Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Aggregation of $*$–transitive fuzzy relations by quasi - linear means

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Abstract

The aim of this article is to investigate the correlation between the properties of aggregated fuzzy relation R and the individual fuzzy relations R_i. The problem originates from multicriteria decision making, where aggregation procedures realized the way for compensation between some evaluations. The quasi–linear means will be taken as aggregation functions. The author will make an attempt at answering two questions: does relation R obtained by aggregation of relations R_i have the same kind of $*$–transitivity as basic relations? If we don't obtain a positive answer, we will try to investigate which $*$–transitive class of relation do the results of aggregation belong to? T–norms will be taken as $*$.

Keywords: aggregation operators, fuzzy relations, $\sup -*$ transitive relations.

1 Introduction

Problems of aggregation are important in multi criteria decision making (see [2], [7], [8], [9]). Authors examine a finite set of alternatives (which a decision maker has to choose from), a finite set of criteria on the basis of which the alternatives are evaluated, and this leads them to the matrices corresponding to the fuzzy relations by each criterion. The properties of fuzzy relations during aggregation of

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finite families of these relations are studied in [2], [3], [4], [7], [8], [10]. We will aggregate n fuzzy relations R_i by an aggregation function F. A fuzzy relation on $X \neq \emptyset$ is an arbitrary function $R\colon X \times X \to [0,1]$. The family of all such functions will be denoted by $FR(X)$. Since fuzzy relations have values in $[0,1]$ we use a real function $F\colon [0,1]^n \to [0,1]$ for their transformation. To shorten some expressions for the family of the above functions we will use the notation $\mathcal{F}_n$. An aggregation operator is a function $F \in \mathcal{F}_n$ which is increasing and idempotent. Let us formalize our considerations:

Definition 1 ([6])**.** *Let $F \in \mathcal{F}_n, R_1, \dots, R_n \in FR(X), n \in \mathbb{N}, n \geq 2$. The fuzzy relation $R_F \in FR(X)$ is an aggregation of fuzzy relations $R_1, \dots, R_n$ by the function F, when*

$$R_F(x,y) = F(R_1(x,y), \dots R_n(x,y)), \qquad x,y \in X. \tag{1}$$

The quasi–linear means, described in Section 3, will be regarded as an aggregation function F. $*$–transitive relations, presented in Section 4, will be taken as relations $R_i, 1 \leq i \leq n$. We will denote t–norms by $*$. In the last section of the paper, the answers the following questions are provided:

- Does the relation R_F obtained by the aggregation of relations R_i have the same kind of $*$–transitivity as these relations?
- Which $*$–transitive class of relations do the results of aggregation belong to?

2 T–norms

The aim of this chapter is to recall formal definitions and basic properties, as well as to show some examples of t–norms. This kind of functions serves as a basis for defining intersections of fuzzy relations. The monograph [5] was very helpful to prepare this part of the article.

Definition 2. *A function $T \in \mathcal{F}_2$ is called a triangular norm (t–norm) when it is commutative, associative, increasing in each component, and has 1 as a neutral element.*

It is easy to show that 0 is the zero element for t-norms, so

Lemma 1. *For every triangular norm T we have*

$$\underset{x \in [0,1]}{\forall} T(x,0) = 0. \tag{2}$$

In our investigation continuous and Archimedean t–norms play a very useful role, so let recall a definition and representation theorem.

Definition 3. *A t-norm T is said to be*

- *continuous if T as a function is continuous on the unit interval,*
- *Archimedean if $T(x,x) < x$ for all $x \in (0,1)$.*

Theorem 1 ([5], Theorem 5.1)**.** *A t-norm T is continuous and Archimedean iff there exists a strictly decreasing and continuous function $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$ such that*

$$T(x,y) = f^{-1}\left(\min(f(x) + f(y), f(0))\right), \quad x, y \in [0,1]. \tag{3}$$

Moreover the representation (3) *is unique up to a positive multiplicative constant.*

Under the assumption of Theorem 1 if t–norm T has the representation (3) we say that T is generated by the function f.

