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J. Kacprzyk, M. Krawczak,
G. Szkatuła

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.:(+48) (22) 3810100

fax:(+48) (22) 3810105

Kierownik Zakładu zgłaszający pracę:
Prof. dr hab. inż. Janusz Kacprzyk

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On bilateral matching between fuzzy sets

Janusz Kacprzyk[1], Maciej Krawczak[1, 2], Grażyna Szkatuła[1]

[1]*Systems Research Institute, Polish Academy of Sciences, Newelska 6, Warsaw, Poland*
[2]*Warsaw School of Information Technology, Newelska 6, Warsaw, Poland*

{kacprzyk, krawczak, szkatulg}@ibspan.waw.pl

Abstract. In the paper we describe the new measure of matching fuzzy sets. The introduced *measure of perturbation of one fuzzy set by another fuzzy set* is considered instead of commonly used distance between two fuzzy sets. The operations known in the fuzzy set theory are used and the perturbation of one fuzzy set by another fuzzy set is understood as a measure describing changes of the first fuzzy set after adding the second one. Obviously, the opposite case can also be considered wherein the second fuzzy set is perturbed by the first one. The values of such measures of fuzzy sets' perturbation are ranged between 0 and 1, and in general, are not symmetric. Therefore, the described measure of perturbation cannot be considered as a distance between fuzzy sets. In this paper several mathematical properties of the measure of fuzzy sets' perturbation are studied, and the meaning of sets' proximity is explained by comparison of selected measures.

Keywords: Perturbation of fuzzy sets, fuzzy sets' matching, sets' perturbation, nominal-valued attributes.

1 Introduction

Comparing two objects some kind of similarity or dissimilarity assessment between them is applied. The role of similarity or dissimilarity of two objects is fundamental in many theories of cognitive knowledge as well as behavior, and for comparison of objects there are commonly used different measures of objects' similarity. If the measure of similarity is normalized, many researches assume that dissimilarity is the inverse of similarity, however others consider that the dissimilarity and a non-similarity (i.e., the difference between a number 1 and the similarity) are not synonymous. For example, in the paper by Wang, Meng and Li (2008), the authors consider that the similarity cannot be limited to a number between 0 and 1.

In general, there are two classes of proximity representation between objects. In the first class, each object is represented by a point in the adequate multidimensional Cartesian coordinates, and an appropriate measure of the proximity between two such objects is specified just by the distance between these two corresponding points in that space. Naturally, the metric axioms of non-negativity, symmetry and triangle inequality must be satisfied.

In the second class, each object is represented as a collection of some features or attributes. Usually similarity between objects is expressed as a matching function of their common and distinctive features (Tversky, 1977), however similarity can also be expressed as structural compatibility or simple features matching (R. L. Goldstone, D. L. Medin, D. Gentner, 1991). For instance, Goldstone gave the following simple example, which illustrates these two concepts, see Fig. 1.

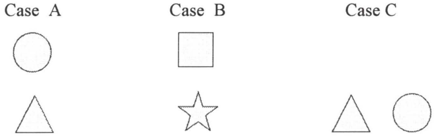


Fig.1. Relational similarity and attribute similarity

In Fig. 1, there are three cases, the case A shows a pair of two figures (circle and triangle) allocated vertically, the case B shows another pair of two figures (square and star) also allocated vertically, and the case C with another pair of two figures (triangle and circle) allocated horizontally. Thus, the case A and C have a better matching of the attribute because of the same attribute's values (circle and triangle), but they have different relational similarities because of the different arrangements (vertical vs. horizontal). The case A and B have the same relational similarities (vertical allocation) but they have different values of the attribute (circle and triangle vs. square and star) (Li, Fonseca, 2006).

The objects' similarity may also be referred as a transformational distance between two objects, such a distance is understood as a smallest number of operations (the minimum cost) that a computer program needs to transform the first object's representation to the representation of the second object. This concept is known as Levenshtein's distance (Levenshtein, 1966). The sets' perturbation concept presented in this paper, sometimes can be regarded to a certain extent as related in spirit to Levenshtein's concept, however our concept is much more general because is bidirectional and concerns nominal-valued attributes.

In the majority of theoretical works of objects' similarity there is an essential assumption about symmetry, i.e., the similarity of one object A to another object B is equals to the similarity of B to A. However, some research, e.g. in psychological literature, does not follow this assumption, and it is believed that the objects' similarity can be asymmetric. Goodman (1972) claimed that the concept of similarity of an object A to another object B is ill-defined, because does not include the concept "under what term". It seems to be obvious that objects are similar with respect to "something". In order to support this reasoning, Fig. 2 demonstrates mutual impact of colors. In the picture, both inner circles are exactly the same, while colors of the backgrounds around are different. However, the circle on the left side seems to be somewhat darker than the circle on the right side of the picture. The perception of colors and their intensity may substantially depend on the background color.

Fig.2. Mutual affecting of colors

In a similar way, asymmetry of the objects' similarity appears due to the considered task context or modifications of attributes, or relations. The issue of symmetry was extensively analyzed by Amos Tversky (1977). He considered objects represented by sets of features or attributes, and proposed the measure of similarity as a comparison of the feature space, which is rather inferred from the general context. Tversky claims that similarity should not be considered as a symmetric relation between objects, and the commonality increases greater than the difference decreases similarity. Additionally, the similarity focuses on matching of relations between objects while difference focuses on mismatching of attributes. Therefore, when the object A is more similar to the object B than to the object C, then it is possible that the object A is more different from the object B than from the object C. The similarity concept of Tversky just describes the second class of paradigm of proximity measures between objects, although his model does not define a uniform scale of similarity.

In this papers, we consider a finite, non-empty set of objects (concepts or patterns) described by a set of nominal attributes and the values of the attributes are not precisely known. In other words, it means that there is some information about possible or acceptable, as well as some information about impossible or unacceptable attributes values. Methodology of the fuzzy set theory allows us to model such imperfect, incomplete and inconsistent data. In the literature one can find numerous different ways of defining various forms of the fuzzy sets theory, and various numerical scales are used to represent the positive and negative information by membership degree and non-membership degree. Usually, a membership degree lies within the range from 0 up to 1.

There are presented in the literature quite many measures of similarity between objects as well as some formulas to calculate them (Baccour, Alimi and John, 2014; Dubois and Prade, 1980; Pappis and Karacapilidis, 1993; Weken, Nachtegael and Kerre, 2004; Wang, Meng and Li, 2008; Cross and

Sudkamp, 2002). Generally, in the fuzzy set theory, the value of a similarity measure of two fuzzy sets is defined in terms of a membership function which is obtained by comparing corresponding membership degrees for each element, and provides a value between 0 and 1.

Our present work is motivated by the need to develop effective procedures for comparing objects wherein each is described by a set of incomplete and inconsistent nominal-valued attributes, and the attributes have a fuzzy sets form. Additionally, following Tversky's suggestions about possible asymmetric nature of similarities between objects we want just to verify symmetry of objects' proximity. Therefore, comparing objects described by such kind of attributes we are obliged to develop several efficient mathematical tools which ensure satisfactory comparisons of two fuzzy sets of attributes. Here in this paper, we focus our attention on data described by nominal-valued fuzzy sets. This paper is a continuation as well as extension of the authors' previous papers related to perturbation of sets (Krawczak and Szkatuła, 2013 a, b, 2014 a, b, 2015).

The term "perturbation" is used here in the general sense and should not be confused with that known in mathematics or physics. Let us illustrate meaning of the used here term "perturbation". Namely, let us consider two baskets, the first, say A, contains a single red apple, while the second basket, say B, contains 100 green apples. Now, we will perform two experiments. In the first experiment 100 green apples are added to the basket A with the lonely red apple, then the contents of the basket A was perturbed by the contents of the basket B. As a result, 100 green apples completely dominated the single red apple. Within the second experiment, let us put the single red apple to the basket B with 100 green apples, it means that the contents of the basket B was perturbed by the contents of the basket A. As a result, the added red apple is almost indistinguishable within 100 green apples. We claim that in the first case 100 green apples perturbed the contents of the basket A significantly, while in the second case the single red apple perturbed the contents of the basket B negligible. In this way the term perturbation of one set by another set corresponds to Tversky's considerations about objects' similarities.

Here, instead of the sets, we examine fuzzy sets and then we introduce an innovative measure of proximity between two fuzzy sets. The consideration is based on the fuzzy set theory and its basic operations. We do not consider commonly used similarity or dissimilarity between two fuzzy sets, but we introduce *a new measure of perturbation of one fuzzy set by another fuzzy set*. The proposed measure identifies occurred changes of the first set after adding the second set or vice versa occurred changes of the second set after adding the first set. After normalization the measure of perturbation of fuzzy sets is ranged from 0 up to 1, where 0 is the lowest value of perturbation while 1 is the highest value of perturbation. It is shown that this measure is not always symmetrical, it means that a value of the measure of perturbation of the first fuzzy set by the second fuzzy set can be different then a value of the measure of perturbation of the second fuzzy set by the first fuzzy set. Therefore, it should not be considered as the distance between the fuzzy sets. We can however say, that the sum of these measures gives an equivalent interpretation of dissimilarity, i.e., can be regarded as a dissimilarity measure between two fuzzy sets. Also we show some special properties for the proposed measure of perturbation between two fuzzy sets.

For a better understanding of the proposed concept of the fuzzy sets' perturbation, the geometrical interpretations in 2D and 3D space are presented.

In the paper we gave a short suggestion for application of sets perturbation measure to solve a short illustrative example. The proposed measure of perturbation of one fuzzy set by another fuzzy set is compared with the selected measures of similarity. This short example is intended to emphasize the differences between the "classical" measures which are symmetric and the fuzzy sets' perturbations, which are not necessarily symmetric. In our opinion the concept of perturbation of fuzzy sets can find a wide applications to solve problems based on comparison of fuzzy sets, when "direction" of comparing sets have significant meaning.

It must be emphasized that there are known quite many proximity measures and mostly there are developed especially for considered real data and stated problems. In our case, the concept of the measure of set's perturbation between two fuzzy sets is another proposal and seems to be more general.

This paper is organized as follows: in Section 2, we present the basis definitions and notations of fuzzy sets. The bilateral matching methodology as well as the mathematical properties of the perturbation measure are studied in Section 3. In Section 4 the proposed measure is compared to the selected measures of non-similarity. The extended geometrical interpretation is provided in Appendix.

2. Preliminaries and some generic definitions

Let us consider a non-empty and finite set V of nominal elements, denoted by $V=\{v_1,v_2,...,v_N\}$. In 1965, L. A. Zadeh introduced the concept of fuzzy set theory as an extension of the classical set theory (Zadeh, 1965). *A fuzzy set* A *in the set* V may be represented by a collection of ordered pairs written in the following form

$$\mathrm{A}=\{(v,\ \mu_{\mathrm{A}}(v))\mid \forall v\in V\}. \tag{1}$$

where $\mu_{\mathrm{A}}: V\rightarrow[0,1]$ is *the membership function*. There are three main and exclusive conditions: the condition $\mu_{\mathrm{A}}(v)=1$ means, that the element v for sure belongs to the fuzzy set A; the condition $\mu_{\mathrm{A}}(v)=0$ means, that the element v for sure does not belong to the fuzzy set A; the condition $0<\mu_{\mathrm{A}}(v)<1$ means, that the element v belongs to the fuzzy set A with the membership degree $\mu_{\mathrm{A}}(v)$.

Zadeh also introduced the fundamental operations for fuzzy sets, namely union, intersection and complementation. Let us consider a collection of all fuzzy sets in the set V of nominal values, denoted by $FS(V)$ in short.

2.1. Measure of the fuzzy sets similarity

A measure of the fuzzy similarity or dissimilarity defines the proximity between two fuzzy sets. There is no unique definition of sets proximity thus rather axiomatic definition is explored. In 1983 P. Z. Wang first introduced the concept of *the similarity measure of two fuzzy sets* A_i and A_j in V as a function $S: FS(V)\times FS(V)\rightarrow[0,1]$ which assigns to every pair of the considered fuzzy sets a nonnegative number (Wang, 1983), and the following axiomatic conditions are satisfied:

(1) $S(V,\varnothing)=0$;

(2) $S(\mathrm{A}_i,\mathrm{A}_i)=1$;

(3) $S(\mathrm{A}_i,\mathrm{A}_j)=S(\mathrm{A}_j,\mathrm{A}_i)$;

(4) $\forall \mathrm{A}_i,\mathrm{A}_j,\mathrm{A}_k\in FS(V)$, if $\mathrm{A}_i\subseteq\mathrm{A}_j\subseteq\mathrm{A}_k$ then the inequalities $S(\mathrm{A}_i,\mathrm{A}_k)\le S(\mathrm{A}_i,\mathrm{A}_j)$ and $S(\mathrm{A}_i,\mathrm{A}_k)\le S(\mathrm{A}_j,\mathrm{A}_k)$.

There are many various proposed properties of a fuzzy similarity measure $S(\mathrm{A}_i,\mathrm{A}_j)$ between two fuzzy sets A_i and A_j in V (e.g. Baccour, Alimi and John, 2014). For the fuzzy similarity measure based on proximity measures these properties and their modifications, as well as properties of distance measures are discussed in many works (e.g., Pappis and Karacapilidis, 1993; Wang, 1997; Liu, 1992; Raha, Hossain and Ghosh, 2008). Below, there are recalled selected properties of the fuzzy similarity measure $S(\mathrm{A}_i,\mathrm{A}_j)$.

