Polska Akademia Nauk • Instytut Badań Systemowych

AUTOMATYKA STEROWANIE ZARZĄDZANIE

Książka jubileuszowa
z okazji
70-lecia urodzin

PROFESORA KAZIMIERZA MAŃCZAKA

pod redakcją
Jakuba Gutenbauma

Polska Akademia Nauk • Instytut Badań Systemowych

AUTOMATYKA
STEROWANIE
ZARZĄDZANIE

Książka jubileuszowa
z okazji
70-lecia urodzin

PROFESORA KAZIMIERZA MAŃCZAKA

pod redakcją
Jakuba Gutenbauma

Warszawa 2002

Książka jubileuszowa z okazji
70-lecia urodzin
Profesora Kazimierza MAŃCZAKA

Redaktor
prof. dr hab. inż. Jakub Gutenbaum

Warszawa 2002

ISBN 83-85847-78-2

Wydawca: Instytut Badań Systemowych PAN
ul. Newelska 6 01-447 Warszawa
http://www.ibspan.waw.pl

Opracowanie składopisu: Anna Gostyńska, Jadwiga Hartman

Druk: KOMO-GRAF, Warszawa
nakład 200 egz., 34 ark. wyd., 31 ark. druk.

ON THE ADJUSTMENT PROBLEM IN COMBINATORIAL OPTIMIZATION

Marek Libura

Systems Research Institute, Polish Academy of Sciences
Newelska 6, 01-447 Warszawa, <libura@ibspan.waw.pl>

Abstract: *In this paper we introduce the* adjustment problem *corresponding to the generic combinatorial optimization problem. It consists in finding less costly perturbations of weights in the original problem, which guarantee that the optimal solution of the perturbed problem belongs to the specified subset of feasible solutions. We study properties of the adjustment problem an its relations to standard inverse problem in combinatorial optimization.*

Keywords: *combinatorial optimization, adjustment problem, inverse problems, sensitivity analysis.*

1. Introduction

Let $E = \{e_1, \dots, e_n\}$ be an arbitrary finite set called the *ground set*. For any subset $F \subseteq E$, $\xi(F) = (\xi_1(F), \dots, \xi_n(F))^T \in \{0,1\}^n$ denotes the characteristic vector of F, i.e., $\xi_i(F) = \lfloor e_i \in F \rceil$, $i = 1, \dots, n$, where for any sentence Q, $\lfloor Q \rceil = 1$ if and only if the logical value of Q is *truth*.

Let $w : \mathbf{R}^n \times 2^E \to \mathbf{R}$ denote the real valued function, which we will call the *weight function*. In this paper we assume, that for $F \subseteq E$

$$w(c, F) = c^T \cdot \xi(F), \tag{1}$$

where $c \in \mathbf{R}^n$ is a vector of the so-called *weights* of elements of the ground set.

For a family of subsets $\mathcal{G} \subseteq 2^E$ and $c \in \mathbf{R}^n$ let

$$\mu(c, \mathcal{G}) = \min\{w(c, F) : F \in \mathcal{G}\},$$

with standard convention that for arbitrary vector $c \in \mathbf{R}^n$, $\mu(c, \mathcal{G}) = \infty$ if $\mathcal{G} = \emptyset$.

Given the weight vector $c \in \mathbf{R}^n$ and a family $\mathcal{F} \subseteq 2^E$ of the so-called *feasible subsets* (*feasible solutions*), the generic *combinatorial optimization problem* is defined as follows:

$$\text{Find } F^* \in \mathcal{F} \text{ such that } w(c, F^*) = \mu(c, \mathcal{F}).$$

In this paper we will use also a more standard notation for the combinatorial optimization problem:

$$\min_{F \in \mathcal{F}} w(c, F). \tag{P}$$

Sometimes it is required to find not only a single set F^* satisfying the condition $w(c, F^*) = \mu(c, \mathcal{F})$, but the family of all such sets. Given $\mathcal{F} \subseteq 2^E$ and $c \in \mathbf{R}^n$, we will denote this family by $\Omega(c, \mathcal{F})$, and we will call any of its element an *optimal solution* of the problem (P).