Example 1 ([5], Example 1.2, Theorem 3.23, Example 3.28, Remark 4.6)**.** *The most frequently used t–norms with their additive generators are listed below.*

type	*t-norm*	*additive generator f(x)*
minimum	$T_M(x,y) = \min\{x,y\}$	
Hamacher	$T_H(x,y) = \begin{cases} 0, & x = y = 0 \\ \frac{xy}{x+y-xy}, & \textit{otherwise} \end{cases},$	$\frac{1-x}{x}$
product	$T_P(x,y) = xy$	$-logx$
Einstein	$T_E(x,y) = \frac{xy}{2-(x+y-xy)},$	$\log \frac{2-x}{x}$
Łukasiewicz	$T_L(x,y) = \max(0, x+y-1),$	$1-x$
drastic	$T_D(x,y) = \begin{cases} y, & x = 1 \\ x, & y = 1 \\ 0, & \textit{otherwise} \end{cases},$	$\begin{cases} 2-x, & x \in [0,1), \\ 0 & x = 1 \end{cases}$

The drastic t–norm is Archimedean, the minimum t–norm is continuous. Hamacher, product, Einstein and Łukasiewicz t–norms display both of the above properties. Referring to [5] we can write

Theorem 2 ([5], Remark 1.5, Remark 4.6)**.** *T–norms from Example 1 are comparable, in particular we have*

$$T_D \leq T_L \leq T_E \leq T_P \leq T_H \leq T_M.$$

3 Aggregation operators

In this chapter some facts about an aggregation operator F are presented. Let us take into consideration n objects.

Definition 4 ([3])**.** *Let $n \geq 2$. By n–ary mean we call a function $F \in \mathcal{F}_n$ which fulfills the following properties:*

$$\underset{s_1,\ldots,s_n,t_1,\ldots,t_n\in[0,1]}{\forall} \left(\underset{1\leq k\leq n}{\forall} s_k \leq t_k \right) \Rightarrow F(s_1,\ldots,s_n) \leq F(t_1,\ldots,t_n); \quad (4)$$

$$\underset{x\in[0,1]}{\forall} F(x,\ldots,x) = x. \quad (5)$$

By [2] the mean should have one of the following additional properties:

- continuous
- strictly increasing iff

$$\underset{s,t_1,\ldots,t_n\in[0,1]}{\forall} \Big(\underset{1\leq k\leq n}{\forall} (s < t_k \Rightarrow F(t_1,\ldots,t_{k-1},s,t_{k+1},\ldots,t_n) < F(t_1,\ldots,t_n))\Big); \quad (6)$$

- bisymetrical iff

$$\underset{t_{11},\ldots,t_{1n},t_{n1},\ldots,t_{nn}}{\forall} F(F(t_{11},\ldots,t_{1n}),\ldots,F(t_{n1},\ldots,t_{nn})) = F(F(t_{11},\ldots,t_{n1}),\ldots,F(t_{1n},\ldots,t_{nn})) \quad (7)$$

In this paper will be investigated a specific class of means namely quasi–linear means. Below we present the theorem, obtained by Aczél (1948), which characterizes this function.

Theorem 3 ([1], p. 394)**.** *A function $F\colon [a,b]^n \to [a,b]$ is continuous, symmetric, strictly increasing, idempotent, and bisymmetrical iff F represents a quasi–linear mean, i. e. (there exists an strictly monotonic and continuous function $f\colon [a,b] \to R$ such that)*

$$\underset{x_1,\dots,x_n}{\forall} F(x_1,\dots,x_n) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right), \tag{8}$$

where $w_i > 0, i \in \{1,\dots,n\}$ and $\sum_{i=1}^{n} w_i = 1$.

Quasi–linear means constitute a wide group of functions. They include arithmetic, quadratic, geometric, harmonic means, as it can be seen in Example 2.