A normal measure of similarity $S(\mathrm{A}_i,\mathrm{A}_j)$ between two fuzzy sets A_i and A_j in V satisfies the following conditions (Liu, 1992):

(1) $S(\mathrm{A}_i,\mathrm{A}_j)=S(\mathrm{A}_j,\mathrm{A}_i)$, $\forall \mathrm{A}_i,\mathrm{A}_j\in FS(V)$;

(2) $S(\mathrm{A}_i,\mathrm{A}_i)=1$;

(3) $S(\mathrm{A}_i,\mathrm{A}_i^C)=0$ if and only if A_i is a crisp set;

(4) if the conditions $\mathrm{A}_i \subseteq \mathrm{A}_j \subseteq \mathrm{A}_k$, $\forall \mathrm{A}_i,\mathrm{A}_j,\mathrm{A}_k \in FS(V)$ are satisfied then $S(\mathrm{A}_i,\mathrm{A}_j)\geq S(\mathrm{A}_i,\mathrm{A}_k)$ and $S(\mathrm{A}_j,\mathrm{A}_k)\geq S(\mathrm{A}_i,\mathrm{A}_k)$.

Whereas, in the paper by Raha, Hossain and Ghosh (2008) the following definition of the measure of similarity between two fuzzy sets is given:

(1) $S(\mathrm{A}_i,\mathrm{A}_j)=S(\mathrm{A}_j,\mathrm{A}_i)$, $\forall \mathrm{A}_i,\mathrm{A}_j \in FS(V)$;

(2) $0\leq S(\mathrm{A}_i,\mathrm{A}_j)\leq 1$, $\forall \mathrm{A}_i,\mathrm{A}_j \in FS(V)$;

(3) $Sim(\mathrm{A}_i,\mathrm{A}_j)=1$ if and only if $\mathrm{A}_i=\mathrm{A}_j$;

(4) if $\forall \mathrm{A}_i,\mathrm{A}_j,\mathrm{A}_k \in FS(V)$ the conditions $\mathrm{A}_i \subseteq \mathrm{A}_j \subseteq \mathrm{A}_k$ are satisfied then $S(\mathrm{A}_i,\mathrm{A}_k)\leq \min\{S(\mathrm{A}_i,\mathrm{A}_j),S(\mathrm{A}_j,\mathrm{A}_k)\}$;

(5) for two fuzzy sets $\mathrm{A}_i,\mathrm{A}_j$, where $\mathrm{A}_i\neq\emptyset$ and $\mathrm{A}_j\neq\emptyset$, if $S(\mathrm{A}_i,\mathrm{A}_j)=0$ then $\min\{\mu_{\mathrm{A}_i}(v),\mu_{\mathrm{A}_j}(v)\}=0 \quad \forall v\in V$.

In the literature there are proposed lots of measures of similarity as well as some formulas to calculate them. Several authors have proposed similarity measures for fuzzy sets that can be viewed as generalizations of the classical set similarity measures. Then, we would like to present a few selected similarity measures already defined in the literature (Baccour, Alimi and John, 2014; Dubois and Prade, 1980). In general, we can distinguish three main groups of definitions of similarity measures.

Within the first group, similarity measures are based on the basic operations on fuzzy sets, namely on union, intersection and cardinality, and here they are recalled below (Pappis and Karacapilidis, 1993; Weken, Nachtegael and Kerre, 2004; Wang, Meng and Li, 2008): e.g.

$$S_1(\mathrm{A}_i,\mathrm{A}_j)=\frac{card(\mathrm{A}_i\cap \mathrm{A}_j)}{card(\mathrm{A}_i\cup \mathrm{A}_j)}=\frac{\sum_{n=1}^{N}\min\{\mu_{\mathrm{A}_i}(v_n),\mu_{\mathrm{A}_j}(v_n)\}}{\sum_{n=1}^{N}\max\{\mu_{\mathrm{A}_i}(v_n),\mu_{\mathrm{A}_j}(v_n)\}} \quad \text{for } \mathrm{A}_i\cup \mathrm{A}_j\neq\emptyset,$$

$$\text{if } \mathrm{A}_i\cup \mathrm{A}_j=\emptyset \text{ then } S_1(\mathrm{A}_i,\mathrm{A}_j)=1;$$

$$S_2(\mathrm{A}_i,\mathrm{A}_j)=\sum_{n=1}^{N}\frac{\min\{\mu_{\mathrm{A}_i}(v_n),\mu_{\mathrm{A}_j}(v_n)\}}{\max\{\mu_{\mathrm{A}_i}(v_n),\mu_{\mathrm{A}_j}(v_n)\}} \quad \text{for } \mathrm{A}_i\cup \mathrm{A}_j\neq\emptyset, \text{ if } \mathrm{A}_i\cup \mathrm{A}_j=\emptyset \text{ then } S_2(\mathrm{A}_i,\mathrm{A}_j)=1;$$

$$S_3(\mathrm{A}_i,\mathrm{A}_j)=\frac{card(\mathrm{A}_i^C\cap \mathrm{A}_j^C)}{card(\mathrm{A}_i^C\cup \mathrm{A}_j^C)}=\frac{\sum_{n=1}^{N}\min\{(1-\mu_{\mathrm{A}_i}(v_n)),(1-\mu_{\mathrm{A}_j}(v_n))\}}{\sum_{n=1}^{N}\max\{(1-\mu_{\mathrm{A}_i}(v_n)),(1-\mu_{\mathrm{A}_j}(v_n))\}};$$

$$S_4(\mathrm{A}_i,\mathrm{A}_j)=\frac{card(\mathrm{A}_i\cap \mathrm{A}_j)}{\max\{card(\mathrm{A}_i),card(\mathrm{A}_j)\}}=\frac{\sum_{n=1}^{N}\min\{\mu_{\mathrm{A}_i}(v_n),\mu_{\mathrm{A}_j}(v_n)\}}{\max\{\sum_{n=1}^{N}\mu_{\mathrm{A}_i}(v_n),\sum_{n=1}^{N}\mu_{\mathrm{A}_j}(v_n)\}};$$

$$S_5(\mathrm{A}_i,\mathrm{A}_j)=\frac{card(\mathrm{A}_i^C\cap \mathrm{A}_j^C)}{\max\{card(\mathrm{A}_i^C),card(\mathrm{A}_j^C)\}}=\frac{\sum_{n=1}^{N}\min\{(1-\mu_{\mathrm{A}_i}(v_n)),(1-\mu_{\mathrm{A}_j}(v_n))\}}{\max\{\sum_{n=1}^{N}(1-\mu_{\mathrm{A}_i}(v_n)),\sum_{n=1}^{N}(1-\mu_{\mathrm{A}_j}(v_n))\}};$$

$$S_6(A_i,A_j)=\frac{card(A_i \cap A_j)}{\min\{card(A_i),card(A_j)\}}=\frac{\sum_{n=1}^{N}\min\{\mu_{A_i}(v_n),\mu_{A_j}(v_n)\}}{\min\{\sum_{n=1}^{N}\mu_{A_i}(v_n),\sum_{n=1}^{N}\mu_{A_j}(v_n)\}};$$

$$S_7(A_i,A_j)=\frac{card(A_i^C \cap A_j^C)}{\min\{card(A_i^C),card(A_j^C)\}}=\frac{\sum_{n=1}^{N}\min\{(1-\mu_{A_i}(v_n)),(1-\mu_{A_j}(v_n))\}}{\min\{\sum_{n=1}^{N}(1-\mu_{A_i}(v_n)),\sum_{n=1}^{N}(1-\mu_{A_j}(v_n))\}};$$

$$S_8(A_i,A_j)=\frac{1}{N}\sum_{n=1}^{N}\frac{2\cdot\min\{\mu_{A_i}(v_n),\mu_{A_j}(v_n)\}}{\mu_{A_i}(v_n)+\mu_{A_j}(v_n)} \quad \text{for } A_i \cup A_j \neq \emptyset,$$

$$\text{if } A_i \cup A_j = \emptyset \text{ then } S_8(A_i,A_j)=1;$$

$$S_9(A_i,A_j)=\max_{n=1,\ldots,N}\{\min\{\mu_{A_i}(v_n),\ \mu_{A_j}(v_n)\}\}.$$

Other group of similarity measures based on symmetric difference operations on fuzzy sets (Weken, Nachtegael and Kerre, 2004) are presented below.

$$S_{10}(A_i,A_j)=\frac{card((A_i \Delta A_j)^C)}{\max\{card((A_i \Delta A_j)^C),card((A_j \Delta A_i)^C)\}};$$

$$S_{11}(A_i,A_j)=\frac{card((A_i \Delta A_j)^C)}{\min\{card((A_i \Delta A_j)^C),card((A_j \Delta A_i)^C)\}}.$$

Next, similarity measures can be based on distance measures (Pappis and Karacapilidis, 1993; Wang, Meng and Li, 2008), e.g.

$$S_{12}(A_i,A_j)=1-D_{12}(A_i,A_j)=1-\frac{card(A_i \Delta A_j)}{card(A_i \oplus A_j)}=1-\frac{\sum_{n=1}^{N}\left|\mu_{A_i}(v_n)-\mu_{A_j}(v_n)\right|}{\sum_{n=1}^{N}(\mu_{A_i}(v_n)+\mu_{A_j}(v_n))};$$

$$S_{13}(A_i,A_j)=1-D_{13}(A_i,A_j)=1-\frac{1}{N}\sum_{n=1}^{N}\left|\mu_{A_i}(v_n)-\mu_{A_j}(v_n)\right|;$$

$$S_{14}(A_i,A_j)=1-D_{14}(A_i,A_j)=1-\max_{n=1,\ldots,N}\left|\mu_{A_i}(v_n)-\mu_{A_j}(v_n)\right|.$$

Next, let us consider two fuzzy sets A_i and A_j in the set V, and present a short review of the most important properties of the normalized measure of similarity.

Corollary 1. *The following properties of the normalized measure of similarity are true:*

(1) $S(A_i,A_j)=S(A_j,A_i)$;

(2) $S(A_i,A_j)=1$ *if and only if* $A_i=A_j$;

(3) $S(A_i,A_j)=0$ *if and only if* $A_i \cap A_j=\emptyset$;

(4) $S(A_i,A_i^C)=1$ *if and only if* $\forall v \in V,\ \ \mu_{A_i}(v)=0.5$;

(5) $S(A_i,A_i^C)=0$ *if and only if* $\forall v \in V,\ \ \mu_{A_i}(v)=1$ *or* $\forall v \in V,\ \ \mu_{A_i}(v)=0$.

Thus, the similarity measures between two fuzzy sets A_i, A_j are calculated by using values of the membership functions $\mu_{A_i}(v)$ and $\mu_{A_j}(v)$ for every member of set V, so it has been assumed that the similarity is typically symmetric.

2.2. Measure of the fuzzy sets dissimilarity

The distance measure between two sets can also be considered as the measure of dissimilarity between two fuzzy sets and is given as a function $D: FS(V)\times FS(V)\rightarrow[0,1]$ that satisfies certain properties (axioms). In the paper by Fan and Xie (1999), the following definition of *a measure of the distance between two fuzzy sets* is given:

(1) $D(\mathrm{A}_i,\mathrm{A}_j)=D(\mathrm{A}_j,\mathrm{A}_i)$, $\forall \mathrm{A}_i,\mathrm{A}_j\in FS(V)$;

(2) $D(\mathrm{A}_i,\mathrm{A}_i)=0$, $\forall \mathrm{A}_i\in FS(V)$;

(3) $D(\mathrm{A}_i,\mathrm{A}_i^C)=1$;

(4) if $\mathrm{A}_i\subseteq \mathrm{A}_j\subseteq \mathrm{A}_k$ $\forall \mathrm{A}_i,\mathrm{A}_j,\mathrm{A}_k\in FS(V)$ then the inequalities $D(\mathrm{A}_i,\mathrm{A}_j)\le D(\mathrm{A}_i,\mathrm{A}_k)$ and $D(\mathrm{A}_j,\mathrm{A}_k)\le D(\mathrm{A}_i,\mathrm{A}_k)$ are satisfied.

In the literature one can find several measures of distance between the fuzzy sets $\mathrm{A}_i,\mathrm{A}_j\in FS(V)$ as well as some formulas for calculating these measures, let us recall a few the most popular (Kacprzyk J, 1997).