For $\mathcal{F} \subseteq E$ and arbitrary family $\mathcal{G} \subseteq \mathcal{F}$ we will define the set $S(\mathcal{G})$ of all weight vectors, for which any solution belonging to $\mathcal{G}$ is an optimal solution of the problem (P). Namely,

$$S(\mathcal{G}) = \{c \in \mathbf{R}^n : w(c, F) = \mu(c, \mathcal{F}) \text{ for any } F \in \mathcal{G}\}.$$

The set of vectors $S(\mathcal{G})$ is called *the optimality region* with respect to the family $\mathcal{G}$.

The optimality region with respect to the family of feasible solutions generalizes in a natural way the notion of so called *stability region* with respect to a single solution $F^o \in \mathcal{F}$ e.g.: (Greenberg 1998, Libura 1996, Libura et al. 1998, Sotskov et al. 1995).

It is well known that for any $F^o \in \mathcal{F}$ the stability region $S(\{F^o\})$ is a polyhedral convex cone in $\mathbf{R}^n$. This implies that also an optimality region with respect to the family $\mathcal{G}$ forms a polyhedral convex cone in $\mathbf{R}^n$, which is simply an intersection of stability regions with respect to all solutions belonging to the family $\mathcal{G}$.

Most of discrete optimization problems can be stated in the above form or – at least – reformulated as problem (P). In this paper we will use as an example the following combinatorial optimization problem:

Example
Consider the symmetric undirected graph G shown in Figure 1. Let E be the set of all edges of the graph G, i.e., $E = \{e_1, \ldots, e_7\}$, and let $\mathcal{T}$ denote the set of all spanning trees in the graph G. From the theorem by Kirchhoff (e.g. Graham et al. 1995) it is easy to calculate that $|\mathcal{T}| = 21$. Figure 2 presents all the spanning trees belonging to $\mathcal{T}$.

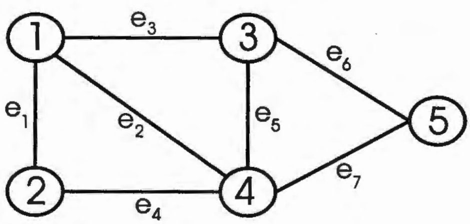

Fig. 1. Graph $G = (V, E)$ from the Example.

Assume now that in the formulation of the combinatorial optimization problem (P) we take $\mathcal{F} = \mathcal{T}$ and $c = c^o = (4, 4, 1, 5, 3, 7, 8)^T$. Thus, we are faced with a well known (e.g. Nemhauser and Woolsey 1988) *minimum spanning tree problem* on the graph G with lengths of edges given by the vector c^o. This problem has a single optimal solution $F^* = \{e_1, e_3, e_5, e_6\}$, so we have $\Omega(c^o, F) = \{F^*\}$ and $w(c^o, F^*) = 15$.

□

2. The adjustment problem

Consider the combinatorial optimization problem (P) stated for $\mathcal{F} \subseteq 2^E$ and $c^o \in \mathbf{R}^n$

$$\min_{F \in \mathcal{F}} w(c^o, F). \tag{2}$$

Given an arbitrary subset of feasible solutions $\overline{\mathcal{F}} \subseteq \mathcal{F}$, a set of vectors of weights $C \subseteq \mathbf{R}^n$, and a real valued function $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$,