Example 2 ([2], p.114)**.** *Let us assume that $\overline{x} = [x_1,\dots,x_n] \in [0,1]$, $w_i > 0$, $i \in \{1,\dots,n\}$ and $\sum_{i=1}^{n} w_i = 1$. The most popular members of the family of quasi–linear means on $[0,1]^n$ are presented in the table.*

type	*weighted mean*	$f(x)$
arithmetic	$A(\overline{x}) = \sum_{k=1}^{n} w_k x_k$,	$1-x$
quadratic	$Q(\overline{x}) = \sqrt{\sum_{k=1}^{n} w_k x_k^2}$,	x^2
geometric	$G(\overline{x}) = \prod_{k=1}^{n} x_k^{w_k}$,	$-logx$
harmonic	$H(\overline{x}) = \begin{cases} 0, & \exists_{1\le k\le n}\, x_k = 0 \\ (\sum_{k=1}^{n} \frac{w_k}{x_k})^{-1}, & otherwise \end{cases}$,	$\frac{1-x}{x}$

4 Transitive fuzzy relations

As it was mentioned in the introduction, a fuzzy relation is a function $R\colon X \times X \to [0,1]$. In the set of fuzzy relations we are able to perform some operations i. e. a sum or an intersection of fuzzy relations (for details see [11]), but for us the most interesting operation will be a composition of relations.

Definition 5 ([11])**.** *Let $* \in \mathcal{F}_2$. A $\sup -*$ composition of fuzzy relations $R, S \in FR(X)$ is a fuzzy relation $R \circ S \in FR(X)$ such that,*

$$(R \circ S)(x,z) = \sup_{y\in X} R(x,y) * S(y,z), \quad (x,z) \in X \times X. \tag{9}$$

One of the above mentioned t–norms will be taken as a generator $*$.

Definition 6. *Let $* \in \mathcal{F}_2$. The relation $R \in FR(X)$ is $*$–transitive if*

$$\forall_{x,y,z \in X} R(x,y) * R(y,z) \leq R(x,z). \tag{10}$$

Sometimes the property (10) is written as $R^2 \subseteq R$. Let $\mathcal{R}_*$ denote the family of $*$–transitive relations, $\mathcal{R}_M$ corresponds to a very popular class of $\sup - \min$ transitive relations, and $\mathcal{R}_L$ stands for T_L-transitive relations. On the ground of Definition 6 it is easy to prove

Theorem 4. *If $*_1, *_2 \in \mathcal{F}_2$ and $*_1 \leq *_2$, then $\mathcal{R}_{*_2} \subseteq \mathcal{R}_{*_1}$.*

By Theorems 2 and 4 we have

Corollary 1. *Families of $*$–transitive relations create the following chain*

$$\mathcal{R}_M \subseteq \mathcal{R}_H \subseteq \mathcal{R}_P \subseteq \mathcal{R}_E \subseteq \mathcal{R}_L \subseteq \mathcal{R}_D. \tag{11}$$

The theorem presented below proves to be very useful to check which of the functions preserves the $*$–transitivity. It will be used to build counterexamples.

Theorem 5 ([4], Theorem 8)**.** *Let $card X \geq 3$, $* \in \mathcal{F}_2$ be an operation with zero element $z = 0$. An increasing function $F \in \mathcal{F}_n$ preserves $*$–transitivity iff it fulfills the following condition*

$$\forall_{(s_1,\ldots,s_n),(t_1,\ldots t_n) \in [0,1]^n} F(s_1 * t_1, \ldots s_n * t_n) \geq F(s_1, \ldots, s_n) * F(t_1, \ldots t_n). \tag{12}$$

It is obvious that in the above theorem a t–norm could be taken as $*$, and the function F can denote an aggregation function.

5 Aggregation of relations

In this part answers to the above raised questions will be presented. We will aggregate a finite number of fuzzy relations into a single output fuzzy relation. As it was stated in the introduction R_F will denote the result of aggregation. Arithmetic, quadratic, geometric or harmonic means will be denoted by R_A, R_Q, R_G and R_H respectively. First we present known results. In [2] we can find the following lemma.

Lemma 2 ([2], Lemma 2.3). *If $*$ is a continuous Archimedean t–norm with the additive generator f, then $*$–transitivity condition* (10) *is equivalent to*

$$f(R(x,y)) + f(R(y,z)) \geq f(R(x,z) \tag{13}$$

for all $x, y, z \in X$.