Let us assume, that a measure of distance is defined by a mapping $D: FS(V)\times FS(V)\rightarrow R^+\cup\{0\}$. The *Hamming distance* between the fuzzy sets A_i and A_j, is given by

$$D_{HFS}(\mathrm{A}_i,\mathrm{A}_j)=\sum_{l=1}^{K}\left|\mu_{\mathrm{A}_i}(v_l)-\mu_{\mathrm{A}_j}(v_l)\right|. \tag{2}$$

An extension of the *Hamming distance* between the fuzzy sets A_i and A_j, proposed by Kacprzyk (Kacprzyk, 1976), is given by

$$D_{2HFS}(\mathrm{A}_i,\mathrm{A}_j)=\sum_{l=1}^{K}\left|\mu_{\mathrm{A}_i}(v_l)-\mu_{\mathrm{A}_j}(v_l)\right|^2, \tag{3}$$

while the generalization of the distance (3) (Klir, Fogler, 1988) introduced a one-parameter class of distance functions, is given by

$$D_{rHFS}(\mathrm{A}_i,\mathrm{A}_j)=\left(\sum_{l=1}^{K}\left|\mu_{\mathrm{A}_i}(v_l)-\mu_{\mathrm{A}_j}(v_l)\right|^r\right)^{\frac{1}{r}},\ r\ge1. \tag{4}$$

and the *Euclidean distance* is defined as follows

$$D_{EFS}(\mathrm{A}_i,\mathrm{A}_j)=\sqrt{\sum_{l=1}^{K}((\mu_{\mathrm{A}_i}(v_l)-\mu_{\mathrm{A}_j}(v_l))^2}\ . \tag{5}$$

Now, let us assume, that a measure of distance is defined by the following mapping $D: FS(V)\times FS(V)\rightarrow[0,1]$. Thus, the *normalized Hamming distance* between two fuzzy sets A_i and A_j, is defined by

$$D_{NHFS}(\mathrm{A}_i,\mathrm{A}_j)=\frac{1}{K}\sum_{l=1}^{K}\left|\mu_{\mathrm{A}_i}(v_l)-\mu_{\mathrm{A}_j}(v_l)\right|, \tag{6}$$

while the *normalized Euclidean distance* between two intuitionistic fuzzy sets A_i and A_j, is defined by

$$D_{NEFS}(\mathrm{A}_i,\mathrm{A}_j)=\frac{D_{EFS}(\mathrm{A}_i,\mathrm{A}_j)}{\sqrt{K}}=\sqrt{\frac{1}{K}\sum_{l=1}^{K}((\mu_{\mathrm{A}_i}(v_l)-\mu_{\mathrm{A}_j}(v_l))^2}\ , \tag{7}$$

and for example *the supremum distance* (Nowakowska, 1977), is given by

$$D_{\infty FS}(\mathrm{A}_i, \mathrm{A}_j) = \sup_{v \in V} \left| \mu_{\mathrm{A}_i}(v) - \mu_{\mathrm{A}_j}(v) \right|. \qquad (8)$$

It should be mentioned that in other science area like psychology there are very interesting contributions to description of objects' proximity and the pioneering research by Tversky (1977).

2.3. Tversky's Model

Tversky extensively analyzed the issue of symmetry of relations between objects and he has given an axiomatic theory of objects' similarity. He provided empirical evidence that objects' similarity should not always be treated as a symmetric relation, and then the axiom of symmetry has been released in similarity measures and used in the field of psychology. According to such approach, each considered object is represented by a set of features or attributes. Thus, similarity among two objects is expressed as a linear combination of the measure of their common and distinctive features (Tversky, 1977).

For two sets of attributes A_i and A_j Tversky derives the following family of similarity measures (*Tversky Contrast Model*, *Tversky* in short)

$$Tversky(\mathrm{A}_i, \mathrm{A}_j; \theta, \alpha, \beta) = \theta \cdot f(\mathrm{A}_i \cap \mathrm{A}_j) - \alpha \cdot f(\mathrm{A}_i \setminus \mathrm{A}_j) - \beta \cdot f(\mathrm{A}_j \setminus \mathrm{A}_i) \qquad (9)$$

for some parameters θ, α, $\beta \geq 0$. This model is characterized by different values of the parameters θ, α and β, and by the function $f(.)$.

Another matching function of two objects (sets of attributes) proposed by Tversky (1977) is *the ratio model of objects' similarity*. It is the measure of degree to which two objects (viewed as sets of features) match each other. The matching between objects is expressed as a linear combination of the measures of common and distinctive features. The matching value is normalized to a value ranged from 0 to 1 and *the generalized Tversky's index* is defined as follows

$$Tversky(\mathrm{A}, \mathrm{B}; \alpha, \beta) = \frac{f(\mathrm{A}_i \cap \mathrm{A}_j)}{f(\mathrm{A}_i \cap \mathrm{A}_j) + \alpha \cdot f(\mathrm{A}_i \setminus \mathrm{A}_j) + \beta \cdot f(\mathrm{A}_j \setminus \mathrm{A}_i)} \qquad (10)$$

for some parameters $\alpha, \beta \geq 0$. The interpretation of the respective terms is following. The term $\mathrm{A}_i \cap \mathrm{A}_j$ represents the features that set A_i and set A_j have in common, the term $\mathrm{A}_i \setminus \mathrm{A}_j$ represents the features that set A_i has but set A_j not, the term $\mathrm{A}_j \setminus \mathrm{A}_i$ analogously. The values of parameters α, β determine relative importance of the distinctive terms in the similarity assessment, if $\alpha \neq \beta$ we get a unsymmetrical similarity measure that focuses on the distinctive features.

It easy to notice that the measure of similarity (Eq. 10) has a very general form. Selection of different values of the parameters α and β leads to different types of similarity measures proposed already in the literature. Considering two sets there are two cases, in the first the set A_i can be treated as the reference set while the set A_j is compared to A_i, or the set A_j is treated as the reference set while the set A_i is compared to A_j. The similarity measures in these two cases can be not the same (not symmetrical). Thus, assuming that two sets A_i and A_j have the same status then the measures of similarities are symmetrical. Note, that a symmetry of Tversky's index is satisfied only if $\alpha = \beta$.

The similarity is based on a function $f(.)$ called *the measure of a set*. Finite sets can be measured by the number of elements, i.e., the cardinality of a set, however may be measured by any function that satisfies feature additively, i.e., any function satisfying $f(\mathrm{A}_i \cup \mathrm{A}_j) = f(\mathrm{A}_i) + f(\mathrm{A}_j)$ for disjoint sets A_i and A_j.

In our approach, we define the measure of fuzzy set *f*(.) as the function $f: FS(V) \to R^+ \cup \{0\}$ that assigns a non-negative real number of the cardinality to each finite fuzzy set $A_i \in FS(V)$, i.e., $f(A_i) = card(A_i) = \sum_{v\in V} \mu_{A_i}(v)$, and $card(V) = N$.

In fuzzy space, we can consider different operations of the difference of fuzzy sets. The fuzzy sets' difference represents the feature which the first fuzzy set has but the second fuzzy set has not (Zadeh, 1975). Let us consider $A_i, A_j, A_k \in FS(V)$. A function on $FS(V) \times FS(V)$ is called *a difference of fuzzy sets,* denoted by $\backslash$, and satisfying the following conditions:

(1) if $A_i \subseteq A_j$, then $A_i \backslash A_j = \emptyset$,

(2) if $A_j \subseteq A_k$ then $A_j \backslash A_i \subseteq A_k \backslash A_i$,

$\forall A_i \in FS(V)$, $\forall A_j \in FS(V)$ (Bouchon-Meunier, Rifqi and Bothorel, 1996).

Here, the following three definitions of the sets' difference, namely 1) *the difference between two fuzzy sets* $A_i, A_j \in FS(V)$, denoted by $A_i \backslash A_j$, 2) *the arithmetic subtraction*, denoted by $A_i \backslash_1 A_j$, and 3) *the difference operation*, denoted by $A_i \backslash_2 A_j$, are considered (Bouchon-Meunier, Rifqi and Bothorel, 1996), and then these difference measures will be applied for Tversky's measure (10).

1) *The difference between two fuzzy sets* $A_i, A_j \in FS(V)$, denoted by $A_i \backslash A_j$, can be written as $A_i \backslash A_j = A_i \cap A_j^C$ and the condition $\mu_{A_j^C}(v) = 1 - \mu_{A_j}(v)$, $\forall v \in V$ is satisfied. So, we obtain $card(A_i \backslash A_j) = \sum_{v\in V} \mu_{A_i \backslash A_j}(v) = \sum_{v\in V} \min\{\mu_{A_i}(v), 1 - \mu_{A_j}(v)\}$. This way, Tversky's measure (10) takes a form of the function *Tversky*: $FS(V) \times FS(V) \to [0,1]$ which is defined as follows

$$Tversky(A_i, A_j; \alpha, \beta) =$$

$$= \frac{\sum_{n=1}^{N} \min\{\mu_{A_i}(v_n), \mu_{A_j}(v_n)\}}{\sum_{n=1}^{N} (\min\{\mu_{A_i}(v_n), \mu_{A_j}(v_n)\} + \alpha \cdot \min\{\mu_{A_i}(v_n), 1 - \mu_{A_j}(v_n)\} + \beta \cdot \min\{1 - \mu_{A_i}(v_n), \mu_{A_j}(v_n)\})}. \quad (11)$$

2) *The arithmetic subtraction,* denoted by $A_i \backslash_1 A_j$ is defined as $A_i \backslash_1 A_j = A_i \Theta A_j = \{(v, \mu_{A_i \Theta A_j}) | \mu_{A_i \Theta A_j} = \max\{\mu_{A_i}(v) - \mu_{A_j}(v), 0\}\}$. So we obtain $card(A_i \backslash_1 A_j) = \sum_{v\in V} \mu_{A_i \Theta A_j}(v) = \sum_{v\in V} (\max\{\mu_{A_i}(v) - \mu_{A_j}(v), 0\}) = \sum_{v\in V} (\mu_{A_i}(v) - \mu_{A_i \cap A_j}(v))$. Thus, Tversky's measure, Eq. 10, is rewritten for $V = \{v_1, v_2, \dots, v_N\}$ in the following way

$$Tversky(A_i, A_j; \alpha, \beta) =$$

$$= \frac{\sum_{n=1}^{N} \min\{\mu_{A_i}(v_n), \mu_{A_j}(v_n)\}}{\sum_{n=1}^{N} (\min\{\mu_{A_i}(v_n), \mu_{A_j}(v_n)\} + \alpha \cdot \max\{\mu_{A_i}(v_n) - \mu_{A_j}(v_n), 0\} + \beta \cdot \max\{\mu_{A_j}(v_n) - \mu_{A_i}(v_n), 0\})} =$$

$$= \frac{\sum_{n=1}^{N} \mu_{A_i \cap A_j}(v_n)}{\sum_{n=1}^{N} \mu_{A_i \cap A_j}(v_n) + \alpha \sum_{n=1}^{N} (\mu_{A_i}(v_n) - \mu_{A_i \cap A_j}(v_n)) + \beta \cdot \sum_{n=1}^{N} (\mu_{A_j}(v_n) - \mu_{A_i \cap A_j}(v_n))} =$$

$$=\frac{\sum_{n=1}^{N}\mu_{A_i\cap A_j}(v_n)}{(1-\alpha-\beta)\sum_{n=1}^{N}\mu_{A_i\cap A_j}(v_n)+\alpha\sum_{n=1}^{N}\mu_{A_i}(v_n)+\beta\cdot\sum_{n=1}^{N}\mu_{A_j}(v_n)} \tag{12}$$

Selection of different values for the parameters leads to different types of similarity measures. It is easy to notice that Tversky's similarity measure (Eq. 12) for parameters $\alpha=\beta=1$ can be seen as the Jaccard extended coefficient

$$Tversky(A_i,A_j;1,1)=\frac{\sum_{n=1}^{N}\min\{\mu_{A_i}(v_n),\mu_{A_j}(v_n)\}}{\sum_{n=1}^{N}\max\{\mu_{A_i}(v_n),\mu_{A_j}(v_n)\}}=\frac{\sum_{n=1}^{N}\mu_{A_i\cap A_j}(v_n)}{\sum_{n=1}^{N}(\mu_{A_i}(v_n)+\mu_{A_j}(v_n)-\mu_{A_i\cap A_j}(v_n))}. \tag{13}$$

If $\alpha=\beta=\frac{1}{2}$, then Tversky's similarity measure is reduced to

$$Tversky(A_i,A_j;\frac{1}{2},\frac{1}{2})=\frac{2\cdot\sum_{n=1}^{N}\min\{\mu_{A_i}(v_n),\mu_{A_j}(v_n)\}}{\sum_{n=1}^{N}(\mu_{A_i}(v_n)+\mu_{A_j}(v_n))}=\frac{2\sum_{n=1}^{N}\mu_{A_i\cap A_j}(v_n)}{\sum_{n=1}^{N}\mu_{A_i}(v_n)+\sum_{n=1}^{N}\mu_{A_j}(v_n)}. \tag{14}$$

If $\alpha=1$ and $\beta=0$ then Tversky's similarity measure is rewritten as

$$Tversky(A_i,A_j;1,0)=\frac{\sum_{n=1}^{N}\min\{\mu_{A_i}(v_n),\mu_{A_j}(v_n)\}}{\sum_{n=1}^{N}\mu_{A_i}(v_n)}=\frac{2\sum_{n=1}^{N}\mu_{A_i\cap A_j}(v_n)}{\sum_{n=1}^{N}\mu_{A_i}(v_n)}. \tag{15}$$

If $\alpha=0$ and $\beta=1$, then

$$Tversky(A_i,A_j;0,1)=\frac{\sum_{n=1}^{N}\min\{\mu_{A_i}(v_n),\mu_{A_j}(v_n)\}}{\sum_{n=1}^{N}\mu_{A_j}(v_n)}=\frac{2\sum_{n=1}^{N}\mu_{A_i\cap A_j}(v_n)}{\sum_{n=1}^{N}\mu_{A_j}(v_n)}. \tag{16}$$

3) *The difference operation,* denoted by $A_i\setminus_2 A_j$ is defined as follows

$$\mu_{A_i\setminus_2 A_j}(v)=\begin{cases}\mu_{A_i}(v) & if\ \ \mu_{A_j}(v)=0\\ 0 & if\ \ \mu_{A_j}(v)>0\end{cases},\ \forall v\in V.$$

Let us assume, that *the support of a fuzzy set* A_j, denoted by $Supp(A_j)$, is the set $Supp(A_j)=\{v\in V \mid \mu_{A_j}(v)>0\}$ (i.e., the membership function of the elements is non-zero). Thus, the membership function $\mu_{A_i\setminus_2 A_j}(v)$ can be rewritten as $\mu_{A_i\setminus_2 A_j}(v)=\begin{cases}\mu_{A_i}(v) & if\ \ v\notin Supp(A_j)\\ 0 & if\ \ v\in Supp(A_j)\end{cases}$, $\forall v\in V$.
In this way, Tversky's measure (Eq. 10) can be rewritten, for $V=\{v_1,v_2,...,v_N\}$, in the following way

$$Tversky(\mathrm{A}_i, \mathrm{A}_j; \alpha, \beta) = \frac{\sum_{n=1}^{N} \mu_{\mathrm{A}_i \cap A_j}(v_n)}{\sum_{v \in V} \mu_{\mathrm{A}_i \cap A_j}(v) + \alpha \sum_{v \notin Supp(A_j)} \mu_{\mathrm{A}_i}(v) + \beta \cdot \sum_{v \notin Supp(A_i)} \mu_{\mathrm{A}_j}(v_n)} \tag{17}$$

Thus, in this paper, instead of considering the similarity and distance measures between fuzzy sets, we introduce a new asymmetric measure of proximity between two fuzzy sets, i.e., the measure of perturbation of one fuzzy set by another fuzzy set. Details of the proposed approach are presented in the forthcoming sections.