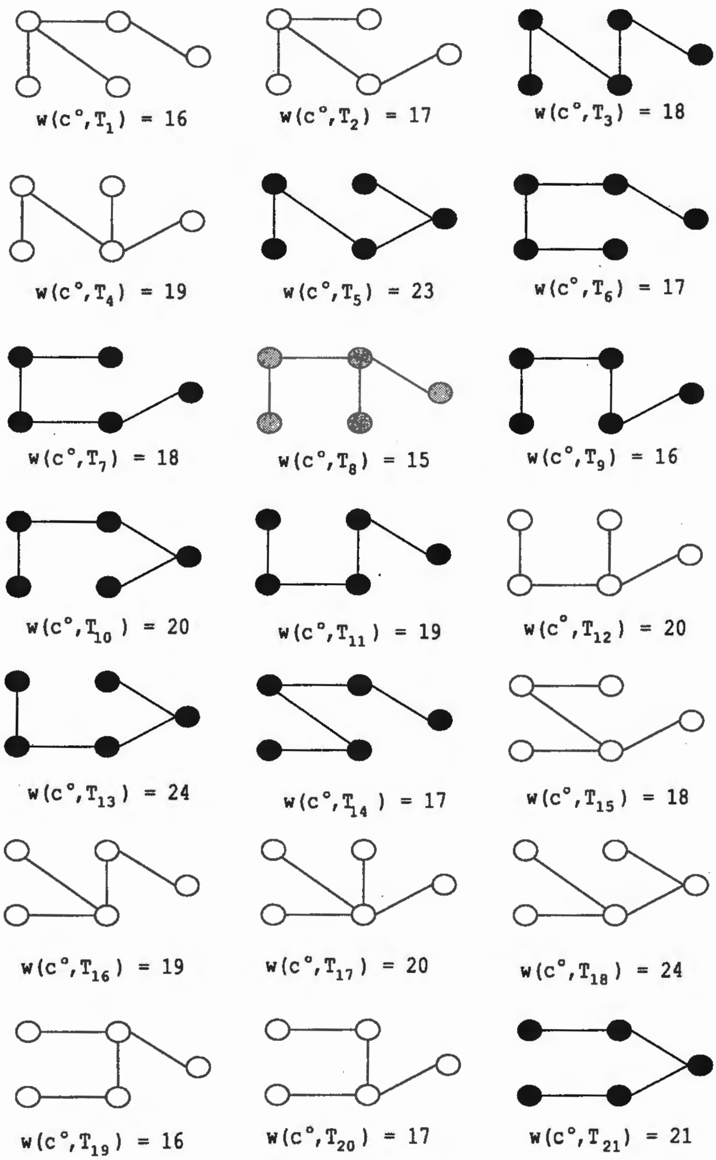

Fig. 2. All spanning trees in the graph G and its weights for $c = c^o$.

we define the *adjustment problem* related to problem (2) in the following form:

Find $c^* \in \mathcal{C}$ such that

$$f(c^*, c^o) = \min\{f(c, c^o) : \ c \in \mathcal{C}\}$$

and

$$\mu(c^*, \overline{\mathcal{F}}) = \mu(c^*, \mathcal{F}).$$

We can shortly denote the adjustment problem as follows:

$$\begin{aligned} \min_{c \in \mathcal{C}} \ & f(c, c^o) \\ & \mu(c, \overline{\mathcal{F}}) = \mu(c, \mathcal{F}). \end{aligned} \tag{3}$$

Let $a(\overline{\mathcal{F}})$ denote the optimal value of the problem (3); we will call this value the *adjustment cost* related to the subset $\overline{\mathcal{F}}$.

The adjustment problem may be interpreted in the following way:

For a given combinatorial optimization problem (P) and an initial vector of weights c^o we want to find a new vector of weights c^*, belonging to a specified set $\mathcal{C}$, and such that some optimal solution of the problem (P) modified in this way, belongs to the set $\overline{\mathcal{F}}$. Moreover, we want to minimize the adjustment cost equal to $f(c^*, c^o)$.

The set $\mathcal{C}$ in the formulation of the adjustment problem is called the *restriction set* and the function f is called the *cost function*. Frequently,

$$f(c, c^o) = ||c - c^o||,$$

where $|| \cdot ||$ denotes some norm in $\mathbf{R}^n$. Moreover, typically $\mathcal{C} = \mathbf{R}^n$ or $\mathcal{C} = \mathbf{R}^n_+$. The subset $\overline{\mathcal{F}}$ of the family of feasible solutions $\mathcal{F}$ is called the *required solutions set.*

One can interpret the adjustment problem as an optimization problem consisting in finding the „cheapest" (measured by the value of cost function) and admissible (i.e., belonging to the set $\mathcal{C}$) perturbation of the original vector of weights, which guarantees that the solution of the original problem becomes a solution of the restricted problem with the feasible set $\overline{\mathcal{F}}$.

Observe that when the restriction set $\mathcal{C}$ contains a vector of weights c^c in which all components are equal to zero, then the adjustment problem (3) has a feasible solution c^c. This follows simply from the fact that in this case all feasible solutions of the problem (P) have the same weight. In particular, a solution of the adjustment problem always exists if $\mathcal{C} = \mathbf{R}^n$ or $\mathcal{C} = \mathbf{R}^n_+$.

