FEM MODELING OF FGM THERMO-MECHANICAL CYLINDER

D. Szubartowski, S. Hernik, and A. Ganczarski

Institute of Applied Mechanics, Cracow University of Technology, Kraków, Poland

e-mail: dszubartowski@pk.edu.pl, hernik@mech.pk.edu.pl, artur.ganczarski@pk.edu.pl

1. Introduction

Functionally graded materials (FGMs) provide thermal insulation and mechanical toughness at high temperature by varying the composition of thermal conductivity coefficient, thermal expansion coefficient and Young's modulus from high temperature side to low temperature side continuously and simultaneously by removing the discontinuity of layered structures.

When the classical FEM based on homogeneous elements is used for FGMs, the material properties stay the same for all integration points belonging to one finite element. This means that material properties may vary in a piecewise continuous manner, from one element to the other and a unique possibility to model FGM structure is approximation by use of appropriately fine mesh. On the other hand, a too coarse mesh may lead to unrealistic stresses at the interface between the subsequent sub-layers. To overcome this difficulty a special graded element has been introduced by Kim and Paulino [3] to discretize FGM properties. The material properties at Gauss quadrature points are interpolated there from the nodal material properties by the use of isoparametric interpolation functions. Contrary to the classical FEM formulation, the stiffness matrix of an element is expressed by the integral in which constitutive matrix is a function of the coordinates.

2. Formulation of FGM thermo-elastic cylinder

The system of equations of uncoupled thermo-elasticity expressed by stress function formulation [6] is as follows

$$\begin{aligned} &\mathcal{F}_1[\theta(\rho)] = \theta'' + (\tfrac{\lambda'}{\lambda} + \tfrac{1}{\rho})\theta' = 0 \\ &\mathcal{F}_2[F(\rho)] = F'' + (\tfrac{1}{\rho} - \tfrac{E'}{E})F' + (\tfrac{\nu}{1-\nu}\tfrac{E'}{E} - \tfrac{1}{\rho})\tfrac{F}{\rho} = -\tfrac{AE}{1-\nu}(\alpha\theta)' \end{aligned} \tag{1}$$

where: θ stands for increment of temperature, F denotes stress function, λ, E, α are coefficient of thermal conductivity, Young's modulus and coefficient of thermal expansion, respectively, whereas format of constant A depends on plane strain state type according to following scheme: in case of plane strain state imposed on both mechanical and thermal deformation $A = 1$, whereas in case of plane strain state imposed on mechanical deformation only $A = \frac{1}{1+\nu}$. The Poisson ratio ν is not subjected to any change.

All thermo-mechanical properties of the FGM such as α, λ and E are arbitrary functions of radius ρ, subsequent global approximations of which are presented in Table 1.

Voigt [5]	$f_{\mathrm{V}} = (f_{\mathrm{m}} - f_{\mathrm{c}})\frac{\rho - r_1}{r_{\mathrm{f}} - r_1} + f_{\mathrm{c}}$	
Reuss [5]	$f_{\mathrm{R}} = \dfrac{f_{\mathrm{m}} f_{\mathrm{c}}}{f_{\mathrm{m}}(1 - \frac{\rho - r_1}{r_{\mathrm{f}} - r_1}) + f_{\mathrm{c}}\frac{\rho - r_1}{r_{\mathrm{f}} - r_1}}$	
Hashin-Shtrikman $\zeta = \frac{1+\nu}{3(1-\nu)}$ [1,2]	$f^{+}_{\mathrm{HS}} = \dfrac{3f_{\mathrm{m}}f_{\mathrm{c}} + 2f_{\mathrm{m}}(f_{\mathrm{m}} - f_{\mathrm{c}})\frac{\rho - r_1}{r_{\mathrm{f}} - r_1}}{3f_{\mathrm{m}} + (f_{\mathrm{c}} - f_{\mathrm{m}})\frac{\rho - r_1}{r_{\mathrm{f}} - r_1}}$ $E^{+}_{\mathrm{HS}} = E_{\mathrm{c}} + \dfrac{(E_{\mathrm{m}} - E_{\mathrm{c}})\frac{\rho - r_1}{r_{\mathrm{f}} - r_1}}{1 + \zeta(1 - \frac{\rho - r_1}{r_{\mathrm{f}} - r_1})(\frac{E_{\mathrm{m}}}{E_{\mathrm{c}}} - 1)}$	$f^{-}_{\mathrm{HS}} = \dfrac{f_{\mathrm{m}}f_{\mathrm{c}} + 2f_{\mathrm{c}}^2 + 2f_{\mathrm{m}}(f_{\mathrm{m}} - f_{\mathrm{c}})\frac{\rho - r_1}{r_{\mathrm{f}} - r_1}}{2f_{\mathrm{c}} + f_{\mathrm{m}} + (f_{\mathrm{c}} - f_{\mathrm{m}})\frac{\rho - r_1}{r_{\mathrm{f}} - r_1}}$ $E^{-}_{\mathrm{HS}} = E_{\mathrm{m}} + \dfrac{(E_{\mathrm{c}} - E_{\mathrm{m}})(1 - \frac{\rho - r_1}{r_{\mathrm{f}} - r_1})}{1 + \zeta\frac{\rho - r_1}{r_{\mathrm{f}} - r_1}(\frac{E_{\mathrm{c}}}{E_{\mathrm{m}}} - 1)}$

