Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Fuzzy implications on lattice of ordered fuzzy numbers

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Abstract

Step ordered fuzzy numbers (SOFN) form a subspace of ordered fuzzy numbers (OFN) that were invented by the first author and his two coworkers in the previous decade. The definition of OFN uses the extension of the parametric representation of convex fuzzy numbers. The space of SOFN may be identified with a $2K$-dimensional vector space, where K is responsible for the number of steps. OFN may be equipped with a lattice structure. Then fuzzy implications are defined on OFN and SOFN with the help of algebraic operations defined on OFN. The new objects are proposed that have some similarities with intuitionistic fuzzy sets and fuzzy sets of the 2ed type.

Keywords: ordered fuzzy numbers, partial order, lattice, implications, defuzzification functionals, intuitionistic fuzzy sets.

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1 Introduction

A fuzzy implication (FI), as an extension of the classical binary implication, is commonly defined as a two-place operation on the unit interval [7, 12]. Its generalization needs only a complete lattice structure of the domain. In this chapter we will be concerned with such a generalization. The domain on which we give our main definition will be a subspace of the space of ordered fuzzy numbers $\mathcal{R}$, recently introduced by the second author and his 2 coworkers: Dominik Ślęzak and Piotr Prokopowicz, see [15, 16, 17, 18, 19]. Then a weaker assumption concerning the continuity of parametric representation of fuzzy numbers was assumed in [14], i.e. they may be functions of bounded variation, i.e. belong to BV. Then all convex fuzzy numbers are contained in this new space $\mathcal{R}_{BV} \supset \mathcal{R}$ of OFN. Notice that functions from BV [24] are continuous except for a countable numbers of points. Important consequence of this generalization is a possibility of introducing a subspace of OFN composed of pairs of step functions.

Fuzzy implications play important roles in both mathematical and applied sides of fuzzy set theory. They appear in fuzzy logic (FL), in processes of approximate reasoning (AR), in decision support systems (DSS) in several applications where fuzzy control (FC) is suitable, and in many other areas of modern technology. Many different fuzzy implication operators have been proposed and many papers have been written on the subject. One can consult the recent monograph on fuzzy implications published in 2008 [2].

There are several classes of fuzzy implications defined on the unit square $[0,1] \times [0,1]$ with values in the unit interval $I = [0,1]$. The most known is the fuzzy counterpart of the right handed side of the binary logic tautology $a \Longrightarrow b \equiv b \vee \neg a$, called the Kleene–Dienes operation, and defined by the invention of the negation operator $\neg$ to membership functions. The Kleene–Dienes implication $A \Longrightarrow B$ is given by $\max\{1-\mu_A, \mu_B\}$. To some generalizations of the binary implication belongs the so-called S-implication $\mathcal{I}_s(A,B)$ defined by the formula

$$\mathcal{I}_s(A,B) = \mathcal{S}(1-\mu_A, \mu_B)\,, \tag{1}$$

where $\mathcal{S}$ is any s-norm [6]. The implication invented by Łukasiewicz [25], in his 3-valued logic takes the form $\min\{1, 1-\mu_A+\mu_B\}$. Notice that for particular fuzzy numbers $A = B = 1/2$ (in fact they are crisp) the binary implication takes the value $1/2$ while the Łukasiewicz implication gives the full truth, i.e. the value 1.

Another class of implications form Q-implication $\mathcal{I}_q(A,B)$, and they can written by the use of the formula

$$\mathcal{I}_q(A,B) = \mathcal{S}(1-\mu_A, \mathcal{T}(\mu_A, \mu_B))\,, \tag{2}$$

where $\mathcal{S}$ and $\mathcal{T}$ are general s-norm and t-norm, respectively. The example of Q–implication is the Zadeh implication [33] defined by

$$\max\{\min\{\mu_A, \mu_B\}, 1 - \mu_A\}\,.$$

It is no difficult to check that most of invented fuzzy implications are consistent with the classical binary logic implication. Although the definition of the fuzzy implication defined as a two-place operation on the unit interval requires only 3 of 4 conditions of the binary logic implication (cf. Definition 5).