Example (continued)

We will formulate an example of the adjustment problem related to the minimum spanning tree problem defined for the graph G shown in Figure 1. Let us take, as before, $\mathcal{F} = \mathcal{T}$ and $c = c^o = (4,4,1,5,3,7,8)^T$. Thus, the initial combinatorial optimization problem (P) is stated as follows:

$$\min_{F\in\mathcal{T}} w(c^o, F). \tag{4}$$

Assume now that we are interested in such a solution of the problem (4) which is not only a spanning tree, but also forms a path in the graph G. This means that we are looking for a solution which is a Hamiltonian path in G. Denote by $\mathcal{H}$ the set of all Hamiltonian paths in G. Obviously, $\mathcal{H} \subseteq \mathcal{T}$. In our very small example it is easy to see from Figure 2, that $\mathcal{H} = \{T_3, T_5, T_6, T_7, T_9, T_{10}, T_{11}, T_{13}, T_{14}, T_{21}\}$ (spanning trees belonging to this subset are distinguished in Figure 2).

Our goal is to make the least costly adjustment of the initial weight vector c^o, which would guarantee that the solution of the modified problem (4) is a Hamiltonian path in G.

Assume that the cost of an adjustment is measured by the l_1 norm in $\mathbf{R}^7$ and that $\mathcal{C} = \mathbf{R}^7$. Thus we have the following adjustment problem related to the minimum spanning tree problem (4):

$$\begin{aligned} \min \quad & \sum_{i=1}^{7} |c(e_i) - c^o(e_i)| \\ & \mu(c, \mathcal{H}) = \mu(c, \mathcal{T}). \end{aligned} \tag{5}$$

We will show later that this problem has the following optimal solution:

$$c^* = (4,4,1,5,3,8,8)^T.$$

We have $f(c^*, c^o) = \sum_{i=1}^{7} |c^*(e_i) - c^o(e_i)| = 1$. Thus, the optimal value of the adjustment problem is equal to 1. Comparing the initial vector of weights c^o and a solution c^* of the problem (5) it is easy to see that it is enough to adjust the initial vector of weights $c^o = (4,4,1,5,3,7,8)^T$ by increasing the weight $c^o(e_6)$ by 1 in order to guarantee that the optimal solution of the modified minimum spanning tree problem becomes a Hamiltonian path in the graph G.

□

The adjustment problem is closely related to so-called *inverse problem*, which attracts recently significant attention e.g.: (Burton 1992, Burton and Toint 1994, Cai et al. 1999, Sokkalingam et al. 1999, Xu and Zhang 1995, Zhang and Cai 1998, Zhang and Ma 1999).

3. The adjustment problem and the inverse problem

Given the combinatorial optimization problem (P) we will define the *inverse optimization problem* (I) in the following general form:

For $c^o \in \mathbf{R}^n$, $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $\mathcal{F}^r \subseteq \mathcal{F}$ and $\mathcal{C} \subseteq \mathbf{R}^n$,
find $c^* \in \mathbf{R}^n$ such that

$$c^* \in \arg \min_{c \in S(\mathcal{F}^r) \cap \mathcal{C}} f(c, c^o) \tag{I}$$

As before, the function f in the statement of the problem (I) is called *the cost function*. The family of subsets $\mathcal{F}^r$ is called *the reference solutions set* and the vector c^o – *the reference weight vector*. The set $\mathcal{C}$ is called *the restrictions region*.

The inverse optimization problem may be interpreted as follows:

For an initial combinatorial optimization problem (P) we want to find a weight vector c^*, belonging to the restrictions region $\mathcal{C}$, for which any solution from the reference set $\mathcal{F}^r$ is optimal in the problem (P) and, moreover, the cost of changing weights vector from the reference value c^o to c^*, measured by the cost function f, is minimum.

Thus, the only difference in statements of the inverse problem and the adjustment problem is that in the inverse problem we require that *all* solutions belonging to reference solutions set became optimal after changes of weights, whereas in the adjustment problem we require that *at least one* solution from this set becomes the optimal one.

If the set $\mathcal{F}^r$ contains a single element, i.e., $\mathcal{F}^r = \{F^o\}$, then the inverse problem is stated as follows:

$$\begin{aligned} \min \quad & f(c, c^o) \\ & c \in S(\{F^o\}) \cap C. \end{aligned} \tag{6}$$

Assume now that the initial data of the inverse problem are fixed, i.e., the initial vector of weights c^o, the function f as well as the set C are given.

