefore, in view of (107) and (111), we can conclude convergence $(110)_3$. Similarly, by the assumption on $\mathbf{M}(\cdot)$,

$$\mathbf{M}(\chi^{\nu,m}) \to \mathbf{M}(\chi^{\nu}) \text{ a.e. in } \Omega^T,$$

so that convergence $(110)_4$ holds true. ■

Consequently, under assumptions of Lemma 6.1 we can pass to the limit in (105), (108), (109) for a subsequence $m_0 \le m \to \infty$, to conclude that

$$\begin{aligned}
&\int_0^T < \mathbf{u}^{\nu}_{tt}, \boldsymbol{\eta} >_{\mathbf{V}_0',\mathbf{V}_0} dt + \int_0^T (\mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^{\nu}) - \overline{\boldsymbol{\varepsilon}}(\chi^{\nu})), \boldsymbol{\varepsilon}(\boldsymbol{\eta})_{\mathbf{L}_2(\Omega)}\, dt \qquad (112)\\
&= \int_0^T (\mathbf{b}, \boldsymbol{\eta})_{\mathbf{L}_2(\Omega)}\, dt \quad \forall\, \boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_{m_0}),\\
&\int_0^T [-(\nu\mu^{\nu}, \xi_t)_{L_2(\Omega)} + (\chi^{\nu}_t, \xi)_{L_2(\Omega)} + (\mathbf{M}(\chi^{\nu})\nabla\mu^{\nu} + \mathbf{h}\chi^{\nu}_t, \nabla\xi)_{\mathbf{L}_2(\Omega)}]\, dt\\
&= (\nu\mu_0, \xi(0))_{L_2(\Omega)} \quad \forall\, \xi \in C^1([0,T];V_{m_0}) \text{ with } \xi(T) = 0,
\end{aligned}$$

$$\int_0^T [(\mu^\nu - \mathbf{g}\cdot\nabla\mu^\nu, \zeta)_{L_2(\Omega)} - (\mathbf{\Gamma}\nabla\chi^\nu, \nabla\zeta)_{\mathbf{L}_2(\Omega)} - (\Psi'(\chi^\nu), \zeta)_{L_2(\Omega)}$$
$$+(\overline{\varepsilon}'(\chi^\nu)\cdot \mathbf{A}(\varepsilon(\mathbf{u}^\nu) - \overline{\varepsilon}(\chi^\nu)), \zeta)_{L_2(\Omega)} - (\beta\chi_t^\nu, \zeta)_{L_2(\Omega)}]\, dt = 0$$
$$\forall\ \zeta \in L_2(0,T;V_{m_0}).$$

Now, passing to the limit $m_0 \to \infty$ and taking into account the density of $\mathbf{V}_\infty$ in $\mathbf{H}_0^1(\Omega)$ and V_∞ in $H^1(\Omega)$, we arrive at the identities (36)-(38). A priori estimates (39) are the consequences of the uniform in ν and m estimates (71), (75)-(78), (83) and the weak convergences (97)-(99). Thereby the proof of Theorem 3.1 is completed. ■

7. Proof of Theorem 3.2: Passage to the limit $\nu \to 0$ in $(P)^\nu$

Due to estimates (39) it follows that there exists a triple $(\mathbf{u}, \chi, \mu)$ with

$$\begin{aligned}&\mathbf{u} \in L_\infty(0,T;\mathbf{V}_0), \mathbf{u}_t \in L_\infty(0,T;\mathbf{L}_2(\Omega)), \mathbf{u}_{tt} \in L_2(0,T;\mathbf{V}_0'),\\ &\chi \in L_\infty(0,T;H^1(\Omega)), \chi_t \in L_2(\Omega^T),\\ &\mu \in L_2(0,T;H^1(\Omega)),\end{aligned} \tag{113}$$

and a subsequence of $(\mathbf{u}^\nu, \chi^\nu, \mu^\nu)$ of solutions to $(P)^\nu$ (denoted by the same indices) such that as $\nu \to 0$:

$$\begin{aligned}&\mathbf{u}^\nu \to \mathbf{u} \text{ weakly-* in } L_\infty(0,T;\mathbf{V}_0),\\ &\mathbf{u}_t^\nu \to \mathbf{u}_t \text{ weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)),\\ &\mathbf{u}_{tt}^\nu \to \mathbf{u}_{tt} \text{ weakly in } L_2(0,T;\mathbf{V}_0'),\end{aligned} \tag{114}$$

$$\begin{aligned}&\chi^\nu \to \chi \text{ weakly-* in } L_\infty(0,T;H^1(\Omega)),\\ &\chi_t^\nu \to \chi_t \text{ weakly in } L_2(\Omega^T),\end{aligned} \tag{115}$$

$$\begin{aligned}&\mu^\nu \to \mu \text{ weakly in } L_2(0,T;H^1(\Omega)),\\ &\nu\mu^\nu \to 0 \text{ strongly in } L_\infty(0,T,L_2(\Omega)).\end{aligned} \tag{116}$$

