Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Recent Advances in Fuzzy Sets,
Intuitionistic Fuzzy Sets,
Generalized Nets and Related Topics
Volume I: Foundations

Editors

Krassimir T. Atanassov
Michał Baczyński
Józef Drewniak
Janusz Kacprzyk
Maciej Krawczak
Eulalia Szmidt
Maciej Wygralak
Sławomir Zadrożny

Editors
Krassimir T. Atanassov
Michał Baczyński
Józef Drewniak
Janusz Kacprzyk
Maciej Krawczak
Eulalia Szmidt
Maciej Wygralak
Sławomir Zadrożny

SRI PAS IBS PAN

Recent Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
Volume I: Foundations

Systems Research Institute
Polish Academy of Sciences

Recent Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
Volume I: Foundations

Editors

Krassimir T. Atanassov
Michał Baczyński
Józef Drewniak
Janusz Kacprzyk
Maciej Krawczak
Eulalia Szmidt
Maciej Wygralak
Sławomir Zadrożny

IBS PAN SRI PAS

Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
www.ibspan.waw.pl

ISBN 9788389475350

Scalar and fuzzy cardinalities – tools for intelligent counting under information imprecision

Maciej Wygralak
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87, 61-614 Poznań, Poland
wygralak@amu.edu.pl

Abstract

Counting belongs to the most basic and frequent mental activities of humans. The subject of this paper is the non-trivial case of counting, namely that with imprecisely specified objects of counting. We will show that both scalar and fuzzy cardinalities of fuzzy sets as well as their extensions to Atanassov's intuitionistic fuzzy sets are not sophisticated, artificial constructions. On the contrary, they model and reflect human counting procedures under imprecision possibly combined with incompleteness of information.

Keywords: Counting under information imprecision, sigma *f*–count, fuzzy cardinality, Atanassov's intuitionistic fuzzy set, its cardinality.

1 Introduction

This paper deals with counting processes performed by humans, and with the resulting cardinalities. However, we do not mean the trivial case when the objects of counting are precisely specified and, thus, the problem collapses to the usual counting in a set by means of the natural numbers. Our subject will be the other, more sophisticated situation when the objects of counting are imprecisely (fuzzily) specified, e.g.

"How many *affordable* hotel rooms are there in the *vicinity* of the conference venue?".

What we deal with is then counting in a fuzzy set, *counting under information imprecision*. This can be viewed as *intelligent counting* as the counting person has to think twice and decide what and how to count. We will

Recent Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations *(K.T. Atanassow, M. Baczyński, J. Drewniak, J. Kacprzyk, M.Krawczak, E. Szmidt, M.Wygralak, S.Zadrożny, Eds.), IBS PAN - SRI PAS, Warsaw, 2010.*

discuss two approaches to that counting: the scalar approach and the fuzzy one which lead to scalar and fuzzy cardinalities, respectively. Our aim is to show that they are reflections of real, human counting procedures. Speaking generally, we will look at old, well-known constructions in a fresh way: just from the viewpoint of human counting. Also, we will discuss the case of counting in I–fuzzy sets, Atanassov's intuitionistic fuzzy sets.

2 Main human counting procedures under information imprecision

Looking at human counting procedures under information imprecision, it seems that two of them are fundamental.

● **Counting by thresholding** (*CAC*, *cut-and-count method*).
Assume we ask

"How many *warm* days were there last (calendar) summer?".

The standard human way of doing is then

– to establish a threshold temperature, say, 22ºC,
– to count up all the summer days with temperature $\geq$ 22ºC or $>$ 22ºC.

Speaking more formally, this collapses to defining the cardinality $|A|$ of a fuzzy set A as

$$|A| = |A_t| \quad \text{or} \quad |A| = |A^t|,$$

where A_t and A^t, respectively, denote the usual t-cut set and the sharp t–cut set of A, respectively; $A^t = \{x\colon A(x) > t\}$.

CAC thus leads to *scalar cardinalities* of fuzzy sets (see Section 3). The cardinality of A is then a single nonnegative real number.

● **Counting by multiple thresholding** (*MCAC*, *multiple cut-and-count*).
The answer to queries like "How many *warm* days were there last (calendar) summer?" is now given in a more advanced form which involves more than one threshold, e.g.

"62 days were at least *fairly warm* ($\geq$22ºC), including 40 *definitely warm* days ($\geq$24ºC) of which 18 were *totally warm* ($\geq$26ºC)".

This cardinal information can be modeled as

$$|A| = \textit{totally}/18 + \textit{definitely}/40 + \textit{fairly}/62$$

$= 1/18 + 0.8/40 + 0.4/62$
$= 1/|A_1| + 0.8/|A_{0.8}| + 0.4/|A_{0.4}|$.

The result of counting is now a fuzzy set in {0, 1, 2, …}. So, MCAC leads to *fuzzy cardinalities* which are fuzzy sets of nonnegative integers (see Section 4).

