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Shape optimization problem for coupling of elasticity and Navier-Stokes equations

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Chapter 2

Linearization of the Navier-Stokes equation

2.1 Variational formulation

Let us consider now the variational formulation of the coupled problem. We rewrite the equations (1.10)-(1.11) in the following form for each coordinate of the vector w:

$$\begin{aligned} -\nu\Delta_{\mathrm{y}}\mathrm{w}_1 + \mathbf{w}\nabla_{\mathrm{y}}\mathrm{w}_1 + \nabla_{\mathrm{y}}\mathrm{p}\mathrm{e}_1 &= 0 &(1.1)\\ -\nu\Delta_{\mathrm{y}}\mathrm{w}_2 + \mathbf{w}\nabla_{\mathrm{y}}\mathrm{w}_2 + \nabla_{\mathrm{y}}\mathrm{p}\mathrm{e}_2 &= 0 &(1.2)\\ \mathrm{div}_{\mathrm{y}}\mathbf{w} &= 0 &(1.3)\end{aligned}$$

We associate a test function ξ_1 with w_1, a test function ξ_2 with w_2 and η with p, we denote by $\mathbf{n} = [\mathrm{n}_1, \mathrm{n}_1]^\top$. We recall the boundary conditions for functions w_1 and w_2:

$$\begin{aligned} \mathrm{w}_1|_{\Gamma_{\text{wall}}} &= 0, & \mathrm{w}_2|_{\Gamma_{\text{wall}}} &= 0,\\ \mathrm{w}_1|_{\Gamma_{\text{in}}} &= g_1, & \mathrm{w}_2|_{\Gamma_{\text{in}}} &= g_2,\\ \mathrm{w}_1|_{\Gamma_{\text{int}(u)}} &= 0, & \mathrm{w}_2|_{\Gamma_{\text{int}(u)}} &= 0,\\ \frac{\partial \mathrm{w}_1}{\partial \mathbf{n}} + \mathrm{p}\cdot\mathrm{n}_1 &= 0, & \tfrac{\partial \mathrm{w}_2}{\partial \mathbf{n}} + \mathrm{p}\cdot\mathrm{n}_2 &= 0,\end{aligned}$$

and we set the boundary conditions for test functions ξ_1, ξ_2 as:

$$\begin{aligned} \xi_1 &= 0 \text{ on } \Gamma_{\text{wall}} \cup \Gamma_{\text{in}} \cup \Gamma_{\text{int}}(u),\\ \xi_2 &= 0 \text{ on } \Gamma_{\text{wall}} \cup \Gamma_{\text{in}} \cup \Gamma_{\text{int}}(u).\end{aligned}$$

Multiplying the equation (1.1) by the test function ξ_1 we get:

$$-\int\limits_{\Omega_2(u)} (\xi_1 \nu \Delta_y \mathrm{w}_1 - \xi_1 \mathbf{w} \nabla_y \mathrm{w}_1 - \xi_1 \nabla_y \mathrm{p} e_1) = 0, \qquad (1.4)$$

thus

$$-\int\limits_{\Gamma_{\text{wall}} \cup \Gamma_{\text{in}} \cup \Gamma_{\text{int}}(u)} \nu \xi_1 (\nabla_y \mathrm{w}_1 \cdot \mathbf{n}) - \int\limits_{\Gamma_{\text{out}}} \nu \xi_1 (\nabla_y \mathrm{w}_1 \cdot \mathbf{n}) + \int\limits_{\Omega_2(u)} \nu (\nabla_y \xi_1 \cdot \nabla_y \mathrm{w}_1)$$
$$+ \int\limits_{\Omega_2(u)} \xi_1 (\mathbf{w} \cdot \nabla_y \mathrm{w}_1) - \int\limits_{\partial \Omega_2(u)} \xi_1 \mathrm{p} \mathrm{n}_1 + \int\limits_{\Omega_2(u)} \mathrm{p} \xi_{1/y_1} = 0$$