Now we are able to prove the theorem, which is formulate in [2] for weights $w_i = \frac{1}{n}$, $1 \leq i \leq n$.

Theorem 6 ([2], Theorem 5.11). *If relations $R_i \in FR(X), 1 \leq i \leq n$ are $*$–transitive, where $*$ is a continuous Archimedean t–norm with the generator f and F represents a quasi–linear mean with the same generator f, then the relation $R_F(x,y) = F(R_1(x,y), \dots R_n(x,y))$ is $*$–transitive.*

Proof. Let us assume that relations $R_i, 1 \leq i \leq n$ are $*$–transitive, hence

$$R_i(x,y) * R_i(y,z) \leq R_i(x,z) \qquad x, y, z \in X.$$

By virtue of Lemma 2 the above condition is equivalent to the following inequality

$$f(R_i(x,y)) + f(R_i(y,z)) \geq f(R_i(x,z)) \qquad x, y, z \in X.$$

We know, that weights are positive, so

$$\sum_{i=1}^{n} w_i f(R_i(x,y)) + \sum_{i=1}^{n} w_i f(R_i(y,z)) \geq \sum_{i=1}^{n} w_i f(R_i(x,z)) \qquad x, y, z \in X.$$

According to (8) we have

$$\begin{aligned} f(F(R_1(x,y), \dots, R_n(x,y))) + f(F(R_1(y,z), \dots, R_n(y,z))) \\ \geq f(F(R_1(x,z), \dots, R_n(x,z))) \end{aligned}$$

Now, using the notation (1) we obtain

$$f(R_F(x,y)) + f(R_F(y,z)) \geq f(R_F(x,z))$$

and applying again Lemma 2, it is easy to see that the relation R_F is $*$–transitive. □

Let us consider $F = A$, $F = G$ or $F = H$ in Theorem 6 (cf. Example 1, Example 2).

Theorem 7. *Let F be the weighted arithmetic mean ($F = A$). If fuzzy relations $R_i \in \mathcal{R}_L$ $1 \leq i \leq n$, then $R_A \in \mathcal{R}_L$.*

Theorem 8. *Let F be the weighted geometric mean ($F = G$). If fuzzy relations $R_i \in \mathcal{R}_P$ $1 \leq i \leq n$, then $R_G \in \mathcal{R}_P$.*

Theorem 9. *Let F be the weighted harmonic mean ($F = H$). If fuzzy relations $R_i \in \mathcal{R}_H$ $1 \leq i \leq n$, then $R_H \in \mathcal{R}_H$.*

Similarly results concerning geometric and arithmetic means, but obtained in algebraic manner, we can find in [8]. The last theorem means that harmonic mean preserves the Hamacher–transitivity. It is easy to prove, that

Theorem 10 (cf. [4], Example 12)**.** *Arbitrary quasi linear means preserves T_D transitivity.*

Proof. Let F denotes an arbitrary quasi–linear mean and $*$ stands for a drastic t–norm. According to Theorem 5 we have to check the inequality (12). The right side of this inequality is greater then 0 when $F(s_1, \ldots, s_n) = 1$ or $F(t_1, \ldots, t_n) = 1$. Let us assume that $F(s_1, \ldots, s_n) = 1$. It is easy to see, that

$$F(s_1, \ldots, s_n) = 1 \Leftrightarrow \underset{1 \leq n \leq n}{\forall} s_i = 1. \tag{14}$$

Indeed, $F(1, \ldots, 1) = 1$, by virtue of (5). Conversely, let us suppose that there exists some $1 \leq i \leq n$ such that $s_i < 1$, so by (6) (compare Theorem 3) we obtain that

$$1 = F(1, \ldots, s_i, \ldots, 1) < F(1, \ldots, 1) = 1, \tag{15}$$

which is impossible, so the equivalence (14) is true. Now we are able to finish our prove

$$\begin{aligned} F(s_1 * t_1, \ldots, s_n * t_n) &= F(1 * t_1, \ldots 1 * t_n) = F(t_1, \ldots t_n) \\ &= F(t_1, \ldots, t_n) * 1 = F(t_1, \ldots t_n) * F(s_1, \ldots s_n). \end{aligned}$$