What is unpleasant with all those implications: if we restrict our attention to convex fuzzy numbers (CFN) of compact supports as their original domain they do not lead to convex fuzzy numbers, as their operation results, they have, in general, unbounded supports [28]. The aim of our chapter is to omit this drawback by defining fuzzy implications by restricting our attention to ordered fuzzy numbers, which form a generalization of all CFN with the well defined 4 algebraic operations. Notice, that in OFN only the operation of addition gives the same result as the interval calculation and Zadeh's extension principle.

The organization of the chapter is as follows. First a short repetition on ordered fuzzy numbers and operations defined on them will be made. Then the subspace of step ordered fuzzy numbers will be introduce and defuzzification functionals defined on them. The next section will bring the lattice structure of $\mathcal{R}_K$, with the fixed resolution K. Then the particular subset $\mathcal{N}$ of $\mathcal{R}_K$ will be introduced composed of pairs of binary K-dimensional vectors. The set $\mathcal{N}$ with two operations forms a complete and complemented lattice in which complements are unique; in fact it is a *Boolean algebra*. On this algebra a strong negation is introduced and the counterpart of the Kleene–Dienes operation as a new binary implication on the set $\mathcal{N} \subset \mathcal{R}_{\mathcal{K}}$ of step ordered fuzzy numbers. Two other implications will be proposed as counterparts of the Łukasiewicz and Q-implications. The paper brings different approach for the construction of fuzzy implications on OFN from that presented in our previous publications [20, 21, 32] and the idea of P.Prokopowicz from his Ph.D. thesis [29]. The interpretation of the introduced concept and its relation to the classical fuzzy set objects are given in the next section. A generalization of Atanasov's intuitionistic fuzzy sets in the form of suspected fuzzy sets ends the paper.

2 Ordered fuzzy numbers

Proposed recently by the first author and his two coworkers: P.Prokopowicz and D. Ślęzak [15, 16, 17, 18, 19] an extended model of convex fuzzy numbers [27]

(CFN), called ordered fuzzy numbers (OFN), does not require any existence of membership functions. In this model an ordered fuzzy number is a pair of continuous functions, f and g, say, defined on the interval $[0, 1]$ with values in **R**. To see OFN as an extension of CFN - model, take a look on a parametric representation know since 1986, [8] of convex fuzzy numbers.

The continuity of both parts implies their images are bounded intervals, say UP and $DOWN$, respectively. We may used symbols to mark boundaries for $UP = [l_A, 1_A^-]$ and for $DOWN = [1_A^+, p_A]$. In general, the functions f, g need not to be invertible, only continuity is required. If we add the constant function on the interval $[1_A^-, 1_A^+]$ with its value equal to 1, we might define the membership function

$$\begin{aligned} \mu(x) = \mu_{up}(x), \text{ if } x \in [l_A, 1_A^-] = [f(0), f(1)], \\ \mu(x) = \mu_{down}(x), \text{ if } x \in [1_A^+, p_A] = [g(1), g(0)] \text{ and} \\ \mu(x) = 1 \text{ when } x \in [1_A^-, 1_A^+] \end{aligned} \tag{3}$$

if

1. $f \leq g$ are both invertible, i.e. inverse functions $f^{-1} =: \mu_{up}$ and $g^{-1} =: \mu_{down}$ exist,
2. f is increasing, and g is decreasing, and such that
3. $f \leq g$ (pointwise).

Obtained in this way the membership function $\mu(x), x \in \mathbf{R}$ represents a mathematical object which reminds a convex fuzzy number in the classical sense [5, 11].

On OFN four algebraic operations have been proposed between fuzzy numbers and crisp (real) numbers, in which componentwise operations are present. In particular if $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ are mathematical objects called ordered fuzzy numbers, then the sum $C = A + B$, product $C = A \cdot B$, division $C = A \div B$ and scalar multiplication by real $r \in \mathbf{R}$, are defined in natural way:

$$r \cdot A = (rf_A, rg_A)\,,$$

and

$$f_C(y) = f_A(y) \star f_B(y), \qquad g_C(y) = g_A(y) \star g_B(y)\,, \tag{4}$$

where "$\star$" works for "$+$", "$\cdot$", and "$\div$", respectively, and where $A \div B$ is defined, if the functions $|f_B|$ and $|g_B|$ are bigger than zero. Notice that the subtraction of B is the same as the addition of the opposite of B, i.e. the number $(-1) \cdot B$, and consequently $B - B = 0$. From this follows that any fuzzy algebraic equation

$A + X = C$ with given A and C as OFN possesses a solution, that is OFN, as well. Moreover, to any convex and continuous fuzzy number correspond two OFNs, they differ by the orientation: one has positive, say (f, g) , another (g, f) has negative.