Let $i(F^o)$ denote the optimal value of the problem (6). We will call this value the *inverse cost* with respect to the feasible solution F^o. The inverse cost is simply the minimum adjustment cost necessary to make the feasible solution F^o an optimal solution of the problem (P).

It is now easy to see that the adjustment cost related to an arbitrary subset $\overline{\mathcal{F}}$ is equal to the minimum of the inverse costs with respect to all solutions belonging to the set $\overline{\mathcal{F}}$, i.e.,

$$a(\overline{\mathcal{F}}) = \min\{i(F) : F \in \overline{\mathcal{F}}\}.$$

This fact gives, in principle, the way of solving the adjustment problem through solving a sequence of the inverse problems for all elements of the set $\overline{\mathcal{F}}$. In such a 'brute force' solution one could incorporate simple bounds for the optimal value of the inverse problem. Such bounds can be derived for various functions f.

In the following we will consider the most typical formulation of the inverse problem. Namely we will assume that the adjustment cost is measured by l_1 norm and that the restriction set C is taken as $\mathbf{R}^n$. Observe that in this case for $F \in \mathcal{F}$,

$$w(c^o, F) - \mu(c^o, \mathcal{F}) \;\square\; i(F) \;\square\; ||c^o||_{l_1}$$

and if, moreover, $|F|$ is the same for any $F \in \mathcal{F}$ then

$$i(F) \square u(c^o),$$

where $u(c^o)$ denotes l_1 distance of the vector c^o from the line $c(e_i) =$ const, $i = 1, \ldots, n$.

4. Optimality conditions

It is easy to see that both described problems: the adjustment problem and the inverse problem, are closely related to the optimality condition for the initial combinatorial optimization problem. In fact, this is also the reason for difficulty of solving these problems, because such optimality conditions are rather seldom available in combinatorial optimization (Libura 1996).

Assume for simplicity that $C = \mathbf{R}^n$. Then, the inverse problem (I) is stated as follows:

$$\begin{aligned} &\min f(c, c^o) \\ &c \in S(\mathcal{F}^r). \end{aligned} \tag{7}$$

The set $S(\mathcal{F}^r)$ in the formulation of the above problem is an itersection of stability regions $S(\{F\})$ with respect to all solutions $F \in \mathcal{F}^r$.

For some combinatorial optimization problems we can provide a complete description of the cone $S(\{F\})$ and this leads to efficient algorithms for corresponding inverse optimization problems.

However, in general, the only available optimality conditions are the so-called trivial optimality conditions (Libura 1996). One can hardly expect that these optimality conditions might lead to efficient algorithms for the inverse or adjustment problems, but they are useful in understanding some properties of the problems.

We will state below the trivial optimality conditions with respect to some specified feasible solution F^o in the context of the simplest inverse optimization problem (6) assuming that $C = \mathbf{R}^n$.

Let for $F \in \mathcal{F}$, $I' = \{i : e_i \in F^o \setminus F\}$ and $I'' = \{i : e_i \in F \setminus F^o\}$. Then

$$S(F^o) = \{c \in \mathbf{R}^n : c_i = c_i^o + \delta_i^+ - \delta_i^-,\ \delta_i^+, \delta_i^- \geq 0,\ i = 1, \ldots, n,$$
$$\text{and} \sum_{i \in I'} (\delta_i^+ - \delta_i^-) - \sum_{i \in I''} (\delta_i^+ - \delta_i^-) \,\square\, \sum_{i \in I''} c_i^o - \sum_{i \in I'} c_i^o \qquad (8)$$
$$\text{for any} \quad F \in \mathcal{F} \setminus \{F^o\}\ \}.$$

If the cost function $f(c, c^o)$ is given by the l_1 norm in $\mathbf{R}^n$, then the objective function in (7) can be simply expressed with introduced above variables δ_i^+, δ_i^-, $i = 1, \ldots, n$. Namely, we have

$$f(c, c^o) = \sum_{i=1}^{n} (\delta_i^+ + \delta_i^-). \qquad (9)$$

Thus, the inverse optimization problem may be, in principle, formulated as a (large) linear programming problem. We will illustrate this possibility on the following example.

Example (continued) Consider again the minimum spanning tree problem defined for graph G shown on Figure 1. Appendix contains a *Mathematica* program, which generates linear programming problem for calculating the inverse cost for any spanning tree belonging to the set $\mathcal{T}$.