>From the boundary conditions for ξ_1 and w_1 on Γ_{out} we get

$$-\int\limits_{\Gamma_{\text{out}}} \nu \xi_1 \mathrm{p} \mathrm{n}_1 + \int\limits_{\Omega_2(u)} \nu (\nabla_y \xi_1 \cdot \nabla_y \mathrm{w}_1)$$
$$+ \int\limits_{\Omega_2(u)} \xi_1 (\mathbf{w} \cdot \nabla_y \mathrm{w}_1) - \int\limits_{\Gamma_{\text{out}}} \xi_1 \mathrm{p} \mathrm{n}_1 + \int\limits_{\Omega_2(u)} \mathrm{p} \cdot \xi_{1/y_1} = 0$$

so

$$\int\limits_{\Omega_2(u)} \nu (\nabla_y \xi_1 \cdot \nabla_y \mathrm{w}_1) + \int\limits_{\Omega_2(u)} \xi_1 (\mathbf{w} \cdot \nabla_y \mathrm{w}_1) + \int\limits_{\Omega_2(u)} \mathrm{p} \cdot \xi_{1/y_1} + (\nu - 1) \int\limits_{\Gamma_{\text{out}}} \xi_1 \mathrm{p} \mathrm{n}_1 = 0 \qquad (1.5)$$

2.2 Linearisation

Suppose that $\mathrm{T} : \Omega(0) \to \Omega(u)$ is given by $\mathrm{T}(x) = x + \varphi(x)$, where

$$\varphi(x) = \begin{bmatrix} \varphi_1(x_1, x_2) \\ \varphi_2(x_1, x_2) \end{bmatrix} \qquad (1.6)$$

We denote by $\varphi_{i/j} = \dfrac{\partial \varphi_i}{\partial x_j}$ partial derivatives of function φ, so we have

$$D\varphi = \begin{bmatrix} \varphi_{1/1} & \varphi_{1/2} \\ \varphi_{2/1} & \varphi_{2/2} \end{bmatrix} \qquad (1.7)$$

and the derivative of the mapping T can be written as

$$DT = \begin{bmatrix} 1+\varphi_{1/1} & \varphi_{1/2} \\ \varphi_{2/1} & 1+\varphi_{2/2} \end{bmatrix} \tag{1.8}$$

which means that $DT = I + D\varphi$ and is such, that $\|D\varphi\| << 1$, and $\varphi = 0$ on $\Gamma_{\text{wall}} \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$.

We are looking for a linearised expression of operator $A(\varphi)$ given by:

$$A(\varphi) = J(\varphi) \cdot (DT)^{-1}(DT)^{-T} \tag{1.9}$$

where $J(\varphi)$ is a determinant of the matrix DT, $(DT)^{-1}$ is an inverse of DT and $(DT)^{-T}$ is transpose of DT^{-1}. In linearisation we ignore all terms $(\varphi_{i/j})^k$ for $k > 1$, then the linearised form of determinant $J(\varphi)$ is obtained as follows:

$$\begin{aligned} J(\varphi) = \det(DT) &= \det \begin{bmatrix} 1+\varphi_{1/1} & \varphi_{1/2} \\ \varphi_{2/1} & 1+\varphi_{2/2} \end{bmatrix} \\ &= (1+\varphi_{1/1})(1+\varphi_{2/2}) - \varphi_{1/2}\varphi_{2/1} \\ &\cong 1 + \varphi_{1/1} + \varphi_{2/2} = 1 + \mathrm{div}\varphi \end{aligned} \tag{1.10}$$

Since the inverse $(DT)^{-1}$ and the transpose $(DT)^{-T}$ are given by

$$DT^{-1} = \frac{1}{J} \begin{bmatrix} 1+\varphi_{2/2} & -\varphi_{1/2} \\ -\varphi_{2/1} & 1+\varphi_{1/1} \end{bmatrix} \tag{1.11}$$

and

$$DT^{-T} = \frac{1}{J} \begin{bmatrix} 1+\varphi_{2/2} & -\varphi_{2/1} \\ -\varphi_{1/2} & 1+\varphi_{1/1} \end{bmatrix} \tag{1.12}$$