A relation of partial ordering in the space of all OFN, denoted by $\mathcal{R}$, can be introduced by defining the subset of 'positive' ordered fuzzy numbers: a number $A = (f, g)$ is not less than zero, and by writing

$$A \geq 0 \quad \text{iff} \quad f \geq 0,\ g \geq 0\,. \tag{5}$$

In this way the set $\mathcal{R}$ becomes a partially ordered ring. Notice, that for each two fuzzy numbers $A = (f_A, g_A), B = (f_B, g_B)$ as above, we may define $A \wedge B =: F$ and $A \vee B =: G$, both from $\mathcal{R}$, by the relations:

$$F = (f_F, g_F),\ \text{if}\ \ f_F = \inf\{f_A, f_B\}\,, g_F = \inf\{g_A, g_B\}\,. \tag{6}$$

Similarly, we define $G = A \vee B$.

Notice that in the definition of OFN it is not required that two continuous functions f and g are (partial) inverses of some membership function. Moreover, it may happen that the membership function corresponding to A does not exist; such numbers are called improper.

In any case for $A = (f, g)$ we call f - the up-part and g - the down-part of the fuzzy number A. To be in agreement with further and classical denotations of fuzzy sets (numbers), the independent variable of the both functions f and g is denoted by y (or some times by s), and their values by x.

In dealing with applications of fuzzy numbers we need set of functionals that map each fuzzy number into real, and in such a way that is consistent with operations on reals. Those operations are called defuzzifications. To be more strict we introduce

Definition 1. A map ϕ from the space $\mathcal{R}$ of all OFN's to reals is called a defuzzification functional if is satisfies:

1. $\phi(c^\ddagger) = c\,,$
2. $\phi(A + c^\ddagger) = \phi(A) + c\,,$
3. $\phi(cA) = c\phi(A)\,,$ for any $c \in \mathbf{R}$ and $A \in \mathcal{R}\,.$

where $c^\ddagger(s) = (c, c)\,, s \in [0, 1]$, represents crisp number (a real) $c \in \mathbf{R}$.

From this follow that each defuzzification functional must be homogeneous of order one, restrictive additive, and some how normalized.

2.1 Step ordered fuzzy numbers

It is worthwhile to point out that a class of ordered fuzzy numbers (OFNs) represents the whole class of convex fuzzy numbers that possess continuous membership functions. To include all CFN (with discontinuous membership functions) some generalization of functions f and g is needed. This has been already done by the first author who in [14] assumed they are functions of bounded variation. i.e. they belong to BV. Then all convex fuzzy numbers are contained in this new space $\mathcal{R}_{BV} \supset \mathcal{R}$ of OFN. Then operations are defined $\mathcal{R}_{BV}$ in the similar way, the norm, however, will change into the norm of the Cartesian product of the space of functions of bounded variations. Then all convex fuzzy numbers are contained in this new space $\mathcal{R}_{BV}$ of OFN. Notice that functions from BV [24] are continuous except for a countable numbers of points.

Important consequence of this generalization is the possibility of introducing a subspace of OFN composed of pairs of step functions. If we fix a natural number K and split $[0,1)$ into $K-1$ subintervals $[a_i, a_{i+1})$, i.e. $\bigcup_{i=1}^{K-1} [a_i, a_{i+1}) = [0,1)$, where $0 = a_1 < a_2 < ... < a_K = 1$, and define a step function f of resolution K by putting u_i on each subinterval $[a_i, a_{i+1})$, then each such function f is identified with a K-dimensional vector $f \sim \boldsymbol{u} = (u_1, u_2 ... u_K) \in \mathbf{R}^K$, the K-th value u_K corresponds to $s = 1$, i.e. $f(1) = u_K$. Taking a pair of such functions we have an ordered fuzzy number from $\mathcal{R}_{BV}$. Now we introduce

Definition 2. By a step ordered fuzzy number A of resolution K we mean an ordered pair (f, g) of functions such that $f, g : [0,1] \to \mathbf{R}$ are K-step functions.