3 Scalar cardinalities

A suitable formalization of scalar counting procedures is offered by the notion of the *sigma f–count* $\sigma_f(A)$ of a (finite) fuzzy set A in a universe $\boldsymbol{U}$ (see [3, 4]):

$$\sigma_f(A) = \sum_{x \in \mathrm{supp}(A)} f(A(x)),$$

where f: [0, 1] → [0, 1] is a *weighting function* understood as a non-decreasing function with $f(0) = 0$ and $f(1) = 1$. Each weight $f(a)$ forms a *degree of participation* in the counting process in A assigned to an element x whose membership degree in A is $a \in [0, 1]$. $\sigma_f(A)$ is viewed as the cardinality of A. Obviously, $\sigma_{id}(A)$ collapses to the classical sigma count of A (see e.g. [7, 8]). Basic examples of weighting functions and the resulting sigma f–counts are listed below.

- **Counting by thresholding:** $f(a) = (1 \text{ if } a \geq t, \text{ else } 0)$ with $t \in (0, 1]$. Then

$$\sigma_f(A) = |A_t|.$$

- **Counting by sharp thresholding:** $f(a) = (1 \text{ if } a > t, \text{ else } 0)$ with $t \in [0, 1)$. It gives

$$\sigma_f(A) = |A^t|.$$

- **Counting by joining:** $f(a) = a^p$, $p > 0$. Now

$$\sigma_f(A) = \sum_{x \in \mathrm{supp}(A)} (A(x))^p,$$

and $p = 1$ generates the usual sigma count of A.

- **Counting by thresholding and joining:** $f(a) = (a^p \text{ if } a \geq t, \text{ else } 0)$, where $t \in (0, 1]$ and $p > 0$. So, $p = 1$ gives

$$\sigma_f(A) = \sum_{x \in A_t} A(x) .$$

The reader is referred to [3, 4] for more examples.

4 Fuzzy cardinalities

Let us return to the MCAC method from Section 2. Going to extreme, the counting person can use all possible threshold values from (0, 1] and combine all the results of counting. Consequently, one defines

$$|A| = \sum_{t \in (0,1]} t / |A_t| = \sum_{k \geq 0} \sup\{t\colon |A_t| = k\} / k.$$

What we get is the *basic fuzzy count* BFC(A) of A introduced in [6], the oldest and a bit forgotten type of fuzzy cardinality:

$$\mathrm{BFC}(A) = \sum_{k \geq 0} \sup\{t\colon |A_t| = k\} / k.$$

Let us look at the following simple example with

$$A = 0.6/x_1 + 1/x_2 + 0.9/x_3 + 0.8/x_4 + 1/x_5 + 0.6/x_6 + 0.6/x_7 + 0.2/x_8.$$

Then

$$\mathrm{BFC}(A) = 1/2 + 0.9/3 + 0.8/4 + 0.6/7 + 0.2/8.$$

As one sees, BFC(A) is generally nonconvex and can be viewed as a dynamic and compact piece of information about all possible results of counting in A by thresholding. If $\mathrm{BFC}(A)(k) = 0$, k is impossible as a result of that counting, whereas $\mathrm{BFC}(A)(k) > 0$ means that BFC(A)(k) is a maximum threshold t giving k as a result of counting.

It is easy to notice that a slight modification of BFC(A) leads to convex fuzzy cardinalities:

$$|A| = \sum_{k \geq 0} \sup\{t\colon |A_t| \geq k\} / k = \sum_{k \geq 0} [A]_k / k$$

with $[A]_k$ denoting the kth greatest membership degree in A; $[A]_0 = 1$. Clearly,

$$\mathrm{FGC}(A) = \sum_{k\geq 0} [A]_k / k$$

is nothing else than the *FGCount of* A, the most commonly known type of fuzzy cardinality (see [7]; cf. also [3]). For A from the previous example, we obtain

$$\mathrm{FGC}(A) = 1/0 + 1/1 + 1/2 + 0.9/3 + 0.8/4 + 0.6/5 + 0.6/6 + 0.6/7 + 0.2/8.$$

Speaking generally and practically, FGC(A) forms a ranking list of membership degrees in A. Let us notice that

$$|A_t| = |(\mathrm{FGC}(A))_t| - 1,$$

i.e. FGC(A) can be viewed as a compact piece of information about the results of counting by thresholding in A with all possible thresholds.

A comprehensive study of scalar and fuzzy cardinalities, including many other types of these cardinalities and putting emphasis on relationships with human counting under information imprecision, will be presented in [5].

5 Extensions to I–fuzzy sets

Let $\mathcal{E} = (A^+, A^-)$ with $A^+, A^-\colon \boldsymbol{U} \to [0, 1]$ and $A^+ \subset (A^-)'$ be an I–fuzzy set ([1]; see also [4]). A^+ and A^-, respectively, are understood as a *membership function* and a *nonmembership function*, respectively. Consequently, $A^+(x)$ forms a *membership degree*, whereas $A^-(x)$ is a *nonmembership degree* of x.