3. Matching fuzzy sets

Let us assume that there is a collection of fuzzy sets $FS(V)$ in the set V, where V is a finite set of nominal values, $V = \{v_1, v_2, ..., v_N\}$. In the next subsection we will present the detailed conception of the perturbation of fuzzy sets.

3.1. The concept of perturbation of fuzzy sets

Let us consider two fuzzy sets $\mathrm{A}_i, \mathrm{A}_j \in FS(V)$. Attaching the first fuzzy set A_i to the second fuzzy set A_j we treat as the perturbation of the second set by the first set - in other words the fuzzy set A_i perturbs the fuzzy set A_j with some degree. In this way, we defined a novel *concept of perturbation of a fuzzy set* A_j *by a fuzzy set* A_i, which is denoted by $(\mathrm{A}_i \mapsto \mathrm{A}_j)$, and interpreted as a fuzzy set $\mathrm{A}_i \Theta \mathrm{A}_j$, where

$$(\mathrm{A}_i \mapsto \mathrm{A}_j) = \{(v,\ \mu_{\mathrm{A}_i \mapsto \mathrm{A}_j}(v)) |\ \mu_{\mathrm{A}_i \mapsto \mathrm{A}_j}(v) := \max\{\mu_{\mathrm{A}_i}(v) - \mu_{\mathrm{A}_j}(v),\ 0\}, \forall v \in V\}. \tag{18}$$

In the opposite case, the perturbation of the fuzzy set A_i by the fuzzy set A_j is defined in a similar way

$$(\mathrm{A}_j \mapsto \mathrm{A}_i) = \{(v,\ \mu_{\mathrm{A}_j \mapsto \mathrm{A}_i}(v)) |\ \mu_{\mathrm{A}_j \mapsto \mathrm{A}_i}(v) := \max\{\mu_{\mathrm{A}_j}(v) - \mu_{\mathrm{A}_i}(v),\ 0\}, \forall v \in V\}. \tag{19}$$

The above described operations performed on fuzzy sets will be illustrated in the following example.

Example 1. Let us consider the set $V = \{v_1, v_2, v_3\}$ and two fuzzy sets $\mathrm{A}_1, \mathrm{A}_2 \in FS(V)$, where $\mathrm{A}_1 = \{(v_1, 0.4), (v_2, 0), (v_3, 0)\}$ and $\mathrm{A}_2 = \{(v_1, 1), (v_2, 0.1), (v_3, 0.4)\}$. The perturbation of the fuzzy set A_2 by the fuzzy set A_1 is the empty fuzzy set because the following condition $(\mathrm{A}_1 \mapsto \mathrm{A}_2) = \mathrm{A}_1 \Theta \mathrm{A}_2 = \{(v_1, 0), (v_2, 0), (v_3, 0)\} = \emptyset$ is satisfied. On the other hand, the perturbation of the fuzzy set A_1 by the fuzzy set A_2 is the following fuzzy set $(\mathrm{A}_2 \mapsto \mathrm{A}_1) = \mathrm{A}_2 \Theta \mathrm{A}_1 =$ $= \{(v_1, 0.6), (v_2, 0.1), (v_3, 0.4)\}$.

The geometrical interpretation of the proposed concept of the perturbation in 2D space is presented in the forthcoming subsection.

3.2. Geometrical interpretation of fuzzy sets perturbations

Let us assume that $card(V) = 2$, i.e., $V = \{v_1, v_2\}$. Let us consider two fuzzy sets $\mathrm{A}_i, \mathrm{A}_j \in FS(V)$, denoted by $\mathrm{A}_i = \{(v_1, \mu_{\mathrm{A}_i}(v_1)), (v_2, \mu_{\mathrm{A}_i}(v_2))\}$ and $\mathrm{A}_j = \{(v_1, \mu_{\mathrm{A}_j}(v_1)), (v_2, \mu_{\mathrm{A}_j}(v_2))\}$. Each considered fuzzy set can be represented as a point in Fig. 3 and such points have the following coordinates $(\mu_{A_i}(v_1), \mu_{A_i}(v_2))$ and $(\mu_{A_j}(v_1), \mu_{A_j}(v_2))$, respectively. For simplicity, the values $v_1, v_2 \in V$ can be omitted, when it does not lead to confusion in this paper, i.e., the points have coordinates $(\mu^1_{A_i}, \mu^2_{A_i})$ and $(\mu^1_{A_j}, \mu^2_{A_j})$, respectively.

Point (0,0) represents the values v_1 and v_2 fully not belonging to fuzzy set; point (1,1) represents the values v_1 and v_2 fully belonging to fuzzy set; point (1,0) represents the value v_1 fully belonging to the fuzzy set and the value v_2 fully not belonging to the fuzzy set; point (0,1) represents the value v_1 fully not belonging to the fuzzy set and the value v_2 fully belonging to the fuzzy set. The values v_1 and v_2 belonging to the fuzzy set with some degree, can be represented inside the square.

For a fixed fuzzy set A_j there are four areas of possible location of any arbitrary fuzzy set A_i (i.e., the area I, II, III and IV), and for each point related to the fuzzy set A_i lying within specified area some conditions are satisfied, detailed cases are shown in Fig. 3.

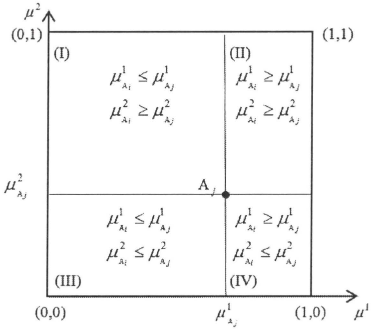


Fig. 3. The conditions for fixed fuzzy set A_j and arbitrary fuzzy sets A_i.

According to Eq. 19, the perturbation of an arbitrary fuzzy set A_i by other fuzzy set A_j is interpreted as a new fuzzy set described as follows:

$$(A_j \mapsto A_i)=\{(v_1, \mu_{A_j\mapsto A_i} := \mu_{A_j}(v_1)-\mu_{A_j\cap A_i}(v_1)), (v_2, \mu_{A_j\mapsto A_i} := \mu_{A_j}(v_2)-\mu_{A_j\cap A_i}(v_2))\} \tag{20}$$

For simplicity, the values $v_1, v_2 \in V$ can be partially omitted and dependence describing fuzzy sets' perturbation can be written in the form $(A_j \mapsto A_i)=\{(v_1, \mu^1_{A_j\mapsto A_i}), (v_2, \mu^2_{A_j\mapsto A_i})\}$. In the opposite case, the perturbation of the fuzzy set A_j by the fuzzy set A_i has the similar definition

$$(A_i \mapsto A_j)=\{(v_1, \mu_{A_i\mapsto A_j} := \mu_{A_i}(v_1)-\mu_{A_i\cap A_j}(v_1)), (v_2, \mu_{A_i\mapsto A_j} := \mu_{A_i}(v_2)-\mu_{A_i\cap A_j}(v_2))\} \tag{21}$$

and in the simplified form is as follows $(A_i \mapsto A_j)=\{(v_1, \mu^1_{A_i\mapsto A_j}), (v_2, \mu^2_{A_i\mapsto A_j})\}$.

The geometrical interpretation of the perturbations of the fixed fuzzy set A_j and the arbitrary fuzzy set A_i are presented in Fig. 4, 5, 6 and 7. Successive figures illustrate different cases related to the area I, II, III or IV, respectively. Within each figure, there are indicated two perturbations, i.e., $(A_j \mapsto A_i)$ and $(A_i \mapsto A_j)$. The used arrows indicate the direction of the perturbations.

Let us describe the distinguished areas in details.

The area I

Let us consider the fixed fuzzy set A_j and arbitrary fuzzy set A_i represented as a point in the area I, i.e., in the selected rectangle, as shown in Fig. 4. For the fuzzy set A_j represented by a point $(\mu^1_{A_j}, \mu^2_{A_j})$ and the fuzzy set A_i represented by a point $(\mu^1_{A_i}, \mu^2_{A_i})$ belonging to the area I, the conditions $\mu^1_{A_j} \geq \mu^1_{A_i}$ and $\mu^2_{A_j} \leq \mu^2_{A_i}$ are satisfied, as shown in Fig. 3.

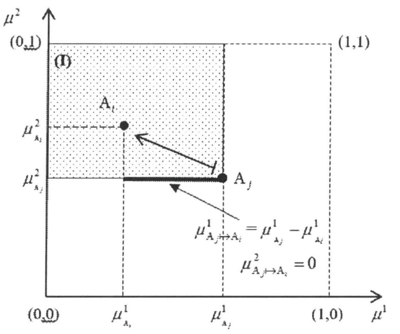
Due to the conditions $\mu^1_{A_j \cap A_i} = \mu^1_{A_i}$ and $\mu^2_{A_j \cap A_i} = \mu^2_{A_j}$, the perturbation $(A_j \mapsto A_i)$ is interpreted as a new fuzzy set described as follows:

$$
\begin{aligned}
(A_j \mapsto A_i) &= \{(v_1, \mu^1_{A_j \mapsto A_i} = \mu^1_{A_j} - \mu^1_{A_j \cap A_i}), (v_2, \mu^2_{A_j \mapsto A_i} = \mu^2_{A_j} - \mu^2_{A_j \cap A_i})\} = \\
&= \{(v_1, \mu^1_{A_j \mapsto A_i} = \mu^1_{A_j} - \mu^1_{A_i}), (v_2, \mu^2_{A_j \mapsto A_i} = \mu^2_{A_j} - \mu^2_{A_j})\} = \\
&= \{(v_1, \mu^1_{A_j \mapsto A_i} = \mu^1_{A_j} - \mu^1_{A_i}), (v_2, \mu^2_{A_j \mapsto A_i} = 0)\}.
\end{aligned}
$$

In the case of the perturbation $(A_i \mapsto A_j)$, a new fuzzy set can be written as follows:

$$
\begin{aligned}
(A_i \mapsto A_j) &= \{(v_1, \mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_j \cap A_i}), (v_2, \mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_j \cap A_i})\} = \\
&= \{(v_1, \mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_i}), (v_2, \mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_j})\} = \\
&= \{(v_1, \mu^1_{A_i \mapsto A_j} = 0), (v_2, \mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_j})\}.
\end{aligned}
$$

A two-dimensional interpretation of the perturbations for the fixed fuzzy set A_j and the arbitrary fuzzy set A_i (according to the above formulas) is presented in Fig. 4.



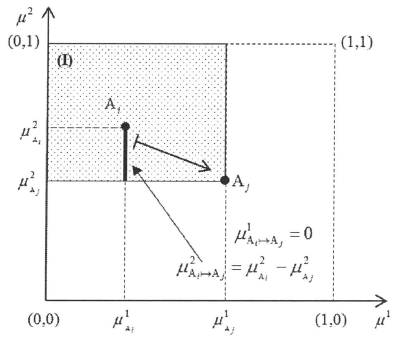


Fig. 4. Geometrical interpretation of the perturbations for fixed fuzzy set A_j and arbitrary fuzzy set A_i.