We use $\mathcal{R}_K$ for denotation the set of elements satisfying Def. 2. The example of a step ordered fuzzy number and its membership function are shown in Fig. 1 and Fig. 2. The set $\mathcal{R}_K \subset \mathcal{R}_{BV}$ has been extensively elaborated by our students in [9] and [22]. We can identify $\mathcal{R}_K$ with the Cartesian product of $\mathbf{R}^K \times \mathbf{R}^K$ since each K-step function is represented by its K values. It is obvious that each element of the space $\mathcal{R}_K$ may be regarded as an approximation of elements from $\mathcal{R}_{BV}$, by increasing the number K of steps we are getting the better approximation. The norm of $\mathcal{R}_K$ is assumed to be the Euclidean one of $\mathbf{R}^{2K}$, then we have a inner-product structure for our disposal.

2.2 Defuzzification functionals on $\mathcal{R}_K$

On the space $\mathcal{R}_K$ a representation formula for a general non-linear defuzzification functional $H : \mathbf{R}^K \times \mathbf{R}^K \to \mathbf{R}$ satisfying the conditions 1.– 3., can be given

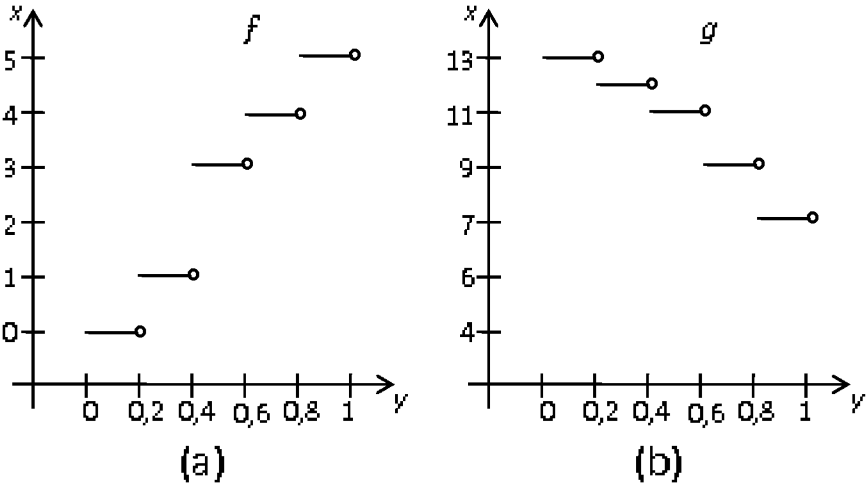

Figure 1: Example of a step ordered fuzzy number $A = (f, g) \in \mathcal{R}_K$, (a) function f, (b) function g.

as a linear composition [30] of arbitrary homogeneous of order one, continuous function G of $2K-1$ variables, with the 1D identity function, i.e.

$$\begin{aligned} H(\underline{u}, \underline{v}) = u_j + \\ G(u_2 - u_j, ..., u_K - u_j, v_1 - u_j, ..., v_K - u_j)\,, \\ \text{with } \underline{u} = (u_1, ..., u_K)\,, \underline{v} = (v_1, ..., v_K)\,, \end{aligned} \tag{7}$$

and some $1 \leq j \leq K$. It is seen that G is given by F in which its j-th argument was put equal to zero.