One can say that I–fuzzy sets are tools for modeling incompletely known fuzzy sets in $\boldsymbol{U}$. Counting in $\mathcal{E}$ thus becomes a task of counting under imprecision combined with incompleteness of information (about the objects of counting). This section presents extensions of the constructions discussed in Sections 3 and 4 to I–fuzzy sets.

A natural extension of the concept of sigma f–count is then of the following form:

$$\sigma_f(\mathcal{E}) = [\sigma_f(A^+), \sigma_f((A^-)')],$$

where f denotes a weighting function. More precisely, we have

$$\sigma_f(\mathcal{E}) = \Big[\sum_{x\in\boldsymbol{U}} f(A^+(x)), \sum_{x\in\boldsymbol{U}} f((A^-)'(x))\Big].$$

$\sigma_f(\mathcal{E})$ is thus an interval of nonnegative reals. In particular, the number $\sigma_f(A^+)$

expresses a minimum possible scalar cardinality $\sigma_f(A)$ of incompletely known fuzzy set A represented by $\mathcal{E}$, whereas $\sigma_f((A^-)')$ forms its maximum possible cardinality. Using $f = id$, one gets the way of counting proposed in [2]. The interested reader is also referred to [4] which shows in a series of model examples that both sigma f–counts and their extensions with various weighting functions have a true technical sense. Those examples can be easily adapted to many problems from the areas of intelligent systems and decision support.

Let us move on to FGCounts from Section 4. Their suitable extension to I–fuzzy sets is

$$\mathrm{FGC}(\mathcal{E}) = (\mathrm{FGC}(A^+), \mathrm{FGC}((A^-)')),$$

an interval-valued fuzzy set of nonnegative integers. Taking into account the observation closing Section 4, we get

$$[\,|(A^+)_t|, |((A^-)')_t|\,] = [\,|(\mathrm{FGC}(A^+))_t| - 1, |(\mathrm{FGC}((A^-)'))_t| - 1]$$

for each $t \in (0, 1]$. On the other hand, $|(A^+)_t|$ seems to be a minimum possible result of counting by thresholding in incompletely known A using threshold t, whereas $|((A^-)')_t|$ forms a maximum possible result of that counting. And, as we see, this cardinal information can be easily derived from $\mathrm{FGC}(\mathcal{E})$. All this means that FGCounts of I–fuzzy sets, too, are closely connected with human counting procedures. $\mathrm{FGC}(\mathcal{E})$ can be viewed as a compact piece of information about all possible results of counting by thresholding in incompletely known A modeled by $\mathcal{E}$. Again, the reader is referred to [5] for a detailed presentation of FGCounts and other types of fuzzy cardinalities extended to I–fuzzy sets.

6 Conclusions

We have shown that scalar and, what is more, fuzzy cardinalities of fuzzy sets are not sophisticated, artificial constructions. The same refers to the presented extensions to I–fuzzy sets. On the contrary, those cardinalities are closely connected with and reflect the results of human counting procedures under imprecision possibly combined with incompleteness of information. Especially, we mean the procedures of counting by thresholding and multiple thresholding.

Acknowledgments
This research was supported by the Ministry of Science and Higher Education Grant N N519 384936.

References

[1] Atanassov K. T. (1999) Intuitionistic Fuzzy Sets – Theory and Applications. Springer–Verlag, Heidelberg.

[2] Szmidt E., Kacprzyk J. (2001) Entropy for Intuitionistic Fuzzy Sets. *Fuzzy Sets and Systems*, 118, 467-477.

[3] Wygralak M. (2003) Cardinalities of Fuzzy Sets. Springer–Verlag, Heidelberg.

[4] Wygralak M. (2008) Model examples of applications of sigma *f*-count to counting in fuzzy sets and I-fuzzy sets. In: Atanassov K. T., Hryniewicz O., Kacprzyk J., Krawczak M., Nahorski Z., Szmidt E., and Zadrożny S., Eds., *Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics*. *Applications*, *Vol. II*. Academic Publishing House EXIT, Warsaw, 187-196.

[5] Wygralak M. (201x) How Many? Intelligent Counting under Imprecision. With Applications to Computer Science and Decision Support (book in preparation).

[6] Zadeh L. A. (1979) A theory of approximate reasoning. In: Hayes J. E., Michie D. and Mikulich L. I., Eds., *Machine Intelligence*, *vol. 9*. John Wiley & Sons, New York, 149-194.

[7] Zadeh L. A. (1983) A Computational Approach to Fuzzy Quantifiers in Natural Languages. *Comput. and Math. with Appl.*, 9, 149-184.

[8] Zadeh L. A. (2002) Toward a perception-based theory of probabilistic reasoning with imprecise probabilities. *Jour. Statistical Planning and Inference*, 105, 233-264.

The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

http://www.ibspan.waw.pl/ifs2010

The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

ISBN-13 9788389475350
ISBN 838947535-9
9 788389 475350