Table 1: Approximations functions of thermo-mechanical properties α, λ and E.

3. FEM formulation

From the FEM point of view both the Fourier equation (1_1) and the mechanical state equation (1_2) are treated as differential equations of variable coefficients describing isotropic material which inhomogeneity is subjected to smooth change form one element to other due to the global FGM approximation functions shown in Table 1). In order to save Euler's type of both equations (1) the following material inhomogeneity shape functions, that approximate the global FGM functions presented in Table 1 at a level of element, are assumed

$$\lambda^{(e)} = \lambda_0 \rho^n \qquad E^{(e)} = E_0 \rho^m \qquad \alpha^{(e)} = \alpha_0 \rho^s \tag{2}$$

Transformation of Eq. (1) to FEM form is done by discretization, use of the Galerkin weighted residual process [3, 4, 7]

$$\int_\Gamma [\mathcal{F}_{1,2}(\phi) + Q] W d\rho = 0 \tag{3}$$

and approximation of unknown function by N_i global shape functions $\phi = \sum_{i=1}^{n} N_i \phi_i$. The weighting functions W_i corresponding to node i are conveniently chosen such that $W_i = N_i$, hence substituting for ϕ and W in Eq. (3) and assembling all elements results in $H_{ij}\Phi_j + Q_i = 0$, in which typical element components of the element stiffness matrices $h_{ij}^{(e)}$ and the element nodal force vectors $q_i^{(e)}$ are

$$\begin{aligned}
h_{ij}^{(e)} &= \frac{\lambda_0}{n+2} \frac{r_{k+1}^{n+2} - r_k^{n+2}}{R^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad q_i^{(e)} = \lambda_0 \begin{bmatrix} -r_k^{n+1} \theta'(r_k) \\ r_{k+1}^{n+1} \theta'(r_{k+1}) \end{bmatrix} \\
h_{ij}^{(e)} &= \begin{bmatrix} \left(1 - \frac{m\nu}{1-\nu}\right) \frac{R}{3\overline{r}} + \frac{\overline{r}}{R} - \frac{m}{2} & \left(1 - \frac{m\nu}{1-\nu}\right) \frac{R}{\overline{r}} \left(\frac{1}{6} - \frac{r_k r_{k+1}}{R^2}\right) - \frac{\overline{r}}{R} + \frac{m}{2} \\ \left(1 - \frac{m\nu}{1-\nu}\right) \frac{R}{\overline{r}} \left(\frac{1}{6} - \frac{r_k r_{k+1}}{R^2}\right) - \frac{\overline{r}}{R} - \frac{m}{2} & \left(1 - \frac{m\nu}{1-\nu}\right) \frac{R}{3\overline{r}} + \frac{\overline{r}}{R} + \frac{m}{2} \end{bmatrix} \\
q_i^{(e)} &= \begin{bmatrix} -r_k F'(r_k) - A \frac{\alpha_0 E_0}{1-\nu} \left(\frac{r_{k+1}^{m+s+3} - r_k^{m+s+3}}{m+s+3} - r_{k+1} \frac{r_{k+1}^{m+s+2} - r_k^{m+s+2}}{m+s+2} \right) \frac{\theta_{k+1} - \theta_k}{R^2} \\ r_{k+1} F'(r_{k+1}) + A \frac{\alpha_0 E_0}{1-\nu} \left(\frac{r_{k+1}^{m+s+3} - r_k^{m+s+3}}{m+s+3} - r_k \frac{r_{k+1}^{m+s+2} - r_k^{m+s+2}}{m+s+2} \right) \frac{\theta_{k+1} - \theta_k}{R^2} \end{bmatrix}
\end{aligned} \tag{4}$$

for thermal and mechanical problems, respectively, and $\overline{r}$ stands for mid radius of an element. For the case of a two-node element with a linear variation of ϕ the shape functions are $N_1^{(e)} = (r_{k+1} - \rho)/R$ and $N_2^{(e)} = (\rho - r_k)/R$, where R is the length of an element. Symbols r_k and r_{k+1} refer to the radii of first and second node of an element, respectively.

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