then we get the following linearization for the multiplication of (1.11) and (1.12) we get

$$\begin{aligned} (DT)^{-1}(DT)^{-T} &\cong \frac{1}{J^2} \begin{bmatrix} 1+2\varphi_{2/2} & -\varphi_{2/1}-\varphi_{1/2} \\ -\varphi_{2/1}-\varphi_{1/2} & 1+2\varphi_{1/1} \end{bmatrix} \\ &= \frac{1}{J^2}(I + B(\varphi)) \end{aligned} \tag{1.13}$$

with

$$B(\varphi) = \begin{bmatrix} 2\varphi_{2/2} & -\varphi_{2/1}-\varphi_{1/2} \\ -\varphi_{2/1}-\varphi_{1/2} & 2\varphi_{1/1} \end{bmatrix} \tag{1.14}$$

Thus

$$A(\varphi) = J \cdot \frac{1}{J^2}(I + B(\varphi)) = \frac{1}{J}(I + B(\varphi)) \tag{1.15}$$

But since

$$\frac{1}{1 + \text{div}\varphi} \cong 1 - \text{div} \tag{1.16}$$

then

$$A(\varphi) = (1 - \text{div})(I + B(\varphi)) \cong I - I\text{div}\varphi + B(\varphi) \tag{1.17}$$

Then the first term in (1.5) can be written as

$$\int\limits_{\Omega_2(u)} \nu(\nabla_y \xi_1 \cdot \nabla_y \mathrm{w}_1)dy = \int\limits_{\Omega_2(0)} (A(\varphi)\nabla_x(\xi_1 \circ \mathrm{T})\nabla_x(\mathrm{w}_1 \circ \mathrm{T}))dx \tag{1.18}$$

The second term in (1.5) can be written as

$$\int\limits_{\Omega_2(u)} \xi_1(\mathbf{w} \cdot \nabla_y \mathrm{w}_1) = \int\limits_{\Omega_2(0)} J\xi_1(\mathbf{w} \cdot (D\mathrm{T})^{-T} \cdot \nabla(\mathrm{w}_1 \circ \mathrm{T}))$$

$$= \int\limits_{\Omega_2(0)} \xi_1 \mathbf{w}(K \cdot \nabla \mathrm{w}_1) \tag{1.19}$$

where

$$K = I + \begin{bmatrix} \varphi_{2/2} & -\varphi_{2/1} \\ -\varphi_{1/2} & \varphi_{1/1} \end{bmatrix} = I + B_1(\varphi) \tag{1.20}$$

with

$$B(\varphi) = B_1(\varphi) + B_1^\top(\varphi)$$

The third term in (1.5) can be rewritten as:

$$\int\limits_{\Omega_2(u)} \mathrm{p} \cdot \xi_{1/y_1} = \int\limits_{\Omega_2} \mathrm{pe}_1^\top \cdot \nabla_y \xi$$

$$= \int\limits_{\Omega_2(0)} \mathrm{pe}_1^\top (D\mathrm{T})^{-\top} \nabla(\xi \circ \mathrm{T}) \cong \int\limits_{\Omega_2(0)} \mathrm{pe}_1^\top K \cdot \nabla \xi \tag{1.21}$$

The third equation in (**??**) can be written as

$$\int\limits_{\Omega_2(u)} \eta \cdot \text{div}\mathbf{w} = 0 \tag{1.22}$$

$$\begin{aligned}
&\int\limits_{\Omega_2(u)} \left[e_1^\top \nabla_y w_1 + e_2^\top \nabla_y w_2\right] \eta \\
&= \int\limits_{\Omega(0)} \eta J \left[e_1^\top (D\mathrm{T})^{-\top} \nabla(w_1 \circ \mathrm{T}) + e_2^\top (D\mathrm{T})^{-\top} \nabla(w_2 \circ \mathrm{T})\right] \\
&\cong \int\limits_{\Omega_2(0)} \eta(e_1^\top K \nabla_x w_1 + e_2^\top K \nabla_x w_2)
\end{aligned} \tag{1.23}$$

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