□

Example 3. *Let $X = 3$. The next table contains values s_1, s_2, t_1, t_2 for which the inequality from Theorem 5 is not preserved. In some cases we put values of weights for which we obtain these results, in other cases we omit it, that means that values could be arbitrary (they must fulfill assumptions of Theorem 3).*

<table>
<tr><th></th><th>T_M</th><th>T_H</th><th>T_P</th><th>T_E</th><th>T_L</th><th>T_D</th></tr>
<tr><td>Q</td><td colspan="5">$s_1 = 0, s_2 = 1, t_1 = 1, t_2 = 0$</td><td>*Thm 10*</td></tr>
<tr><td>A</td><td colspan="4">$s_1 = 0, s_2 = 1, t_1 = 1, t_2 = 0$</td><td>*Thm 7*</td><td>*Thm 10*</td></tr>
<tr><td>G</td><td colspan="2">$s_1 = 0.1, s_2 = 1$
$t_1 = 1, t_2 = 0.1$
$w_1 = w_2 = 0.5$</td><td>*Thm 8*</td><td colspan="2">$s_1 = 0.8, s_2 = 1$
$t_1 = 0.7, t_2 = 1$
$w_1 = w_2 = 0.5$</td><td>*Thm 10*</td></tr>
<tr><td>H</td><td>$s_1 = 0.3, s_2 = 1$
$t_1 = 0.8, t_2 = 0.9$
$w_1 = w_2 = 0.5$</td><td>*Thm 9*</td><td colspan="3">$s_1 = 0.7, s_2 = 1$
$t_1 = 0.6, t_2 = 1$
$w_1 = 0.25, w_2 = 0.75$</td><td>*Thm 10*</td></tr>
</table>

Results presented in the table help us answer the question posed in the introduction. Let us investigate matrices built according to the rule

$$R_i = \begin{pmatrix} 0 & s_i & s_i * t_i \\ 0 & 0 & t_i \\ 0 & 0 & 0 \end{pmatrix} \quad i = 1, 2.$$

It is obvious (according to Definition 6) that relations $R_1, R_2 \in \mathcal{R}_*$ for an arbitrary t-norm $*$. Our goal is to demonstrate that for values displayed in the above table the relation $R_F = F(R_1, R_2) \notin \mathcal{R}_*$ where $* \in \{T_M, T_H, T_P, T_E, T_L\}$ for the quadratic mean, $* \in \{T_M, T_P, T_E, T_L\}$ for the harmonic mean, $* \in \{T_M, T_H, T_E, T_L\}$ for the geometric mean and $* \in \{T_M, T_H, T_P, T_E\}$ for the arithmetic mean. Let us conduct a detailed analysis of the arithmetic mean ($F = A$). In this case matrices are as follows:

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_A = \begin{pmatrix} 0 & w_2 & 0 \\ 0 & 0 & w_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_A^2 = \begin{pmatrix} 0 & 0 & w_1 * w_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We know that $R_1, R_2 \in \mathcal{R}_M \subseteq \mathcal{R}_H \subseteq \mathcal{R}_P \subseteq \mathcal{R}_E$, but the result of the aggregation of the above matrices $R_A \notin \mathcal{R}_E$ for arbitrary values of weight. We have to verify that $R_A^2 \subseteq R_A$ is not true, hence the necessity to verify the following inequality (according to the definition of Einstein's t-norm)

$$\frac{w_1 \cdot w_2}{2 - (w_1 + w_2 - w_1 \cdot w_2)} > 0. \tag{16}$$

But we know that $w_1 + w_2 = 1$ and $w_i > 0,\ i = 1, 2$, therefore the obtained denominator values imply that $w_1 \cdot w_2 > -1$, which is always true. For that reason the inequality (16) is true for arbitrary values of weight w_1, w_2. It has been demonstrated that $R_A \notin \mathcal{R}_E$ for arbitrary weights, now using Corollary 1 we discover that $R_A \notin \mathcal{R}_P$ neither $R_A \notin \mathcal{R}_H$ nor $R_A \notin \mathcal{R}_M$. Now we can prove