Furthermore, by the compactness results, as in (100), it follows that

$$\begin{aligned}&\mathbf{u}^\nu \to \mathbf{u} \text{ strongly in } L_2(0,T,\mathbf{L}_q(\Omega)) \cap C([0,T];\mathbf{L}_q(\Omega))\\ &\qquad \text{and a.e. in } \Omega^T,\\ &\mathbf{u}_t^\nu \to \mathbf{u}_t \text{ strongly in } C([0,T],\mathbf{V}_0'),\\ &\chi^\nu \to \chi \text{ strongly in } L_2(0,T,L_q(\Omega)) \cap C([0,T];L_q(\Omega)) \text{ and a.e. in } \Omega^T,\end{aligned} \tag{117}$$

where $q < q_n$. Hence, the convergences

$$\mathbf{u}^\nu(0) \to \mathbf{u}(0) \text{ strongly in } \mathbf{L}_q(\Omega),$$
$$\mathbf{u}_t^\nu(0) \to \mathbf{u}_t(0) \text{ strongly in } \mathbf{V}_0',$$
$$\chi^\nu(0) \to \chi(0) \text{ strongly in } L_q(\Omega),$$

imply that

$$\mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}_t(0) = \mathbf{u}_1,\ \chi(0) = \chi_0. \tag{118}$$

Due to convergences (114)-(117) it follows by repeating the arguments of Lemma 6.1 that for a subsequence $\nu \to 0$,

$$\begin{aligned} &\Psi'(\chi^\nu) \to \Psi'(\chi) \text{ weakly-* in } L_\infty(0,T;L_2(\Omega)), \\ &\mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^\nu) - \overline{\boldsymbol{\varepsilon}}(\chi^\nu)) \to \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \text{ weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)), \\ &\overline{\boldsymbol{\varepsilon}}'(\chi^\nu) \cdot \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^\nu) - \overline{\boldsymbol{\varepsilon}}(\chi^\nu)) \to \overline{\boldsymbol{\varepsilon}}'(\chi) \cdot \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \\ &\qquad \text{weakly-* in } L_\infty(0,T;L_2(\Omega)), \\ &\mathbf{M}(\chi^\nu)\nabla\mu^\nu \to \mathbf{M}(\chi)\nabla\mu \text{ weakly in } \mathbf{L}_2(\Omega^T). \end{aligned} \tag{119}$$

In view of the convergences (114)-(117) and (119) we can pass to the limit in the identities (36)-(38) to conclude that $(\mathbf{u}, \chi, \mu)$ satisfy (40)-(42). A priori estimate (43) results directly from (39). The proof is completed. ■

8. Proof of Theorem 3.3: Passage to the limit $m \to \infty$ in $(P)^m$

By virtue of estimates (89)-(95) it follows that there exists a triple $(\mathbf{u}, \chi, \mu)$ with

$$\begin{aligned} &\mathbf{u} \in L_\infty(0,T;\mathbf{V}_0), \mathbf{u}_t \in L_\infty(0,T;\mathbf{L}_2(\Omega)), \mathbf{u}_{tt} \in L_2(0,T;\mathbf{V}_0'), \\ &\chi \in L_\infty(0,T;H^1(\Omega)), \chi_t \in L_2(0,T;V'), \\ &\mu \in L_2(0,T;H^1(\Omega)), \end{aligned}$$

and a subsequence $(\mathbf{u}^m, \chi^m, \mu^m)$ of solutions to $(P)^m$ (denoted by the same indices) such that as $m \to \infty$:

$$\begin{aligned} &\mathbf{u}^m \to \mathbf{u} \text{ weakly-* in } L_\infty(0,T;\mathbf{V}_0), \\ &\mathbf{u}_t^m \to \mathbf{u}_t \text{ weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)), \\ &\mathbf{u}_{tt}^m \to \mathbf{u}_{tt} \text{ weakly in } L_2(0,T;\mathbf{V}_0'), \end{aligned} \tag{120}$$

$$\begin{aligned} &\chi^m \to \chi \text{ weakly-* in } L_\infty(0,T;H^1(\Omega)), \\ &\chi_t^m \to \chi_t \text{ weakly in } L_2(0,T;V'), \end{aligned} \tag{121}$$

$$\mu^m \to \mu \text{ weakly in } L_2(0,T;H^1(\Omega)). \tag{122}$$

Further, by the compactness results,

$$\begin{aligned}&\mathbf{u}^m \to \mathbf{u} \text{ strongly in } L_2(0,T,\mathbf{L}_q(\Omega)) \cap C([0,T];\mathbf{L}_q(\Omega)) \\ &\qquad \text{and a.e. in } \Omega^T, \\ &\mathbf{u}_t^m \to \mathbf{u}_t \text{ strongly in } C([0,T],\mathbf{V}_0'), \\ &\chi^m \to \chi \text{ strongly in } L_2(0,T,L_q(\Omega)) \cap C([0,T];L_q(\Omega)) \text{ and a.e. in } \Omega^T, \\ &\qquad \text{for } q < q_n.\end{aligned} \tag{123}$$