Due to the fact that $\mathcal{R}_K$ is isomorphic to $\mathbf{R}^K \times \mathbf{R}^K$ we conclude, from the Riesz theorem and the condition 1. that a general linear defuzzification functional on $\mathcal{R}_K$ has the representation

$$H(\underline{u}, \underline{v}) = \underline{u} \cdot \underline{b} + \underline{v} \cdot \underline{d}\,, \tag{8}$$

with arbitrary $\underline{b}\,, \underline{d} \in \mathbf{R}^K$, such that $\underline{1} \cdot \underline{b} + \underline{1} \cdot \underline{d} = 1$, where $\cdot$ denotes the inner (scalar) product in $\mathbf{R}^K$ and $\underline{1} = (1, 1, ..., 1) \in \mathbf{R}^K$ is the unit vector in $\mathbf{R}^K$, while the pair $(\underline{1}, \underline{1})$ represents a crisp one in $\mathcal{R}_K$. It means that such a functional is represented by the vector $(\underline{b}, \underline{d}) \in \mathbf{R}^{2K}$.

Let us take $\underline{b} = \underline{c}$ and such, that all their components are equal to $1/2K$, and denote such defuzzification functional by ψ_K.

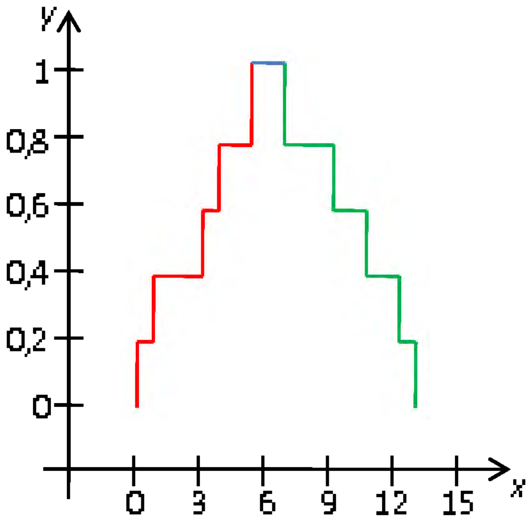

Figure 2: Membership function of the above step ordered fuzzy number $A = (f, g) \in \mathcal{R}_K$.

Now let $\mathcal{B}$ be the set of two binary values: $0, 1$ and let us introduce the particular subset $\mathcal{N}$ of $\mathcal{R}_K$

$$\mathcal{N} = \{A = (\underline{u}, \underline{v}) \in \mathcal{R}_K : \underline{u} \in \mathcal{B}^K, \underline{v} \in \mathcal{B}^K\} \tag{9}$$

It means such that each component of the vector $\underline{u}$ as well as of $\underline{v}$ has value 1 or 0. Since each element of $\mathcal{N}$ is represented by a $2K$-dimensional binary vector the cardinality of the set $\mathcal{N}$ is 2^{2K}. Then after defuzzification even if we apply the linear functional ψ_K, we may have all possible fractional numbers $i/2K$, with $i = 0, 1, ..., 2K$, as its values on $\mathcal{N}$. The set $\mathcal{N}$ will play fundamental roles in the next section.

3 Lattice structure on $\mathcal{R}_K$

Let us consider the set $\mathcal{R}_K$ of step ordered fuzzy numbers with operations $\vee$ and $\wedge$ such that for $A = (f_A, g_A)$ and $B = (f_B, g_B)$,

$$A \vee B = (sup\{f_A, f_B\}, sup\{g_A, g_B\})$$

and

$$A \wedge B = (inf\{f_A, f_B\}, inf\{g_A, g_B\}).$$

In [10] we have shown that the algebra $(\mathcal{R}_K, \vee, \wedge)$ defines a lattice structure and proved the following theorem.

Theorem 1. The algebra $(\mathcal{R}_K, \vee, \wedge)$ is a lattice.

It is easy to observe that all subsets of $\mathcal{N}$ have both a join and a meet in $\mathcal{N}$. In fact, for every pair of numbers from the set $\{0, 1\}$ we can determine *max* and *min* and it is always 0 or 1. Therefore $\mathcal{N}$ creates a *complete lattice*. In such a lattice we can distinguish the greatest element $\underline{1}$ represented by the vector $= (1, 1, ..., 1)$ and the least element $\underline{0}$ represented by the vector $(0, 0, ..., 0)$.

Theorem 2. The algebra $(\mathcal{N}, \vee, \wedge)$ is a complete lattice.

