Polska Akademia Nauk • Instytut Badań Systemowych

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Książka jubileuszowa
z okazji
70-lecia urodzin

PROFESORA KAZIMIERZA MAŃCZAKA

pod redakcją
Jakuba Gutenbauma

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FUZZY DYNAMIC PROGRAMMING: A BRIEF INTRODUCTION AND SURVEY

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Abstract: *We briefly present basic aspects of fuzzy dynamic programming that is an effective tool for dealing with fuzzy multistage decision making and optimization problems. We discuss cases of deterministic, stochastic, and fuzzy state transitions, and of the fixed and specified, implicitly given, fuzzy, and infinite termination time. We briefly mention some more relevant applications.*

Keywords: *multistage decision making under fuzziness, multistage optimization under fuzziness, fuzzy dynamic programming, fuzzy set, fuzzy system.*

1. Introduction

A convenient general model to deal with decision making (control) problems in which both dynamics and fuzziness jointly occur is the model of multistage decision making (control) under fuzziness – or fuzzy multistage decision making (control), for short – that has been proposed in Bellman and Zadeh's (1970) seminal paper, and then considerably extended by many authors; Kacprzyk's books (1983b, 1983c, 1997, 2001) provide here a full account of the area.

Dynamic programming was extended to the fuzzy case very early (Chang 1969, Bellman and Zadeh 1970). For reviews, (e.g., Esogbue and Bellman 1984, Esogbue, Fedrizzi and Kacprzyk 1988, Esogbue and Kacprzyk 1998, Kacprzyk 1994, Kacprzyk and Esogbue 1996, etc.), and four books by Kacprzyk (1983b, 1983c, 1997, 2001). The purpose of this paper is to provide a short and readable survey of fuzzy dynamic programming which would include a review of main problem classes, foundations, developments, and more relevant applications.

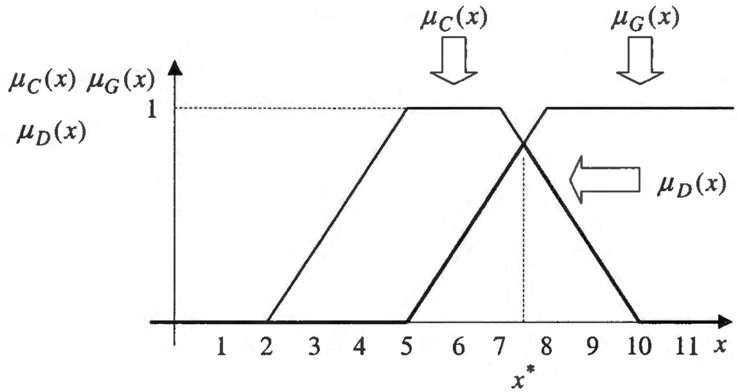

Fig. 1. Fuzzy goal, fuzzy constraint, fuzzy decision, and the optimal (maximizing) decision

2. Multistage decision making under fuzziness

Virtually all works related to decision making under fuzziness, including the multistage decision making case, have as a point of departure the Bellman and Zadeh's (1970) framework. Its basic elements are: a fuzzy goal G in X, a fuzzy constraint C in X, and a fuzzy decision D in X; X is a (nonfuzzy) space of options (alternatives, variants, decisions, ...).

This approach may well be illustrated as in Figure 1. Suppose that the fuzzy goal G is "x should be much more than 6" and the fuzzy constraint C is "x should be about 5", and both are defined as fuzzy sets with piecewise linear membership functions, for simplicity. The fuzzy constraint C = "about 6" is understood as: the numbers between 5 and 7 are certainly *about 6*, those between 2 and 5, and 7 and 10 are *about 6* to some extent (between 0 and 1), and those below 2 and above 10 are certainly not *about 6*; and similarly for the fuzzy goal G = "much more than 6". Notice that this interpretation is clearly implied by an aspiration-level-based attitude which is convenient in our context.

Notice that if $f : X \to R$ is a conventional objective (performance) function, then $\mu_G(x) = \frac{f(x)}{\sup_{x \in X} f(x)}$ is a plausible choice provided that $0 \neq \sup_{x \in X} f(x) < \infty$; thus, the above framework may be viewed as a generalization of the conventional (nonfuzzy) one.

The goal is clearly to

"satisfy C and attain G"

which leads to the fuzzy decision

$$\mu_D(x) = \mu_C(x) \wedge \mu_G(x), \forall x in X \tag{1}$$

that yields the "goodness" of an $x \in X$ as a solution to the decision making problem considered: from 1 for definitely good (perfect) to 0 for definitely bad (unacceptable), through all intermediate values. The "$a \wedge b = \min(a, b)$" operation is commonly used and is assumed throughout this paper, though it may be replaced by, e.g., a t-norm, weighted average or any suitable operation (Kacprzyk 1983b, 1983c, 1997, 2001).