Analyzing Fig. 4, we can notice that for any fuzzy set $A_i \in FS(V)$ which belongs to the area I, the following conditions $\mu^1_{A_j \mapsto A_i} = \mu^1_{A_j} - \mu^1_{A_i}$, $\mu^2_{A_j \mapsto A_i} = 0$ as well as $\mu^1_{A_i \mapsto A_j} = 0$, $\mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_j}$ are satisfied. The segment marked in bold indicate positive values $\mu^1_{A_j \mapsto A_i}$ and $\mu^2_{A_i \mapsto A_j}$, respectively. For the perturbation $(A_j \mapsto A_i)$, the beginning of the segment is the point $A_j = (\mu^1_{A_j}, \mu^2_{A_j})$, and the end of the segment is the point ($\mu^1_{A_i}, \mu^2_{A_j}$). For another perturbation $(A_i \mapsto A_j)$, the beginning of the segment is the point $A_i=(\mu^1_{A_i}, \mu^2_{A_i})$, and the end of the segment is the point ($\mu^1_{A_i}, \mu^2_{A_j}$).

The area II

Now, let us consider the fixed fuzzy set A_j and arbitrary fuzzy set A_i represented as a point in the area II, i.e., in the selected rectangle, as shown in Fig. 5. For the fuzzy set A_j represented as a point $(\mu^1_{A_j}, \mu^2_{A_j})$ and the fuzzy set A_i represented as a point $(\mu^1_{A_i}, \mu^2_{A_i})$ both belonging to the area II, the conditions $\mu^1_{A_j} \le \mu^1_{A_i}$ and $\mu^2_{A_j} \le \mu^2_{A_i}$ are satisfied, as shown in Fig. 3. Due to the conditions $\mu^1_{A_j \cap A_i} = \mu^1_{A_j}$ and $\mu^2_{A_j \cap A_i} = \mu^2_{A_j}$, the perturbation $(A_j \mapsto A_i)$ is interpreted as a new fuzzy set described as follows:

$$(A_j \mapsto A_i)=\{(v_1, \mu^1_{A_j \mapsto A_i} = \mu^1_{A_j} - \mu^1_{A_j \cap A_i}), (v_2, \mu^2_{A_j \mapsto A_i} = \mu^2_{A_j} - \mu^2_{A_j \cap A_i})\}=$$
$$=\{(v_1, \mu^1_{A_j \mapsto A_i} = \mu^1_{A_j} - \mu^1_{A_j}), (v_2, \mu^2_{A_j \mapsto A_i} = \mu^2_{A_j} - \mu^2_{A_j})\}=\{(v_1, \mu^1_{A_j \mapsto A_i} = 0), (v_2, \mu^2_{A_j \mapsto A_i} = 0)\}.$$

In the case of the perturbation $(A_i \mapsto A_j)$, a new fuzzy set can be written as follows:

$$(A_i \mapsto A_j)=\{(v_1, \mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_j \cap A_i}), (v_2, \mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_j \cap A_i})\}=$$
$$=\{(v_1, \mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_j}), (v_2, \mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_j})\}.$$

A two-dimensional interpretation of the perturbations for the fixed fuzzy set A_j and the arbitrary fuzzy set A_i (according to the above formulas) is presented in Fig. 5.

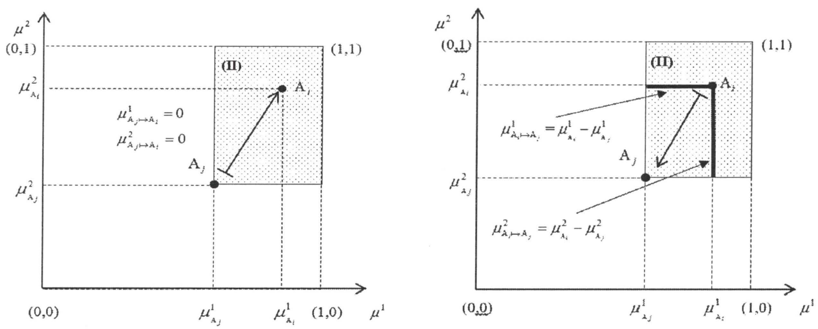


Fig. 5. Geometrical interpretation of the perturbations for fixed fuzzy set A_j and arbitrary fuzzy set A_i.

In the case of the area II, the following conditions $\mu^1_{A_j \mapsto A_i} = 0$, $\mu^2_{A_j \mapsto A_i} = 0$ and $\mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_j}$, $\mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_j}$ are satisfied, as shown in Fig. 5. It is easy to notice, that for the arbitrary fuzzy set A_i belonging to this area, the perturbation $(A_j \mapsto A_i)$ is an empty set. For the perturbation $(A_i \mapsto A_j)$, the segments marked in bold indicate positive values $\mu^1_{A_i \mapsto A_j}$ and $\mu^2_{A_i \mapsto A_j}$, respectively. The beginning of the segment is the point $A_i=(\mu^1_{A_i}, \mu^2_{A_i})$ and the end of the segment is the point ($\mu^1_{A_i}, \mu^2_{A_j}$) and ($\mu^1_{A_j}, \mu^2_{A_i}$), respectively.

The area III

Let us consider the fixed fuzzy set A_j and arbitrary fuzzy set A_i represented as a point in the area III, i.e., in the selected rectangle, as shown in Fig. 6. For the fuzzy set A_j represented as a point $(\mu^1_{\mathrm{A}_j}, \mu^2_{\mathrm{A}_j})$ and the fuzzy set A_i represented as a point $(\mu^1_{\mathrm{A}_i}, \mu^2_{\mathrm{A}_i})$, both belonging to the area III, the conditions $\mu^1_{\mathrm{A}_j} \geq \mu^1_{\mathrm{A}_i}$ and $\mu^2_{\mathrm{A}_j} \geq \mu^2_{\mathrm{A}_i}$ are satisfied, as shown in Fig. 3. Because the conditions $\mu^1_{\mathrm{A}_j \cap \mathrm{A}_i} = \mu^1_{\mathrm{A}_i}$ and $\mu^2_{\mathrm{A}_j \cap \mathrm{A}_i} = \mu^2_{\mathrm{A}_i}$ are satisfied, then the perturbation $(\mathrm{A}_j \mapsto \mathrm{A}_i)$ is interpreted as a new fuzzy set described as follows:

$$(\mathrm{A}_j \mapsto \mathrm{A}_i)=\{(v_1, \mu^1_{\mathrm{A}_j \mapsto \mathrm{A}_i} = \mu^1_{\mathrm{A}_j} - \mu^1_{\mathrm{A}_j \cap \mathrm{A}_i}), (v_2, \mu^2_{\mathrm{A}_j \mapsto \mathrm{A}_i} = \mu^2_{\mathrm{A}_j} - \mu^2_{\mathrm{A}_j \cap \mathrm{A}_i})\}=$$
$$=\{(v_1, \mu^1_{\mathrm{A}_j \mapsto \mathrm{A}_i} = \mu^1_{\mathrm{A}_j} - \mu^1_{\mathrm{A}_i}), (v_2, \mu^2_{\mathrm{A}_j \mapsto \mathrm{A}_i} = \mu^2_{\mathrm{A}_j} - \mu^2_{\mathrm{A}_i})\}.$$

In the case of the perturbation $(\mathrm{A}_i \mapsto \mathrm{A}_j)$, a new fuzzy set can be written as follows:

$$(\mathrm{A}_i \mapsto \mathrm{A}_j)=\{(v_1, \mu^1_{\mathrm{A}_i \mapsto \mathrm{A}_j} = \mu^1_{\mathrm{A}_i} - \mu^1_{\mathrm{A}_j \cap \mathrm{A}_i}), (v_2, \mu^2_{\mathrm{A}_i \mapsto \mathrm{A}_j} = \mu^2_{\mathrm{A}_i} - \mu^2_{\mathrm{A}_j \cap \mathrm{A}_i})\}=$$
$$=\{(v_1, \mu^1_{\mathrm{A}_i \mapsto \mathrm{A}_j} = \mu^1_{\mathrm{A}_i} - \mu^1_{\mathrm{A}_i}), (v_2, \mu^2_{\mathrm{A}_i \mapsto \mathrm{A}_j} = \mu^2_{\mathrm{A}_i} - \mu^2_{\mathrm{A}_i})\}=\{(v_1, \mu^1_{\mathrm{A}_j \mapsto \mathrm{A}_i} = 0), (v_2, \mu^2_{\mathrm{A}_j \mapsto \mathrm{A}_i} = 0)\}.$$

A two-dimensional interpretation of the perturbations for the fixed fuzzy set A_j and the arbitrary fuzzy set A_i (according to the above formulas) is presented in Fig. 6.

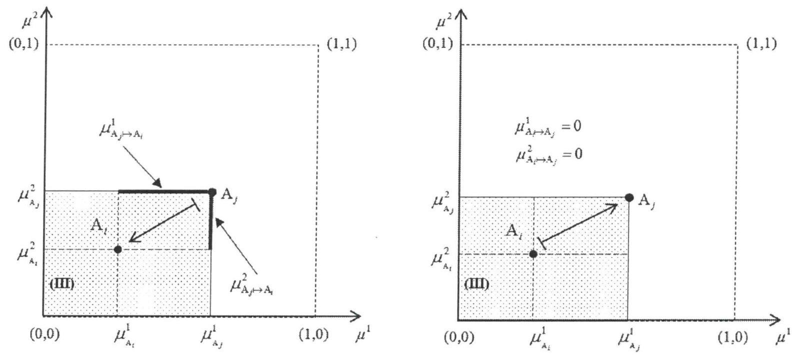


Fig. 6. Geometrical interpretation of the perturbations for fixed fuzzy set A_j and arbitrary fuzzy set A_i

In the case of the area III, the following conditions $\mu^1_{\mathrm{A}_j \mapsto \mathrm{A}_i} = \mu^1_{\mathrm{A}_j} - \mu^1_{\mathrm{A}_i}$, $\mu^2_{\mathrm{A}_j \mapsto \mathrm{A}_i} = \mu^2_{\mathrm{A}_j} - \mu^2_{\mathrm{A}_i}$ and $\mu^1_{\mathrm{A}_i \mapsto \mathrm{A}_j} = 0$, $\mu^2_{\mathrm{A}_i \mapsto \mathrm{A}_j} = 0$ are satisfied, as shown in Fig. 6. For the perturbation $(\mathrm{A}_j \mapsto \mathrm{A}_i)$, the segments marked in bold indicate positive values $\mu^1_{\mathrm{A}_j \mapsto \mathrm{A}_i}$ and $\mu^2_{\mathrm{A}_j \mapsto \mathrm{A}_i}$, respectively. The beginning of the segment is the point $\mathrm{A}_j = (\mu^1_{\mathrm{A}_j}, \mu^2_{\mathrm{A}_j})$ and the end of the segment is the point $(\mu^1_{\mathrm{A}_j}, \mu^2_{\mathrm{A}_i})$ and $(\mu^1_{\mathrm{A}_i}, \mu^2_{\mathrm{A}_j})$, respectively. It is easy to notice, that for the arbitrary fuzzy set A_i belonging to this area, the perturbation $(\mathrm{A}_i \mapsto \mathrm{A}_j)$ is an empty set.

The area IV

Let us consider the fixed fuzzy set A_j and arbitrary fuzzy set A_i represented as a point in the area IV, i.e., in the selected rectangle, as shown in Fig. 7. For the fuzzy set A_j represented as a point

$(\mu^1_{A_j}, \mu^2_{A_j})$ and the fuzzy set A_i represented as a point $(\mu^1_{A_i}, \mu^2_{A_i})$ belonging to the area IV, the conditions $\mu^1_{A_j} \le \mu^1_{A_i}$ and $\mu^2_{A_j} \ge \mu^2_{A_i}$ are satisfied, as shown in Fig. 7. Due to the conditions $\mu^1_{A_j \cap A_i} = \mu^1_{A_j}$ and $\mu^2_{A_j \cap A_i} = \mu^2_{A_i}$, the perturbation $(A_j \mapsto A_i)$ is interpreted as a new fuzzy set described as follows:

$$
\begin{aligned}
(A_j \mapsto A_i) &= \{(v_1, \mu^1_{A_j \mapsto A_i} = \mu^1_{A_j} - \mu^1_{A_j \cap A_i}), (v_2, \mu^2_{A_j \mapsto A_i} = \mu^2_{A_j} - \mu^2_{A_j \cap A_i})\} = \\
&= \{(v_1, \mu^1_{A_j \mapsto A_i} = \mu^1_{A_j} - \mu^1_{A_j}), (v_2, \mu^2_{A_j \mapsto A_i} = \mu^2_{A_j} - \mu^2_{A_i})\} = \\
&= \{(v_1, \mu^1_{A_j \mapsto A_i} = 0), (v_2, \mu^2_{A_j \mapsto A_i} = \mu^2_{A_j} - \mu^2_{A_i})\}.
\end{aligned}
$$

In the case of the perturbation $(A_i \mapsto A_j)$, a new fuzzy set can be rewritten as follows:

$$
\begin{aligned}
(A_i \mapsto A_j) &= \{(v_1, \mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_j \cap A_i}), (v_2, \mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_j \cap A_i})\} = \\
&= \{(v_1, \mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_j}), (v_2, \mu^2_{A_i \mapsto A_j} = \mu^2_{A_i} - \mu^2_{A_i})\} = \\
&= \{(v_1, \mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_j}), (v_2, \mu^2_{A_i \mapsto A_j} = 0)\}.
\end{aligned}
$$

A two-dimensional interpretation of the perturbations for the fixed fuzzy set A_j and the arbitrary fuzzy set A_i (according to the above formulas) is presented in Fig. 7.