Theorem 11. *Let F be the arithmetic mean ($F = A$). If fuzzy relations $R_i \in \mathcal{R}_*$, $1 \leq i \leq n$ and $*$ is an arbitrary t–norm such that $T_L < *$, then $R_A \in \mathcal{R}_L \setminus \mathcal{R}_E$.*

Proof. Let us assume that relations $R_i \in \mathcal{R}_*$ for $* \in \{T_M, T_H, T_P, T_E\}$, $1 \leq i \leq n$. We know, by Corollary 1, that $\mathcal{R}_* \subseteq \mathcal{R}_L$, hence $R_i \in \mathcal{R}_L$. Now using Theorem 7, it is obvious that the relation R_A obtained from relations R_i is T_L–transitive and by virtue of the above example $R_A \notin \mathcal{R}_*$ for $* \in \{T_M, T_H, T_P, T_E\}$. □

Let us consider the quadratic mean $F = Q$. According to the table from Example 3, matrices R_1 and R_2 are the same as in the case of the arithmetic mean. As the result of the aggregation of the above matrices, using the quadratic mean, we have

$$R_Q = \begin{pmatrix} 0 & \sqrt{w_2} & 0 \\ 0 & 0 & \sqrt{w_1} \\ 0 & 0 & 0 \end{pmatrix}, \quad R_Q^2 = \begin{pmatrix} 0 & 0 & \sqrt{w_1} * \sqrt{w_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As it was already mentioned above $R_1, R_2 \in \mathcal{R}_M \subseteq \mathcal{R}_H \subseteq \mathcal{R}_P \subseteq \mathcal{R}_E \subseteq \mathcal{R}_L$, however $R_Q \notin \mathcal{R}_L$. Using the Łukasiewicz t–norm as $*$ we will prove that $\max(0, \sqrt{w_1} + \sqrt{w_2} - 1) > 0$ for arbitrary weights w_1, w_2. Taking into account the fact that $w_1 + w_2 = 1$ we obtain $\sqrt{w_1} + \sqrt{1 - w_1} > 1$, from which we have $\sqrt{w_1(1 - w_1)} > 0$. The last inequality is true for the arbitrary w_1 and w_2. Now by Theorem 10 we see that

Theorem 12. *Let F be the quadratic mean ($F = Q$). If fuzzy relations $R_i \in \mathcal{R}_*, 1 \leq i \leq n$ and $*$ is an arbitrary t–norm, then $R_Q \in \mathcal{R}_D \setminus \mathcal{R}_L$.*

Let us focus now on the geometric mean ($F = G$). For $* = T_M$ or $* = T_H$ matrices are as follows:

$$R_1 = \begin{pmatrix} 0 & 0.1 & 0.1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 0.1 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{pmatrix}.$$

As the result of the aggregation using the geometric mean with $n = 2, w_1 = w_2 = 0.5$ we obtain

$$R_G = \begin{pmatrix} 0 & \sqrt{0.1} & 0.1 \\ 0 & 0 & \sqrt{0.1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Verifying does the matrix R_G belong to $\mathcal{R}_H$, we obtain negative answer because

$$(r_G^2)_{13} = \frac{\sqrt{0.1}\cdot\sqrt{0.1}}{\sqrt{0.1}+\sqrt{0.1}-\sqrt{0.1}\cdot\sqrt{0.1}} \approx 0.18.$$

Remaining values of the matrix R_G^2 are equal 0. On the strength of Definition 6 we see that $R_G \notin \mathcal{R}_H$ for $* \in \{T_M, T_H\}$. Now let us take as $*$ Einstein or Łukasiewicz t–norms. In this case, according to Example 3, matrices are as follows

$$R_1 = \begin{pmatrix} 0 & 0.8 & 0.8 * 0.7 \\ 0 & 0 & 0.7 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(r_1)_{13} = 0.8 * 0.7 = \frac{28}{53} \approx 0.5283 \text{ for } * = T_E$$

$$(r_1)_{13} = 0.8 * 0.7 = \max\{0.8 + 0.7 - 1, 0\} = 0.5 \text{ for } * = T_L.$$