All spanning trees in graph G are given explicitly in table T by the incidence vectors of corresponding subsets of edges. Vector c^o denotes an initial vector of weights.

First, the set of spanning trees is sorted according to nondecreasing weights and a table S of sorted weights is produced. Then, for any spanning tree the linear programming problem defined by matrix *constr* and right-hand-side vector *rhs*, corresponding to inequalities (8) and the objective function (9) is solved. Last line calculates the vector *inv* of inverse costs for all spanning trees, sorted according to nondecreasing weights.

In Figure 3 optimal solutions of inverse problems for all spanning trees in G are shown. Each row of the table contains values of the perturbations δ_i^+, δ_i^-, $i = 1, \ldots, 7$, for weights of graph edges, which correspond to the minimum inverse costs.

ni Below, the vector *inv* of all minimum inverse costs is given:

	δ_1^+	δ_2^+	δ_3^+	δ_4^+	δ_5^+	δ_6^+	δ_7^+	δ_1^-	δ_2^-	δ_3^-	δ_4^-	δ_5^-	δ_6^-	δ_7^-
T_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
T_9	0	0	0	0	0	1	0	0	0	0	0	0	0	0
T_{19}	1	0	0	0	0	0	0	0	0	0	0	0	0	0
T_2	0	0	0	0	1	1	0	0	0	0	0	0	0	0
T_6	0	0	0	0	1	0	0	0	0	0	1	0	0	0
T_{14}	1	0	0	0	1	0	0	0	0	0	0	0	0	0
T_{20}	1	0	0	0	0	1	0	0	0	0	0	0	0	0
T_3	0	0	0	3	0	0	0	0	0	0	0	0	0	0
T_7	0	0	0	0	1	1	0	0	0	0	1	0	0	0
T_{15}	1	0	0	0	1	1	0	0	0	0	0	0	0	0
T_4	0	0	3	0	0	1	0	0	0	0	0	0	0	0
T_{11}	0	0	3	0	0	0	0	0	0	0	1	0	0	0
T_{16}	1	0	3	0	0	0	0	0	0	0	0	0	0	0
T_{10}	0	0	0	0	1	0	0	0	0	0	0	0	3	4
T_{12}	0	0	3	0	0	1	0	0	0	0	1	0	0	0
T_{17}	1	0	3	0	0	1	0	0	0	0	0	0	0	0
T_{21}	1	1	0	0	2	0	0	0	0	0	0	0	2	3
T_5	0	0	6	0	4	0	0	0	0	0	1	0	0	1
T_{13}	0	0	6	0	4	0	0	0	0	0	0	0	0	1
T_{18}	1	0	6	0	4	0	0	0	0	0	0	0	0	1

Fig. 3. Solutions of the inverse problems for all spanning trees from the Example.

$inv = (0, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 8, 5, 5, 9, 11, 12, 12).$

Observe that the spanning trees are ordered according to the non-increasing weights and that there is no corresponding monotonicity in the corresponding inverse costs.

The minimum inverse cost, corresponding to the tree, which forms also a Hamiltonian path, gives the optimal solution of the adjustment problem. In this example this minimum value is equal to 1 and it is achieved for the tree T_9.

□

5. Conclusions

In this paper we introduced the adjustment problem related to the generic combinatorial optimization problem. It can be regarded as a generalization of the standard inverse problem. But the structure of feasible solutions sets for both problems is quite different. Instead of rather simple convex set in the case of inverse problem, we are faced with disjunctive feasible solutions set in the case of adjustment problem, which may lead to its significant difficulty.

6. Appendix

$$\begin{aligned} T = \{ \quad & \{1,1,1,0,0,1,0\}, \{1,1,1,0,0,0,1\}, \{1,1,0,0,0,1,1\}, \\ & \{1,1,0,1,0,0,1\}, \{1,1,0,0,1,1,0\}, \{1,1,0,1,0,1,0\}, \\ & \{1,1,0,0,1,0,1\}, \{1,0,1,0,1,1,0\}, \{1,0,1,0,1,0,1\}, \\ & \{1,0,1,0,0,1,1\}, \{1,0,0,1,1,1,0\}, \{1,0,0,1,1,0,1\}, \\ & \{1,0,0,1,0,1,1\}, \{0,1,1,1,0,1,0\}, \{0,1,1,1,0,0,1\}, \\ & \{0,1,0,1,1,1,0\}, \{0,1,0,1,1,0,1\}, \{0,1,0,1,0,1,1\}, \\ & \{0,0,1,1,1,1,0\}, \{0,0,1,1,1,0,1\}, \{0,0,1,1,0,1,1\}\} \end{aligned}$$