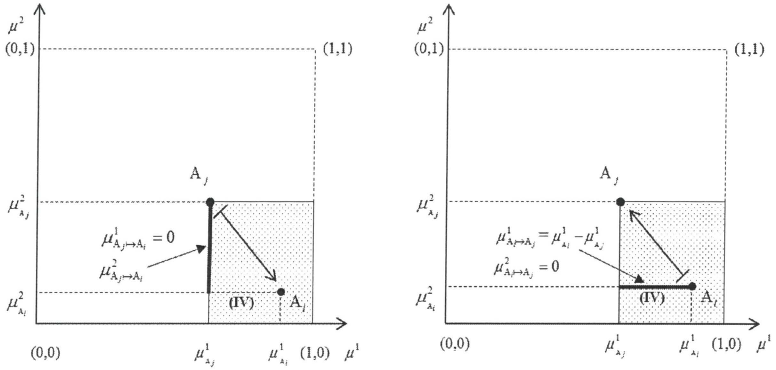


Fig. 7. Geometrical interpretation of the perturbations for fixed fuzzy set A_j and arbitrary fuzzy set A_i.

At the end, the IV area is discussed, namely for the arbitrary fuzzy set $A_i \in FS(V)$ belonging to the area IV, fulfilling the following conditions $\mu^1_{A_j \mapsto A_i} = 0$, $\mu^2_{A_j \mapsto A_i} = \mu^2_{A_j} - \mu^2_{A_i}$ and $\mu^1_{A_i \mapsto A_j} = \mu^1_{A_i} - \mu^1_{A_j}$, $\mu^2_{A_i \mapsto A_j} = 0$ describe cases of this area, as shown in Fig. 7. The segments marked in bold indicate positive values $\mu^2_{A_j \mapsto A_i}$ and $\mu^1_{A_i \mapsto A_j}$, respectively. For the perturbation $(A_j \mapsto A_i)$, the beginning of the segment is the point $A_j = (\mu^1_{A_j}, \mu^2_{A_j})$ and the end of the segment is the point $(\mu^1_{A_j}, \mu^2_{A_i})$. For the perturbation $(A_i \mapsto A_j)$, the beginning of the segment is the point $A_i = (\mu^1_{A_i}, \mu^2_{A_i})$ and the end of the segment is the point $(\mu^1_{A_j}, \mu^2_{A_i})$.

Thus, we claim that the measure of perturbation can be understood as a measure of degree to which two fuzzy sets match each other, however it is important which set is matched to another. Such measures can be calculated by using value of the membership functions for every member of set V. In order to make the sets' perturbation concept more familiar the extended geometrical interpretation in 3D space is provided in Appendix.

In order to range the measure of such sets' perturbation between 0 and 1 we propose some way of normalization within the next subsection.

3.3. Measure of the perturbation of fuzzy sets

Here, we give the following proposal of normalization of the measure of the perturbation of one fuzzy set by another fuzzy set. As it was said, the measure describes changes of one set after adding the other set.

Definition 1. *Let us assume, that* $A_i, A_j \in FS(V)$, $V = \{v_1, v_2, ..., v_N\}$. *The measure of perturbation of the fuzzy set* A_j *by the fuzzy set* A_i, denoted by $Per_{FS}(A_i \mapsto A_j)$, *is defined by a mapping* $Per_{FS} : FS(V) \times FS(V) \to [0,1]$, *in the following manner*:

$$Per_{FS}(A_i \mapsto A_j) = \frac{card(A_i \Theta A_j)}{card(A_i \oplus A_j)}. \qquad (22)$$

In other words, the perturbation of one fuzzy set by another fuzzy set is measured as a ratio of two cardinal numbers of two considered fuzzy sets, one describes the arithmetic subtraction of the fuzzy sets while another the arithmetic addition of these fuzzy sets. In our approach, we define the cardinality of a fuzzy set as a function that assigns to each finite fuzzy set $A_i \in FS(V)$ a non-negative real number, i.e., $card(A_i) = \sum_{v \in V} \mu_{A_i}(v)$. Thus, Eq. (22) can be rewritten as follows

$$Per_{FS}(A_i \mapsto A_j) = \frac{\sum_{n=1}^{N} \max\{\mu_{A_i}(v_n) - \mu_{A_j}(v_n), 0\}}{\sum_{n=1}^{N} (\mu_{A_i}(v_n) + \mu_{A_j}(v_n))} = \frac{\sum_{n=1}^{N} (\mu_{A_i}(v_n) - \mu_{A_i \cap A_j}(v_n))}{\sum_{n=1}^{N} (\mu_{A_i}(v_n) + \mu_{A_j}(v_n))} \qquad (23)$$

In the opposite case, the perturbation of the fuzzy set A_i by the fuzzy set A_j the definition is similar and defined as follows:

$$Per_{FS}(A_j \mapsto A_i) = \frac{card(A_j \Theta A_i)}{card(A_j \oplus A_i)} = \frac{\sum_{n=1}^{N} \max\{\mu_{A_j}(v_n) - \mu_{A_i}(v_n), 0\}}{\sum_{n=1}^{N} (\mu_{A_j}(v_n) + \mu_{A_i}(v_n))} = \frac{\sum_{n=1}^{N} (\mu_{A_j}(v_n) - \mu_{A_j \cap A_i}(v_n))}{\sum_{n=1}^{N} (\mu_{A_j}(v_n) + \mu_{A_i}(v_n))} \qquad (24)$$

Let us illustrate the interpretation of the proposed conception of the perturbation of one fuzzy set by another fuzzy set by the following example.

Example 2. Assume that we have two exemplary fuzzy sets $A_1 = \{(v_1, 0.2), (v_2, 0.2), (v_3, 0.2)\}$ and $A_2 = \{(v_1, 0.3), (v_2, 0.3), (v_3, 0.1)\}$ in the set $V = \{v_1, v_2, v_3\}$. Using Definition 1 we obtain the following values of the measures of perturbation of one fuzzy set by another, namely:

$$Per_{FS}(\mathrm{A}_1 \mapsto \mathrm{A}_2) = \frac{card(\mathrm{A}_1 \Theta \mathrm{A}_2)}{card(\mathrm{A}_1 \oplus \mathrm{A}_2)} = \frac{\sum_{n=1}^{3}(\mu_{\mathrm{A}_1}(v_n) - \mu_{\mathrm{A}_1 \cap \mathrm{A}_2}(v_n))}{\sum_{n=1}^{3}(\mu_{\mathrm{A}_1}(v_n) + \mu_{\mathrm{A}_2}(v_n))} = \frac{1}{13}$$

$$Per_{FS}(\mathrm{A}_2 \mapsto \mathrm{A}_1) = \frac{card(\mathrm{A}_2 \Theta \mathrm{A}_1)}{card(\mathrm{A}_2 \oplus \mathrm{A}_1)} = \frac{\sum_{n=1}^{3}(\mu_{\mathrm{A}_2}(v_n) - \mu_{\mathrm{A}_2 \cap \mathrm{A}_1}(v_n))}{\sum_{n=1}^{3}(\mu_{\mathrm{A}_2}(v_n) + \mu_{\mathrm{A}_1}(v_n))} = \frac{2}{13}$$

which are different, it means this pair of perturbations is not symmetrical.

Now, let us present a short review of the most important features of measure of fuzzy sets perturbation. Again, let us consider that we have two fuzzy sets A_i and A_j in the set *V*.

Corollary 2. *The following properties of the measure of perturbation can be verified.*

(1) $\forall \mathrm{A}_i, \mathrm{A}_j \in FS(V)$, $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) \in [0,1]$;

(2) $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) = 1$ *if and only if* $\mathrm{A}_j = \emptyset$ *and* $\mathrm{A}_i \neq \emptyset$;

(3) $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) = 0$ *if and only if* $\mathrm{A}_i \subseteq \mathrm{A}_j$;

(4) $\forall \mathrm{A}_i, \mathrm{A}_j \in FS(V)$ *we obtain* $0 \le Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) + Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i) \le 1$.

Proof. (1)

1) Let us prove the first inequality $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) \ge 0$. It should be noticed that inequality $\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v) \le \mu_{\mathrm{A}_i}(v)$, $\forall v \in V$, is satisfied, so $\mu_{\mathrm{A}_i}(v) - \mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v) \ge 0$. By Eq. (23) we obtain $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) \ge 0$.

2) Now, we will consider the second inequality, $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) \le 1$. Considering two fuzzy sets A_i and A_j in *V*, it should be noticed that $\mu_{\mathrm{A}_i}(v) - \mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v) \le \mu_{\mathrm{A}_i}(v) + \mu_{\mathrm{A}_j}(v)$, $\forall v \in V$, is satisfied, and then we can obtain the following inequality

$$Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) = \frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n) - \mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n) + \mu_{\mathrm{A}_j}(v_n))} \le \frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n) + \mu_{\mathrm{A}_j}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n) + \mu_{\mathrm{A}_j}(v_n))} = 1.$$

Proof. (2)

1) Let us prove the first implication: *if* $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) = 1$ *then* $A_j = \emptyset$. Let us assume that $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) = 1$. By Eq. (23), the equality $\mu_{\mathrm{A}_i}(v) - \mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v) = \mu_{\mathrm{A}_i}(v) + \mu_{\mathrm{A}_j}(v)$, $\forall v \in V$, is satisfied. It should be noticed that the equality $\mu_{\mathrm{A}_j}(v) = -\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v)$ is satisfied, so $\mu_{\mathrm{A}_j}(v) + \mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v) = 0$, and then we can obtain $A_j = \emptyset$.

2) Consider now the implication: *if* $A_j = \emptyset$, $\mathrm{A}_i \neq \emptyset$ *then* $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) = 1$. Let us assume that $A_j = \emptyset$, thus $\mu_{\mathrm{A}_i}(v) - \mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v) = \mu_{\mathrm{A}_i}(v)$ and $\mu_{\mathrm{A}_i}(v) + \mu_{\mathrm{A}_j}(v) = \mu_{\mathrm{A}_i}(v)$, $\forall v \in V$. This way, we obtained

$$Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j) = \frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n) - \mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n) + \mu_{\mathrm{A}_j}(v_n))} = \frac{\sum_{n=1}^{N}\mu_{\mathrm{A}_i}(v_n)}{\sum_{n=1}^{N}\mu_{\mathrm{A}_i}(v_n)} = 1.$$

Proof. (3)

1) Let us prove the first implication: *if* $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)=0$ *then* $A_i \subseteq A_j$. Let us assume that $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)=0$. By Eq.(23), function $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)$ is non negative, and reaches a minimum if a condition $\mu_{\mathrm{A}_i}(v)-\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v)=0$, $\forall v \in V$, is satisfied. If $\mu_{\mathrm{A}_i}(v)=\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v)$ then condition $A_i \subseteq A_j$ is valid.

2) Consider now the implication: *if* $A_i \subseteq A_j$ *then* $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)=0$. Let us assume that $A_i \subseteq A_j$, thus $\mu_{\mathrm{A}_i}(v) \le \mu_{\mathrm{A}_j}(v)$ and $\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v)=\mu_{\mathrm{A}_i}(v)$, $\forall v \in V$. By Eq. (23), we obtained $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)=0$. This way, the equality $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)=0$ is always verified if $A_i \subseteq A_j$.

Proof. (4)

1) Let us prove the first inequality $0 \le Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)+Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i)$. Due Definition 1 the sum of the following perturbations $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)+Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i)$ is non negative.

2) The right side of inequality can be written as

$$Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)+Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i)=\frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)-\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}+\frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_j}(v_n)-\mu_{\mathrm{A}_j \cap \mathrm{A}_i}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}=$$

$$=\frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)-\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v_n)+\mu_{\mathrm{A}_j}(v_n)-\mu_{\mathrm{A}_j \cap \mathrm{A}_i}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}.$$

It can be noticed that the inequality $\mu_{\mathrm{A}_i}(v)-\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v) \le \mu_{\mathrm{A}_i}(v)$ and $\mu_{\mathrm{A}_j}(v)-\mu_{\mathrm{A}_j \cap \mathrm{A}_i}(v) \le \mu_{\mathrm{A}_j}(v)$, $\forall v \in V$, are satisfied. The right side of inequality can be written as

$$\frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)-\mu_{\mathrm{A}_i \cap \mathrm{A}_j}(v_n)+\mu_{\mathrm{A}_j}(v_n)-\mu_{\mathrm{A}_j \cap \mathrm{A}_i}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))} \le \frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}=1.$$

For a better understanding of the proposed measures of perturbation of one fuzzy set by another fuzzy set some geometrical interpretation is provided in the next subsection.

The problem of similarity degree of two fuzzy sets arises in many theoretical as well as practical considerations. The proposed measure of perturbation of one fuzzy set by another fuzzy set is compared with the selected measures of similarity in the forthcoming section.

4. Particular cases of fuzzy sets perturbation

Let us consider two fuzzy sets $\mathrm{A}_i, \mathrm{A}_j \in FS(V)$ in the set V of nominal values. We are able to prove other interesting properties of the introduced, in this paper, the perturbation of one fuzzy set by another fuzzy set, presented as Corollary 3.

Corollary 3. *The sum of the measures of perturbation of the arbitrary fuzzy set A_j and another fuzzy set A_i satisfies the following equality*

$$Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)+Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i)=\frac{card(\mathrm{A}_i\,\Delta\,\mathrm{A}_j)}{card(\mathrm{A}_i \oplus \mathrm{A}_j)} \tag{25}$$

Proof.