Therefore, owing to the convergences (59), it follows that

$$\mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}_t(0) = \mathbf{u}_1,\ \chi(0) = \chi_0. \tag{124}$$

Moreover, by the same arguments as in Lemma 6.1, we can conclude that for a subsequence $m \to \infty$:

$$\begin{aligned}&\Psi'(\chi^m) \to \Psi'(\chi) \text{ weakly-* in } L_\infty(0,T;L_2(\Omega)), \\ &\mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)) \to \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \text{ weakly-* in } L_\infty(0,T;\mathbf{L}_2(\Omega)), \\ &\overline{\boldsymbol{\varepsilon}}'(\chi^m) \cdot \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)) \to \overline{\boldsymbol{\varepsilon}}'(\chi) \cdot \mathbf{A}(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \\ &\qquad \text{weakly-* in } L_\infty(0,T;L_2(\Omega)), \\ &\mathbf{M}(\chi^m)\nabla\mu^m \to \mathbf{M}(\chi)\nabla\mu \text{ weakly in } \mathbf{L}_2(\Omega^T).\end{aligned} \tag{125}$$

In view of convergences (120)-(122) and (125) we can pass to the limit with $m \to \infty$ in the weak formulation of $(P)^m$ (in a similar fashion as in the proof of Theorem 3.1) to conclude that identities (40), (42) (with $\mathbf{g} = \mathbf{0}$, $\beta = 0$) and (44) are satisfied. The proof is completed. ■

9. Proof of Theorem 3.4: Passage to the limit $\alpha \to 0$ in $(P)^\alpha$

Let $(\mathbf{u}^\alpha, \chi^\alpha, \mu^\alpha)$ be a weak solution to $(P)^\alpha$ in the sense of identities (47)-(49). Due to uniform in α estimates (50) it follows that there exists a triple $(\mathbf{u}, \chi, \mu)$ with

$$\begin{aligned}&\mathbf{u} \in L_\infty(0,T;\mathbf{V}_0), \\ &\chi \in L_\infty(0,T;H^1(\Omega)), \chi_t \in L_2(0,T;V'), \\ &\mu \in L_2(0,T;H^1(\Omega)),\end{aligned}$$

and a subsequence $(\mathbf{u}^\alpha, \chi^\alpha, \mu^\alpha)$ of solutions to $(P)^\alpha$ (denoted by the same indices) such that as $\alpha \to 0$:

$$\begin{aligned}
&\mathbf{u}^\alpha \to \mathbf{u} \text{ weakly-* in } L_\infty(0,T;\mathbf{V}_0), \\
&\alpha \mathbf{u}_t^\alpha \to \mathbf{0} \text{ strongly in } L_\infty(0,T;\mathbf{L}_2(\Omega)), \\
&\chi^\alpha \to \chi \text{ weakly-* in } L_\infty(0,T;H^1(\Omega)), \\
&\chi_t^\alpha \to \chi_t \text{ weakly in } L_2(0,T;V'), \\
&\mu^\alpha \to \mu \text{ weakly in } L_2(0,T;H^1(\Omega)).
\end{aligned} \tag{126}$$

Hence, by the compactness results,

$$\begin{aligned}
&\chi^\alpha \to \chi \text{ strongly in } L_2(0,T;L_q(\Omega)) \cap C([0,T];L_q(\Omega)) \\
&\text{and a.e. in } \Omega^T, \text{ for } q < q_n,
\end{aligned} \tag{127}$$

which, since $\chi^\alpha = \chi_0$, implies that

$$\chi(0) = \chi_0. \tag{128}$$

In view of (126) and (127), recalling the arguments of the proof of Theorem 3.3, we can see that for a subsequence $\alpha \to 0$ the convergences (125) with $(\mathbf{u}^\alpha, \chi^\alpha, \mu^\alpha)$ in place of $(\mathbf{u}^m, \chi^m, \mu^m)$ hold true. Combining the above convergences, in particular taking into account $(126)_2$, we can pass to the limit $\alpha \to 0$ in the identities (47)-(49) to conclude the assertion. ■

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