3.1 Complement and negation

In a lattice in which the greatest and the least elements exist it is possible to define complements. We say that two elements A and B are *complements* of each other if and only if

$$A \vee B = \underline{1}$$

and

$$A \wedge B = \underline{0}.$$

The complement of a number A will be marked with $\neg A$ and is defined as follows:

Definition 3. Let $A \in \mathcal{N}$ be a step ordered fuzzy number represented by a binary vector $(a_1, a_2, \ldots, a_{2K})$. By the complement of A we understand

$$\neg A = (1 - a_1, 1 - a_2, \ldots, 1 - a_{2K}).$$

A bounded lattice for which every element has a complement is called a *complemented lattice*. Moreover, the structure of step ordered fuzzy numbers $\{\mathcal{N}, \vee, \wedge\}$ forms a complete and complemented lattice in which complements are unique. In fact it is a *Boolean algebra*. In the example with $K = 2$ a set of universe is created by binary vectors

$$\mathcal{N} = \{(a_1, a_2, a_3, a_4) \in \mathbf{R}^4 : a_i \in \{0, 1\}, \text{for } i = 1, 2, 3, 4\}.$$

The complements of elements are $\neg(0, 0, 0, 0) = (1, 1, 1, 1)$, $\neg(0, 1, 0, 0) = (1, 0, 1, 1)$, $\neg(1, 1, 0, 0) = (0, 0, 1, 1)$ etc.

Now we can rewrite the definition of the complement in terms of a new mapping.

Definition 4. For any $A \in \mathcal{N}$ we define its negation as

$$N(A) := (1 - a_1, 1 - a_2, \ldots, 1 - a_{2K})\,, \text{if } A = (a_1, a_2, \ldots, a_{2K}).$$

It is obvious, from Definitions 3 and 4, that the negation of given number A is its complement. Moreover, the operator N is a strong negation, because is involutive, i.e.

$$N(N(A)) = A \text{ for any } A \in \mathcal{N}\,.$$

One can refer here to known facts from the theory of fuzzy implications (cf. [2, 3, 7]) and to write the strong negation N in terms of the standard strong negation N_I on the unit interval $I = [0, 1]$ defined by $N_I(x) = 1 - x\,, x \in I$, namely $N((a_1, a_2, \ldots, a_{2K})) = ((N_I(a_1), N_I(a_2), \ldots, N_I(a_{2K}))$.

3.2 Implications

In the classical Zadeh's fuzzy logic the definition of a fuzzy implication on an abstract lattice $\mathcal{L} = (L, \leq_L)$ is based on the notation from the fuzzy set theory introduced in [7].

Definition 5. Let $\mathcal{L} = (L, \leq_L, 0_L, 1_L)$ be a complete lattice. A mapping $\mathcal{I} : L^2 \to L$ is called a fuzzy implication on $\mathcal{L}$ if it is decreasing with respect to the first variable, increasing with respect to the second variable and fulfills the border conditions

$$\mathcal{I}(0_L, 0_L) = \mathcal{I}(1_L, 1_L) = 1_L\,, \mathcal{I}(1_L, 0_L) = 0_L\,. \tag{10}$$

Now, possessing the lattice structure of $\mathcal{R}_\mathcal{K}$ (SOFN) and the Boolean structure of our lattice $\mathcal{N}$, we can repeat most of the definitions know in the Zadeh's fuzzy set theory. The first one is the Kleene–Dienes operation, called a binary implication, already introduced in our previous paper [10] as the new implication (cf. Definition 4 in [10])

$$\mathcal{I}_b(A, B) = N(A) \vee B\,, \text{ for any} A, B \in \mathcal{N}\,. \tag{11}$$

In other words, the result of the binary implication $\mathcal{I}_b(A, B)$, denoted in [10] by $A \to B$, is equal to the result of operation *sup* for the number B and the complement of A:

$$A \to B = sup\{\neg A, B\}.$$

Next we may introduce the Zadeh implication by

$$\mathcal{I}_Z(A, B) = (A \wedge B) \vee N(A)\,, \text{ for any} A, B \in \mathcal{N}\,. \tag{12}$$

Since in our lattice $\mathcal{R}_K$ the arithmetic operations are well defined we may introduce the counterpart of the Łukasiewicz implication by

$$\mathcal{I}_L(A, B) = C\,, \text{where}\, C = 1 \wedge (1 + B - A)\,. \tag{13}$$