The left side of equation can be rewritten as follows

$$Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)+Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i)=\frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)-\mu_{\mathrm{A}_i\cap\mathrm{A}_j}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}+\frac{\sum_{n=1}^{N}(\mu_{\mathrm{A}_j}(v_n)-\mu_{\mathrm{A}_j\cap\mathrm{A}_i}(v_n))}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}=$$

$$=\frac{\sum_{n=1}^{N}\left|\mu_{\mathrm{A}_i}(v_n)-\mu_{\mathrm{A}_j}(v_n)\right|}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}=\frac{card(\mathrm{A}_i\,\Delta\,\mathrm{A}_j)}{card(\mathrm{A}_i \oplus \mathrm{A}_j)}.$$

It seems to be interesting, that the right side of Eq. (25) determines distance measure $D_{12}(\mathrm{A}_i,\mathrm{A}_j)$ (Pappis and Karacapilidis, 1993; Wang, Meng and Li, 2008). Thus the sum of perturbations of any two fuzzy sets may be treated as a distance between these two fuzzy sets, i.e., $Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)+Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i)=D_{12}(\mathrm{A}_i,\mathrm{A}_j)$. We keep the original index marking from Section 2.1.

Here, for the sake of the paper, let us recall one of the existing definition of a measure of fuzzy sets similarity (Arefi and Taheri, 2014) which has the following form:

$$Sim_{FS}(\mathrm{A}_i,\mathrm{A}_j)=1-\frac{card(\mathrm{A}_i\Delta\mathrm{A}_j)}{card(\mathrm{A}_i \oplus \mathrm{A}_j)}=1-\frac{\sum_{n=1}^{N}\left|\mu_{\mathrm{A}_i}(v_n)-\mu_{\mathrm{A}_j}(v_n)\right|}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}. \tag{26}$$

Equipped with such defined measure of similarity of fuzzy sets we can prove a very interesting property. The property combines the newly introduced measures of perturbation of one fuzzy set by another fuzzy set and Eq. 26, presented as Corollary 4.

Corollary 4. *The sum of the measures of perturbation of the arbitrary fuzzy set A_j and another fuzzy set A_i satisfies the following equality*

$$Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)+Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i)=1-Sim_{FS}(\mathrm{A}_i,\mathrm{A}_j) \tag{27}$$

where $Sim_{FS}(\mathrm{A}_i,\mathrm{A}_j)$ described by formula (26) *is the measure of similarity for two fuzzy sets.*

Proof.

The left side of equation can be rewritten by Corollary 3 as follows

$$Per_{FS}(\mathrm{A}_i \mapsto \mathrm{A}_j)+Per_{FS}(\mathrm{A}_j \mapsto \mathrm{A}_i)=\frac{\sum_{n=1}^{N}\left|\mu_{\mathrm{A}_i}(v_n)-\mu_{\mathrm{A}_j}(v_n)\right|}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))}=1-(1-\frac{\sum_{n=1}^{N}\left|\mu_{\mathrm{A}_i}(v_n)-\mu_{\mathrm{A}_j}(v_n)\right|}{\sum_{n=1}^{N}(\mu_{\mathrm{A}_i}(v_n)+\mu_{\mathrm{A}_j}(v_n))})=$$

$$=1-Sim_{FS}(\mathrm{A}_i,\mathrm{A}_j).$$

One can say that the sum of two coupled measures of perturbation of fuzzy sets, namely the measure of perturbation of the set A_j by the set A_i, denoted by $Per_{FS}(A_i \mapsto A_j)$, and the measure of perturbation of the fuzzy set A_i by the set A_j, denoted by $Per_{FS}(A_j \mapsto A_i)$, gives an equivalent interpretation of dissimilarity of two fuzzy sets A_i, A_j. Eq. (27) can be rewritten in order to obtain equivalent definition of the reduced similarity of fuzzy sets as follows $Sim_{FS}(A_i, A_j) = 1 - (Per_{FS}(A_i \mapsto A_j) + Per_{FS}(A_j \mapsto A_i))$, which based on our idea of the fuzzy sets perturbation measures.

In order to make closer the idea of sets perturbation as well as similarity measures for fuzzy sets we will deliver the following illustrative example.

4.1. Illustrative example

The green sea turtles inhabit tropical and subtropical coastal waters around the world. They are listed as endangered (species faces a very high risk of extinction). Despite this, they are still killed for their meat and eggs and their nesting grounds are disturbed by human encroachment. In the protected environment of a marine zoological park or farm, scientists can examine aspects of sea turtle biology that are difficult or impossible to study in the wild, but green turtle ranching and farming are still in the pioneer stages of development. Samples of a turtle's diet can be retrieved by stomach flushing. Green sea turtles change their diet during their ontogeny from omnivorous diet to herbivorous. This change in food preference is not uncommon, since it has been reported in other reptiles (Hirth, 1971). Green turtles are the only sea turtles that are herbivorous in adulthood. Young green sea turtles are mainly carnivorous and eat invertebrates such as sponges, jellyfish, aquatic insects, young crustaceans, crabs and worms. However, they gradually shift to an entirely vegetarian diet as they exit their juvenile stage. The jaws of green sea turtles are finely serrated, an adaptation fit for their diet of sea grasses and algae.

The problem is to calculate degrees of proximity between the diet of the turtles and compare diet in the study population, formulate some conclusions about age of the turtles.

Let us describe the turtles in the form of fuzzy sets describing their diet. Let us describe examples of the two turtles diet as fuzzy sets, namely set

A =“*diet of the adult green turtle #1* ”,

B =“*diet of the juvenile green turtle #2*”

in the following set *V* ={*sea grasses, algae, seaweed, crustaceans, aquatic insects, jellyfish, crabs, sponges, turtles*}. These two fuzzy sets are described in details as follows:

A ={(*sea grass*, 0.9), (*algae*, 1), (*seaweed*, 0.8), (*crustaceans*, 0), (*aquatic insects*, 0), (*jellyfish*, 0)}, (*crabs*, 0), (sponges, 0), (*turtles*, 0);

B ={(*sea grass*, 0.6), (*algae*, 0.8), (*seaweed*, 0.7), (*crustaceans*, 0.4), (*aquatic insects*, 0.5), (*jellyfish*, 0.3)}, (*crabs*, 0.2), (sponges, 0.4), (*turtles*, 0).

In general, there is not one the best measure for choosing of the proximity between two fuzzy sets. Many known in literature proximity measures were developed specially for considered data and stated problems. Let us consider a few selected measures of sets similarity (Baccour, Alimi and John, 2014): $S_1(A,B)$, $S_3(A,B)$, $S_4(A,B)$, $S_5(A,B)$, $S_9(A,B)$, $S_{12}(A,B)$ and $S_{14}(A,B)$ (we keep the original index marking from Section 2.1). Next, for these two fuzzy sets A and B, the measures of dissimilarity: $D_1(A,B)=1-S_1(A,B)$, $D_3(A,B)=1-S_3(A,B)$, $D_4(A,B)=1-S_4(A,B)$, $D_5(A,B)=1-S_5(A,B)$, $D_9(A,B)=1-S_9(A,B)$, $D_{12}(A,B)=1-S_{12}(A,B)$, $D_{14}(A,B)=1-S_{14}(A,B)$ are calculated. The graphic illustration of calculated measures of dissimilarity and measures of perturbation $Per_{FS}(A \mapsto B)$ and $Per_{FS}(B \mapsto A)$ is shown in Fig. 12.

It is easy to notice, for the considered two fuzzy sets, that different measures of sets proximity generate different respective values, in general. Therefore, it seems that it is impossible to indicate which measure is better in general. It means that there does not exist the best measure for evaluation of proximity between two arbitrary fuzzy sets and the choice depends on the nature of data under consideration.

In Fig. 12, the different values of the proximity measures are shown for the considered two fuzzy sets A and B, of course the values of proximity measures will be different for other two fuzzy sets.

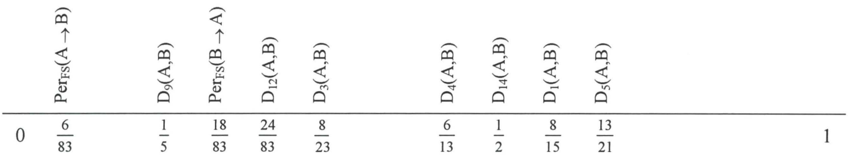


Fig. 12. A graphical illustration of few selected measures

In this example, the proposed measure of perturbation allows us to compare diet of turtles and we can formulate some additional conclusions about their age.

The measure of perturbation of the fuzzy set B by the fuzzy set A is less than the measure of perturbation of the fuzzy set A by the fuzzy set B (6/83 vs. 18/83). So, we can say, that diet of the green turtle *#1* (describe by the fuzzy set A) is less varied than diet of the turtle *#2* (scribe by the fuzzy set B). The diet of the turtle *#2* is more variety diverse and assuredly he is younger than turtle *#1*, what we can't say based on symmetric measures.

Thereby, this short example highlights the differences between the few selected measures (which are obviously symmetric) and the fuzzy set perturbations (which are not necessarily symmetric).

Example 3. Let us consider *assessment of the human behavior*, which we can classify as: *perfect*, *good*, *acceptable* or *wrong*. Their corresponding fuzzy membership we accept as 1, 0.75, 0.5, 0, respectively. Let us consider three fuzzy sets $\mathrm{A}, \mathrm{B}, \mathrm{E} \in FS(\{v\})$, denoted by $\mathrm{A} = \{(v, \mu_{\mathrm{A}}(v))\} = \{(v, 0.75)\}$, $\mathrm{B} = \{(v, \mu_{\mathrm{B}}(v))\} = \{(v, 0.5)\}$ and $\mathrm{E} = \{(v, \mu_{\mathrm{E}}(v))\} = \{(v, 1)\}$.

The problem is to find the most similar fuzzy set to the fuzzy set A, i.e., calculate minimum degree of proximity between the set A and the set B, or E.

Note, that the Hamming distance between A and B will be equal the Hamming distance between A and E, i.e., $D_{HFS}(\mathrm{A}, \mathrm{B}) = |\mu_{\mathrm{A}}(v) - \mu_{\mathrm{B}}(v)| = 0.25$ and $D_{HFS}(\mathrm{A}, \mathrm{E}) = |\mu_{\mathrm{A}}(v) - \mu_{\mathrm{E}}(v)| = 0.25$ showing the same similarity.

According to Eq. 23 and 24 the measures of perturbation of the fuzzy set A by the fuzzy set B, and the fuzzy set E by the fuzzy set A are equal zero, i.e., $Per_{FS}(\mathrm{B} \mapsto \mathrm{A}) = 0$, $Per_{FS}(\mathrm{A} \mapsto \mathrm{E}) = 0$. However, the measure of perturbation of fuzzy set A by the fuzzy set E is less than the measure of the perturbation of the fuzzy set B by the fuzzy set A (1/7 vs. 1/5), i.e., $Per_{FS}(\mathrm{A} \mapsto \mathrm{B}) = \frac{0.25}{1.25} = \frac{1}{5}$ and $Per_{FS}(\mathrm{E} \mapsto \mathrm{A}) = \frac{0.25}{1.75} = \frac{1}{7}$. This way, the sets A and E are less remoteness than the sets A and B, i.e., the set A is more similar to the set E, than to the set B.

Thereby, this short example highlights the difference between the Hamming distance (which is obviously symmetric), and the perturbations (which are not necessarily symmetric).

5. Conclusions

In this paper we propose the new measure of proximity between two fuzzy sets described by nominal values. There are quite many commonly used approaches to describe such measures of sets proximity which are generally based on a distance measure between two fuzzy sets. However here, we propose the new idea of *perturbation of one fuzzy set by another fuzzy set* (and vice versa). This idea is fundamental for introduced definition of *the measure of perturbation* for evaluation of fuzzy sets proximity. Some mathematical properties of the measure of perturbation of fuzzy sets are proved, and the basic property – asymmetrical property - are emphasized. Several examples illustrate the main properties of the fuzzy sets' perturbations as well as the measure of fuzzy sets' perturbations.

In our opinion the methodology presented here is of practical significance because allows to enrich various tasks in data mining, namely it is shown that "direction" of comparing objects can have significant meaning. The proposed measure of perturbation allows us to formulate some additional conclusions relating to proximity of objects due to its asymmetric nature, in general.

We think that the approach presented in this paper can be applied to defining a good measure of remoteness between objects in the case of data represented by nominal values. Of course the presented methodology needs further research.

Appendix A

The geometrical interpretation of the perturbations for fuzzy sets A_j and A_i, are demonstrated below in 3D space. Therefore we will consider a case characterized by $card(V)=3$, i.e., $V=\{v_1,v_2,v_3\}$, so that we will consider the fuzzy set $A_j \in FS(V)$, denoted by $A_j=\{(v_1,\mu_{A_j}(v_1)),(v_2,\mu_{A_j}(v_2)),(v_3,\mu_{A_j}(v_3))\}$, where $\mu_{A_j}: V \to [0,1]$. The fuzzy set A_j can be represented as a point in the three dimensional coordinates space, with coordinates $(\mu_{A_j}(v_1), \mu_{A_j}(v_2), \mu_{A_j}(v_3))$. For simplicity, the elements v_1,v_2,v_3 can be omitted, i.e., the point has coordinates $(\mu^1_{A_j},\mu^2_{A_j},\mu^3_{A_j})$, see Fig. 14.