After aggregation we have

$$(R_G)_{T_E} = \begin{pmatrix} 0 & \sqrt{0.8} & 0.7266 \\ 0 & 0 & \sqrt{0.7} \\ 0 & 0 & 0 \end{pmatrix}, \quad (R_G)_{T_L} = \begin{pmatrix} 0 & \sqrt{0.8} & 0.707 \\ 0 & 0 & \sqrt{0.7} \\ 0 & 0 & 0 \end{pmatrix},$$

where $(R_G)_{T_E}$ and $(R_G)_{T_L}$ denote results of aggregation obtained from matrices R_1 and R_2, where in the matrix R_1 we have $* = T_E$ and $* = T_L$ respectively. To calculate the matrix R^2, only we have to compute the element $(r_G^2)_{13}$, because remaining values will be equal zero. We will verify do above matrices belong to the class $\mathcal{R}_L$, so

$$((r_G^2)_{T_E})_{13} = ((r_G^2)_{T_L})_{13} = \max\{0, \sqrt{0.8} + \sqrt{0.7} - 1\} \approx 0.731,$$

hence $(R_G)_{T_E} \notin \mathcal{R}_L$ and $(R_G)_{T_L} \notin \mathcal{R}_L$. Using Corollary 1, Theorems 8 and 10 we can easily prove

Theorem 13. *Let F be the geometric mean. If fuzzy relations $R_i \in \mathcal{R}_*, 1 \leq i \leq n$, $*$ is an arbitrary t–norm, then $R_G \in \mathcal{R}_D \setminus \mathcal{R}_L$ for $* < T_P$ and $R_G \in \mathcal{R}_P \setminus \mathcal{R}_H$ for $* \geq T_P$.*

Proof. Let us assume that relations $R_i \in \mathcal{R}_*$, $1 \leq i \leq n$. First we will investigate $* \in \{T_M, T_H, T_P\}$. We know, by Corollary 1, that $\mathcal{R}_M \subseteq \mathcal{R}_H \subseteq \mathcal{R}_P$, hence relations $R_i \in \mathcal{R}_P$ $1 \leq i \leq n$. Now using Theorem 8, it is obvious that the relation R_G obtained from relations R_i is T_P–transitive and by virtue of the above example $R_G \notin \mathcal{R}_H$. Now let us take $* < T_P$. Using once more Corollary 1, we can write that $\mathcal{R}_* \subseteq \mathcal{R}_D$, hence $R_i \in \mathcal{R}_D$ $1 \leq i \leq n$. Now using Theorem 10, it is obvious that the relation R_G, obtained as aggregation of relations R_i, is T_D–transitive and by Example 3 $R_G \notin \mathcal{R}_L$. □

Let us consider now the harmonic mean ($F = H$). For $* = T_M$ according to Theorem (5) we have to show that the inequality (12) doesn't hold. Indeed, in the right side of this inequality we have 0.46, while the left side is equals 0.45, so in this case $R_H \notin \mathcal{R}_M$. Now assume that $* < T_H$. Matrices are as follows

$$R_1 = \begin{pmatrix} 0 & 0.7 & 0.7 * 0.6 \\ 0 & 0 & 0.6 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$(r_1)_{13} = 0.42 \text{ for } * = T_P \tag{17}$$

$$(r_1)_{13} = \frac{3}{8} = 0.375 \text{ for } * = T_E \tag{18}$$

$$(r_1)_{13} = 0.3 \text{ for } * = T_L. \tag{19}$$

We will show, that the matrix $R_H = H(R_1, R_2)$ does not belong to $\mathcal{R}_L$. First, the matrix R_H will be compute (in this case $w_1 = 0.25$, $w_2 = 0.75$).