$c^o = \{4,4,1,5,3,7,8\};$

$S = \mathbf{Sort}[T, \mathbf{OrderedQ}[\{\mathbf{Inner}[\mathbf{Times}, \#1, c], \mathbf{Inner}[\mathbf{Times}, \#2, c]\}]\&];$

$w = \mathbf{Table}[\mathbf{Inner}[\mathbf{Times}, S[[i]], c], \{i, 1, 21\}];$

$b = \mathbf{Table}[\mathbf{Table}[w[[i]] - w[[j]], \{j, 21\}], \{i, 21\}];$

$$\begin{aligned} A = {} & \mathbf{Table}[\mathbf{Table}[\mathbf{Join}[(1 - S[[i]])S[[j]] - S[[i]](1 - S[[j]]), \\ & -(1 - S[[i]])S[[j]] + S[[i]](1 - S[[j]])], \{j, 1, 21\}], \{i, 1, 21\}]; \end{aligned}$$

$Inv = \mathbf{Table}[constr = \mathbf{Delete}[A[[i]], i]; rhs = \mathbf{Delete}[b[[i]], i];$
$\quad \mathbf{LinearProgramming}[\{1,1,1,1,1,1,1,1,1,1,1,1,1,1\}, constr, rhs],$
$\quad \{i, 1, 21\}];$

$inv = \mathbf{Apply}[\mathbf{Plus}, Inv, \{1\}]$

References

Burton D., Toint P.L. (1992) On an instance of the inverse shortest path problem. *Mathematical Programming*, **53**, 45-61.

Burton D., Toint P.L. (1994) On the use of an inverse shortest paths algorithms for recovering linearly correlated costs. *Mathematical Programming*, **63**, 1-22.

Cai M., Yang X., Li Y. (1999) Inverse polymatroidal flow problem. *Journal of Combinatorial Optimization*, **3**, 115-126.

Diestel R. (2000) *Graph Theory.* Springer-Verlag, New York, Berlin, Heidelberg.

Graham R.L., Grötsschel M., Lovász L. (1995) *Handbook of Combinatorics*, Elsevier, Amsterdam.

Greenberg H. (1998) An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization. In: D. Woodruff (ed.), *Advances in Computational and Stochastic Optimization, Logic Programming and Heuristic Search.* Kluwer Academic Publishers, Dordrecht, 97-148.

Lawler E.L., Lenstra J.K., Rinnoy Kan A.G., Shmoys D. (1985) *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization.* John Wiley & Sons, New York.

Libura M. (1996) Optimality conditions and sensitivity analysis for combinatorial optimization problems. *Control and Cybernetics*, **25**, 1165-1180.

Libura M. (1999) On accuracy of solutions for combinatorial optimization problems with perturbed coefficients of the objective function. *Annals of Operations Research*, **86**, 53–62.

Libura M., van der Poort E.S., Sierksma G., van der Veen J.A.A. (1998) Stability aspects of the traveling salesman problem based on k-best solutions. *Discrete Applied Mathematics*, **87**, 159-185.

Nemhauser G.L., Woolsey L.A. (1988) *Integer and Combinatorial Optimization.* John Wiley & Sons, New York.

Sokkalingam P.T., Ahuja R.K., Orlin J.B.(1999) Solving inverse spanning tree problems through network flow techniques. *Operations Research*, **47**, 291-298.

Sotskov Y.N., Leontev V.K., Gordeev E.N.(1995) Some concepts of stability analysis in combinatorial optimization. *Discrete Applied Mathematics*, **58**, 169-190.

Xu S., Zhang J. (1995) An inverse problem of weighted shortest path problems. *Japanese Journal of Applied and Industrial Mathematics*, **12**, 47-59.

Zhang J., Cai M. (1998) Inverse problem of minimum cuts. *Mathematical Methods of Operations Research*, **47**, 51-58.

Zhang J., Ma Z. (1999) Solution structure of some inverse combinatorial optimization problems. *Journal of Combinatorial Optimization*, **3**, 127-139.

ISBN 83-85847-78-2