In the calculating the RHS of (13) we have to regard all numbers as elements of $\mathcal{R}_K$, since by adding the ordered fuzzy number A from $\mathcal{N}$ to the crisp number 1 we may leave the subset $\mathcal{N} \subset \mathcal{R}_{\mathcal{K}}$. However, the operation $\wedge$ will take us back to the lattice $\mathcal{N}$. It is obvious that in our notation $1_N = 1$. The explicit calculation will be: if $C = (c_1, c_2, \ldots, c_{2K)})$, $A = (a_1, a_2, \ldots, a_{2K})$, $B = (b_1, b_2, \ldots, b_{2K})$, then $c_i = min\{1, 1 - a_i + b_i\}$, where $1 \leq i \leq 2K$.

It is obvious that all implications $\mathcal{I}_b, \mathcal{I}_Z$ and $\mathcal{I}_L$ satisfy the border conditions (10) as well as the 4th condition of the classical binary implication, namely $\mathcal{I}(0_N, 1_N) = 1_N$.

Since $\mathcal{N}$ is the complete lattice it is obvious that we can define counterparts of t-norms and s-norms. The first example of a t-norm is given as the product, i.e.

$$\mathcal{T}(A, B) = A \cdot B\,, \text{for any } A, B \in \mathcal{N}\,. \tag{14}$$

We call this t-norm a product t-norm. Having t-norms and s-norms introduced we can define (S,N)-implications, since in $\mathcal{N}$ we have the standard strong negation, as well. It will by the subject of further research.

4 Interpretations

Having the described above implication we can apply step ordered fuzzy numbers for evaluation of linguistic statements like “a patient is fat” or “a car is fast” and reasoning on them. Below, we present a method of assigning elements of the set $\mathcal{N}$ to ordered fuzzy numbers representing such statements.

Consider a classical (convex) fuzzy number $Z \in$ CFN with its membership function μ_Z. Let us recall that for Z we may define for each $s \in (0, 1]$ the s-cut (or s-section) of the number (of the membership function) Z as the classical set Z_s by

$$Z_s = \{x \in \mathbf{R} : \mu_Z(x) \geq s\}\,. \tag{15}$$

For each convex fuzzy number Z and two numbers $s_1 \leq s_s$ the following relation $Z_{s_2} \subset Z_{s_1}$ between the corresponding s-sections holds.

Now let us fix the resolution K of step functions defining the $\mathcal{R}_K$ and take the partition of the unit interval into $K-1$ subintervals $\bigcup_{i=1}^{K-1}[a_i, a_{i+1}) \cup \{a_K\} = [0,1]$, with $0 = a_1 < a_2 < ... < a_K = 1$.

Then we may define a mapping

$$val_K : \mathbf{R} \times \text{CFN} \to \mathcal{N} \tag{16}$$

which for given Z and each $x \in Z_{a_i} - Z_{a_{i+1}}$ attaches an element of the set $\mathcal{N}$, a step ordered fuzzy number, in such a way that $\psi_K(val_K(x, Z)) = a_i$, i.e. after defuzzification the value of $val_K(x, Z)$ we get the value of the membership function of Z at the lower end of the s-section to which x belongs. In this way the one-variable function $val_K(\cdot, Z) : \mathbf{R} \to \mathcal{N}$ is piecewise constant: it is constant on each subinterval $Z_{a_i} - Z_{a_{i+1}}$. It means that after defuzzification the correspondence given by the function $val_K(\cdot\,, Z)$ is in agreement with the value of the membership function attached to x by μ_Z, module the assumed finite step-wise approximation of values of the membership function.