$$R_H = \begin{pmatrix} 0 & \frac{28}{31} & x \\ 0 & 0 & \frac{6}{7} \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$x = \frac{84}{113} \approx 0.74 \text{ for } * = T_P \tag{20}$$

$$x = \frac{12}{17} \approx 0.7058 \text{ for } * = T_E \tag{21}$$

$$x = \frac{12}{19} \approx 0.6315 \text{ for } * = T_L. \tag{22}$$

Now the value $(r_H^2)_{13}$ in R_H^2 will be calculate. We will use the Łukasiewicz t–norm.

$$(r_H^2)_{13} = \max\left\{\frac{28}{31} + \frac{6}{7} - 1, 0\right\} = \frac{165}{217} \approx 0.7603$$

According to Definition (6) we have $R_H \notin \mathcal{R}_L$ for $* \in \{T_P, T_E, T_L\}$. Summing up let us record the following

Theorem 14. *Let F be the harmonic mean ($F = H$). If fuzzy relations $R_i \in \mathcal{R}_*, 1 \leq i \leq n$, $*$ is an arbitrary t -norm, then $R_H \in \mathcal{R}_D \setminus \mathcal{R}_L$ for $* < T_H$, and $R_H \in \mathcal{R}_H \setminus \mathcal{R}_M$ for $* \geq T_H$.*

Proof. Let us assume that relations $R_i \in \mathcal{R}_*$, $1 \leq i \leq n$. First we will investigate $* \in \{T_M, T_H\}$. We know, by Corollary 1, that $\mathcal{R}_M \subseteq \mathcal{R}_H$, hence relations $R_i \in \mathcal{R}_H$ $1 \leq i \leq n$. Now using Theorem 9, it is obvious that the relation R_H obtained from relations R_i is T_H–transitive and by virtue of the above example $R_H \notin \mathcal{R}_M$. Now let us take $* < T_H$. Using once more Corollary 1, we can write that $\mathcal{R}_* \subseteq \mathcal{R}_D$, hence $R_i \in \mathcal{R}_D$ $1 \leq i \leq n$. Now using Theorem 10, it is obvious that the relation R_H, obtained as aggregation of relations R_i, is T_D–transitive and by Example 3 $R_H \notin \mathcal{R}_L$. □

6 Conclusion

This article demonstrates which class the result of aggregation belongs to. We have given examples which prove that presented theorems could not be better. Many resent results concerning such aggregations can be found in [3], [4], [9] and [10].

References

[1] Aczél J. (1966), Lectures on functional equations and their applications. Acad. Press, New York.

[2] Fodor J., Roubens M. (1994), Fuzzy preference modeling and multicriteria decision support, Kluwer Academic Publishers, Dordrecht.

[3] Drewniak J., Dudziak U. (2005), Aggregation preserving classes of fuzzy relations, Kybernetica, 41, 265–284.

[4] Drewniak J., Dudziak U. (2007), Preservation of properties of fuzzy relations during aggregation processes, Kybernetica, 43, 115–132.

[5] Klement E.P, Mesiar R., Pap E. (2000), Triangular norms, Kluwer Acad. Publ., Dordrecht.

[6] Ovchinnikov S. (1991), Social choice and Lukasiewicz logic. Fuzzy Sets Syst. 43, 275- 289.

[7] Peneva V., Popchev I. (1998), Aggregation of fuzzy relations, Compt. Rend. Acad. Bulgare Sci., 51 (9-10), 41–44.

[8] Peneva V., Popchev I. (2003), Properties of the aggregation operators related with fuzzy relations, Fuzzy Sets Syst., 139, 615–663.

[9] Peneva V., Popchev I. (2007), Aggregation of fuzzy preference relations to multicriteria decision making, 6, 351–365.

[10] Saminger S., Bodenhofer U., Klement E., Mesiar R, (2003), Aggregation of fuzzy relations and preservation of transitivity. Fuzzy Sets Syst., 139, 615–663.

[11] Zadeh L.A. (1971), Similarity relations and fuzzy orderings. Inform. Sci., 3, 177–200.

The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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