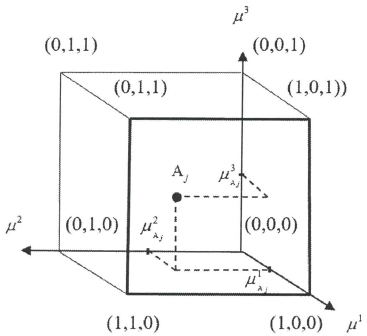


Fig. 14. The fuzzy set A_j represented as a point $(\mu^1_{A_j},\mu^2_{A_j},\mu^3_{A_j})$, marked by●

The point (0,0,0) represents the fuzzy set with the elements v_1, v_2 and v_3 fully not belonging to this fuzzy set; the point (1,1,1) represents the elements v_1, v_2 and v_3 fully belonging to the fuzzy set; the point (1,0,0) represents the element v_1 fully belonging to the fuzzy set and the elements v_2 and v_3 fully not belonging to the fuzzy set; the point (0,1,1) represents the element v_1 fully not belonging to the fuzzy set and the elements v_2 and v_3 fully belonging to the fuzzy set, etc. Any other combination of the values belonging to the fuzzy set with some degree can be represented inside the cube, as shown in Fig. 14.

For the fixed fuzzy set A_j there are eight areas (denoted by the area I, II, ... and VIII), as shown in Fig. 15.

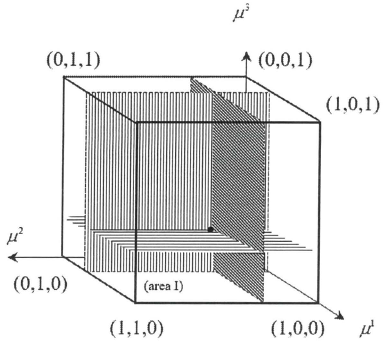


Fig.15. Eight areas which designates a fixed fuzzy set A_j, marked by ●

Let us consider the any arbitrary fuzzy set $A_i \in FS(V)$ represented inside the cube, denoted by $A_i = \{(v_1, \mu_{A_i}(v_1)), (v_2, \mu_{A_i}(v_2)), (v_3, \mu_{A_i}(v_3))\}$, with coordinates $(\mu^1_{A_i}, \mu^2_{A_i}, \mu^3_{A_i})$. Each point related to the fuzzy set A_i lying within specified area and the fuzzy set A_j satisfies some conditions. For example, for the arbitrary fuzzy sets A_i belonging to the area I (see Fig. 15), the conditions $\mu^1_{A_j} \le \mu^1_{A_i}$, $\mu^2_{A_j} \le \mu^2_{A_i}$ and $\mu^3_{A_j} \ge \mu^3_{A_i}$ are satisfied. These conditions can be compared with those from Fig. 3 in Subsection 3.2.

According to Eq. (19), the perturbation of the arbitrary fuzzy set A_i by the fixed fuzzy set A_j is interpreted as a new fuzzy set described as follows:

$$(A_j \mapsto A_i) = \{(v_1, \mu_{A_j \mapsto A_i}(v_1)), (v_2, \mu_{A_j \mapsto A_i}(v_2)), (v_3, \mu_{A_j \mapsto A_i}(v_3))\} =$$
$$= \{(v, \mu_{A_j \mapsto A_i}(v)) \mid \mu_{A_j \mapsto A_i}(v) := \max\{\mu_{A_j}(v) - \mu_{A_i}(v), 0\}, \forall v \in \{v_1, v_2, v_3\}\}.$$

For simplicity, by partial skipping values $v_1, v_2, v_3 \in V$ then the above perturbation can be indicated as follows

$$(A_j \mapsto A_i) = \{(v_1, \mu^1_{A_j \mapsto A_i}), (v_2, \mu^2_{A_j \mapsto A_i}), (v_3, \mu^3_{A_j \mapsto A_i})\}.$$

In the case of the perturbation of the fuzzy set A_j by the fuzzy set A_i, according to Eq. (18), the dependence is similar

$$(A_i \mapsto A_j) = \{(v_1, \mu_{A_i \mapsto A_j}(v_1)), (v_2, \mu_{A_i \mapsto A_j}(v_2)), (v_3, \mu_{A_i \mapsto A_j}(v_3))\} =$$
$$= \{(v, \mu_{A_i \mapsto A_j}(v)) \mid \mu_{A_i \mapsto A_j}(v) := \max\{\mu_{A_i}(v) - \mu_{A_j}(v), 0\}, \forall v \in V\}$$

and

$$(A_i \mapsto A_j) = \{(v_1, \mu^1_{A_i \mapsto A_j}), (v_2, \mu^2_{A_i \mapsto A_j}), (v_3, \mu^3_{A_i \mapsto A_j})\}.$$

The geometrical interpretation of the perturbations for fixed fuzzy set A_j (represented as a point having coordinates $(\mu^1_{A_j}, \mu^2_{A_j}, \mu^3_{A_j})$) and arbitrary fuzzy set A_i (represented as a point having coordinates $(\mu^1_{A_i}, \mu^2_{A_i}, \mu^3_{A_i})$ in the considered area) we will present for selected area III, i.e., in the cubic marked by dotted line, as shown in Fig. 16.

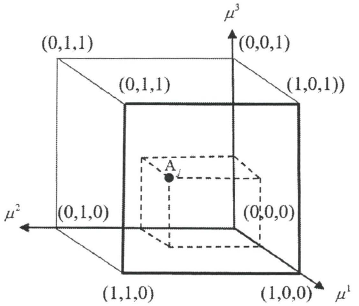


Fig.16. The areas III of possible location of arbitrary fuzzy sets A_i, the cubic marked by dotted line

For the fixed fuzzy set A_j represented as a point $(\mu^1_{A_j}, \mu^2_{A_j}, \mu^3_{A_j})$ and the arbitrary fuzzy set A_i represented as a point $(\mu^1_{A_i}, \mu^2_{A_i}, \mu^3_{A_i})$ belonging to the area III, the conditions $\mu^1_{A_j} \geq \mu^1_{A_i}$, $\mu^2_{A_j} \geq \mu^2_{A_i}$ and $\mu^3_{A_j} \geq \mu^3_{A_i}$ are satisfied. These conditions can be compared with those from Fig. 3 in Subsection 3.2.

Let us consider the following perturbation $(A_j \mapsto A_i)$. Due to the conditions $\mu^1_{A_j \cap A_i} = \mu^1_{A_i}$, $\mu^2_{A_j \cap A_i} = \mu^2_{A_i}$ and $\mu^3_{A_j \cap A_i} = \mu^3_{A_i}$ are satisfied, the perturbation $(A_j \mapsto A_i)$ is interpreted as a new fuzzy set described as follows:

$$(A_j \mapsto A_i) = \{(v_1, \mu^1_{A_j \mapsto A_i} := \mu^1_{A_j} - \mu^1_{A_j \cap A_i}), (v_2, \mu^2_{A_j \mapsto A_i} := \mu^2_{A_j} - \mu^2_{A_j \cap A_i}), (v_3, \mu^3_{A_j \mapsto A_i} := \mu^3_{A_j} - \mu^3_{A_j \cap A_i})\} =$$
$$= \{(v_1, \mu^1_{A_j \mapsto A_i} := \mu^1_{A_j} - \mu^1_{A_i}), (v_2, \mu^2_{A_j \mapsto A_i} := \mu^2_{A_j} - \mu^2_{A_i}), (v_3, \mu^3_{A_j \mapsto A_i} := \mu^3_{A_j} - \mu^3_{A_i})\}.$$

A three-dimensional interpretation of the perturbation $(A_j \mapsto A_i)$ for the fixed fuzzy set A_j and the arbitrary fuzzy set A_i (according to the above formulas) is presented on the left side of the Fig. 17. The arrow indicates direction of the perturbation, and the segments marked in red indicate positive values of the membership functions of the perturbation. The dotted line space from the left side of Fig. 17 is enlarged and shown in details on the right side of Fig. 17. Fig. 17 is a counterpart picture of Fig. 6 in Subsection 3.2.

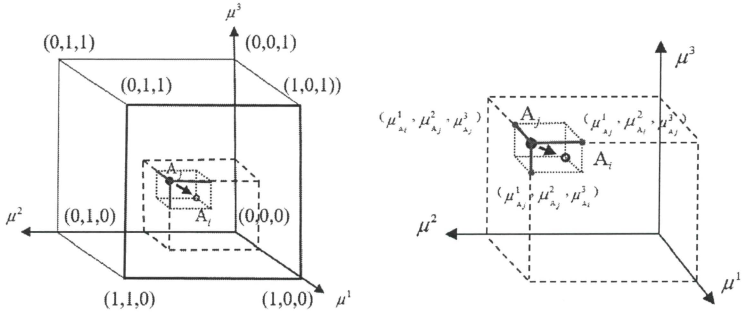


Fig.17. Geometrical interpretation of the perturbation $(A_j \mapsto A_i)$, fuzzy sets A_j and A_i are marked by ● and **o**.

Thus, the following conditions $\mu^1_{A_j\mapsto A_i}=\mu^1_{A_j}-\mu^1_{A_i}$, $\mu^2_{A_j\mapsto A_i}=\mu^2_{A_j}-\mu^2_{A_i}$ and $\mu^3_{A_j\mapsto A_i}=\mu^3_{A_j}-\mu^3_{A_i}$ are satisfied, i.e., the segments marked in red indicate positive values $\mu^1_{A_j\mapsto A_i}$, $\mu^2_{A_j\mapsto A_i}$ and $\mu^3_{A_j\mapsto A_i}$, respectively. The beginning of the segments is the point $A_j=(\mu^1_{A_j},\mu^2_{A_j},\mu^3_{A_j})$ and the ends of the segments are the points $(\mu^1_{A_i},\mu^2_{A_j},\mu^3_{A_j})$, $(\mu^1_{A_j},\mu^2_{A_i},\mu^3_{A_j})$ and $(\mu^1_{A_j},\mu^2_{A_j},\mu^3_{A_i})$, respectively (marked by • in Fig. 17).

Let us consider the perturbation $(A_i \mapsto A_j)$. Because the conditions $\mu^1_{A_j\cap A_i}=\mu^1_{A_i}$, $\mu^2_{A_j\cap A_i}=\mu^2_{A_i}$ and $\mu^3_{A_j\cap A_i}=\mu^3_{A_i}$ are satisfied, a new fuzzy set can be rewritten as follows:

$$(A_i \mapsto A_j)=\{(v_1,\mu^1_{A_i\mapsto A_j}:=\mu^1_{A_i}-\mu^1_{A_j\cap A_i}),(v_2,\mu^2_{A_i\mapsto A_j}:=\mu^2_{A_i}-\mu^2_{A_j\cap A_i}),(v_3,\mu^3_{A_i\mapsto A_j}:=\mu^3_{A_i}-\mu^3_{A_j\cap A_i})\}=$$
$$=\{(v_1,\mu^1_{A_i\mapsto A_j}:=\mu^1_{A_i}-\mu^1_{A_i}),(v_2,\mu^2_{A_i\mapsto A_j}:=\mu^2_{A_i}-\mu^2_{A_i}),(v_3,\mu^3_{A_i\mapsto A_j}:=\mu^3_{A_i}-\mu^3_{A_i})\}=$$
$$=\{(v_1,\mu^1_{A_i\mapsto A_j}=0),(v_2,\mu^2_{A_i\mapsto A_j}=0),(v_3,\mu^3_{A_i\mapsto A_j}=0)\}.$$

The three-dimensional interpretation of the perturbation $(A_i \mapsto A_j)$ for the fixed fuzzy set A_j and the arbitrary fuzzy set A_i (according to the above formulas) is presented on the left side of Fig. 18. The arrow indicates direction of the perturbation. The right side of Fig. 18 shows the enlarged dotted line space from the left side of Fig. 18.

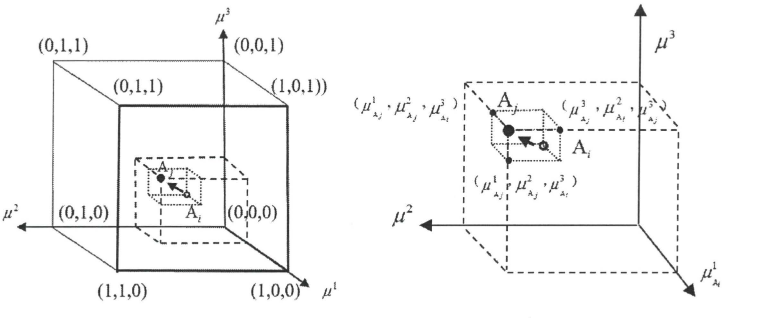


Fig.18. Geometrical interpretation of the perturbation $(A_i \mapsto A_j)$, A_j and A_i are marked by ● and ○.

It is easy to notice, that the perturbation $(A_i \mapsto A_j)$ is an empty set, i.e., the conditions $\mu^1_{A_i\mapsto A_j}=0$, $\mu^2_{A_i\mapsto A_j}=0$ and $\mu^3_{A_i\mapsto A_j}=0$ are satisfied.

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