If we use the so-called parametric representation of convex fuzzy numbers [8] in terms of two left-continuous functions α_1, α_2, the both defined on the interval $[0, 1]$ with values in $\mathbf{R}$, and denote by x_{1^-} and x_{1^+} the points from the support of μ_Z, such that $\mu_Z(x_{1^-}) = \mu_Z(x_{1^+}) = 1$, and at the point x_{1^-} the membership function attains for the first time the value 1, and the point x_{1^+} is the last point with this property , then the condition $x \in Z_{a_i} - Z_{a_{i+1}}$, may be written as $\alpha_1(a_i) \leq x \leq \alpha_2(a_{i+1})$ if $x \leq x_{1^-}$ and $\alpha_1(a_{i+1}) \leq x \leq \alpha_2(a_i)$ if $x \geq x_{1^+}$. This is so, because the function α_1 is non-decreasing and the function α_2 is non-increasing.

Notice that given by (11) the new binary implication $\mathcal{T}_b$ may be implemented in the fuzzy inference if for a classical fuzzy rule [6, 4, 33]: If 'a condition is satisfied' Then 'a consequence follows', when both parts: premise and consequent, are fuzzy, the mapping val_K will be applied to both parts . In the next paper examples of such application will be given.

5 On intuitionistic fuzzy sets of type 2

It seems that the classical Atanasov's intuitionistic fuzzy sets [1] may be generalized to ordered fuzzy numbers from the lattice $\mathcal{N}$.

First we generalized the concept of interval-valued fuzzy set on a universe X.

Definition 6. A mapping $V : X \to \mathcal{R}_{BV}$ is called OFN-valued fuzzy set.

It means that $x_{1^-} = \alpha_1(1)$ and $x_{1^+} = \alpha_2(1)$.

It seems that our approach reminds to some extend the fuzzy sets of type 2.

Now we may confine our interest to those mappings V which have their values in the subspace $\mathcal{R}_K$. In particular in the lattice $\mathcal{N}$. Since $\mathcal{N}$ is isomorphic to $\mathbf{R}^{2K}$, in this way we are generalizing the classical concept of the interval-valued fuzzy set on X (cf. Definition 2 in [3]).

Let us repeat the concept of intuitionistic fuzzy set on X (cf. Definition 1 in [3]).

Definition 7. An intuitionistic fuzzy set A on X is a set

$$A = \{(x, \mu(x), \nu(x)) \, : x \in X\} \tag{17}$$

where $\mu, \nu : X \to [0, 1]$, and $\mu(x) + \nu(x) \leq 1$, for all $x \in X$. The first function is called the membership function and the second - the non-membership function

Since the exponent of $\mathbf{R}^{2K}$ is even, and each element $A \in \mathbf{R}^{2K}$ is represented by a pair step functions of resolution K and , one the other hand, they are uniquely represented by a pair of K-dimensional vectors, $(\underline{u}, \underline{v})$ say, we may attach to the first step function the fuzzy value of the membership while to the second element - the fuzzy value of the non-membership. We need, only, to superpose the condition

$$\underline{u} + \underline{v} \leq \underline{1}.$$

This is the inequality between K-dimensional vectors, hence it should be understood componentwise, of course. The set of those elements of $\mathcal{N}$ which satisfy the last inequality will be denoted $\mathcal{N}_I$.

Definition 8. A mapping $V : X \to \mathcal{T}_I$ is called a suspected fuzzy set.

We can see that by in the general case, i.e. with an arbitrary K superposing any defuzzification operator to the each component of the values V we obtain an intuitionistic fuzzy set. However, in the case $K = 1$ we have the following remark.

Proposition 3. If $K = 1$ then the space of suspected fuzzy set can be identified with the the space of intuitionistic fuzzy set on X.

The proof is obvious. In future work we are going to show application and usefulness of that new concepts especially for modelling uncertain beliefs.

6 Conclusions

So far, ordered fuzzy numbers was applied to deal with optimization problems when data are fuzzy. In this paper we present how they can be used for approximate *reasoning* about uncertainty linguistic propositions. In order to do this a new fuzzy implication on step ordered fuzzy numbers is introduced. In classical two-valued logics only two logical values are applied: 0 or 1. In fuzzy logics it is extended to the values from the interval [0,1]. Our contribution is to enrich these formal systems and use for logical justification step ordered fuzzy numbers. This approach is very innovative and allows for including in logical value more information than that something is true, true with some degree or false. In future work we are going to show application and usefulness of this new reasoning on diverse examples, especially for modelling uncertain beliefs of agents in multi-agent systems.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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