In Figure 1, the membership function of the (min-type) fuzzy decision is given in heavy line for 5 □ x □ 10;. The cases $x < 5$ and $x > 10$ are impossible since $\mu_D(x) = 0$. The value of $\mu_D(x) \in [0, 1]$ may be meant as the degree of satisfaction from the choice of a particular $x \in X$, from 0 for full dissatisfaction (impossibility of x) to 1 for full satisfaction, though all intermediate values; thus, the higher the value of $\mu_D(x)$, the higher the satisfaction from x.

As an optimal (nonfuzzy) solution to this problem, usually an $x \in X$ such that

$$\mu_D(x^*) = \sup_{x \in X} \mu_D(x) = \sup_{x \in X} (\mu_C(x) \wedge \mu_G(x)) \tag{2}$$

is a natural choice. Notice that in Figure 1 $\mu_D(x) < 1$ which indicates some discrepancy between C and G.

In the case of multiple fuzzy constraints and fuzzy goals, suppose that the fuzzy constraint C is defined as a fuzzy set in $X = \{x\}$, and the fuzzy goal G is defined as a fuzzy set in $Y = \{y\}$. Moreover, suppose that a function $f : X \longrightarrow Y$, $y = f(x)$, is known. Typically, X and Y may be sets of decisions and their outcomes, respectively.

Now, the *induced fuzzy goal* G' in X generated by the given fuzzy goal G in Y is defined as

$$\mu_{G'}(x) = \mu_G[f(x)], \qquad \text{for each } x \in X. \tag{3}$$

Notice that by introducing the induced fuzzy goal, both G' and C are defined as fuzzy sets in the same space X, which is evidently a prerequisite.

And now the (min-type) fuzzy decision is

$$\mu_D(x) = \mu_{G'}(x) \wedge \mu_C(x) = \mu_G[f(x)] \wedge \mu_C(x), \qquad \text{for each } x \in X. \quad (4)$$

Thus, if we have n fuzzy constraints defined in X, $C_1, \ldots, C_n$, m fuzzy goals defined in Y, $G_1, \ldots, G_m$, and a function $y = f(x)$, then the (min-type) fuzzy decision is

$$\begin{aligned} \mu_D(x) = \mu_{G_1}[f(x)] \wedge \ldots \wedge \mu_{G_n}[f(x)] \wedge \\ \wedge \mu_{C_1}(x) \wedge \ldots \wedge \mu_{C_m}(x) \end{aligned} \qquad (5)$$

for each $x \in X$.

The above framework can be extended to the multistage decision making case.

2.1. Multistage decision making (control) under fuzziness in Bellman and Zadeh's setting

The dynamics of the problem considered is equated with a deterministic dynamic system described by a state transition equation

$$x_{t+1} = f(x_t, u_t), \qquad t = 0, 1, \ldots \qquad (6)$$

where $x_t, x_{t+1} \in X = \{x\} = \{s_1, \ldots, s_n\}$ are the states (equated with outputs) at time (stage) t and $t+1$, respectively, and $u \in U = \{u\} = \{c_1, \ldots, c_m\}$ is the decision (control or input) at t; the state and decision spaces, X and U, are assumed finite.

The essence of multistage decision making under fuzziness may then be depicted as in Figure 2. We start from an initial state at stage (time) $t = 0$, x_0, make a decision at $t = 0$, u_0, attain a state at time $t = 1$, x_1, make a decision u_1, ... Finally, being at stage $t = N-1$ in state x_{N-1} we make decision u_{N-1} and attain the final state x_N.

The state transitions from state x_t to x_{t+1} under decision u_t are given by (6), the the consecutive decisions applied u_t are subjected to fuzzy constraints C^t, and on the states attained, x_{t+1}, fuzzy goals G^{t+1} are imposed, $t = 0, 1, \ldots, N-1$.

The fuzzy decision is

$$\begin{aligned} \mu_D(u_0, \ldots, u_{N-1} \mid x_0) = \\ = \; & \mu_C^0(u_0) \wedge \mu_{G^1}(x_1) \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N) = \\ = \; & \bigwedge_{t=0}^{N-1} [\mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1})] \end{aligned} \qquad (7)$$

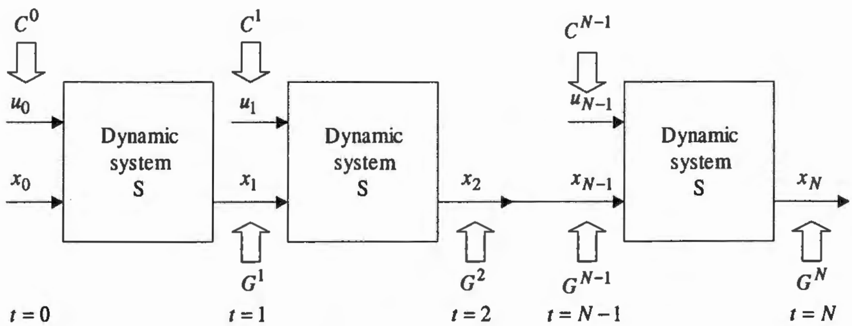

Fig. 2. Multistage decision making under fuzziness

where $x_0 \in X$ is an initial state, the x_t's are given by (6), N is some termination time.

We seek an optimal sequence of decisions (controls) $u_0^*, \ldots, u_{N-1}^*$ such that

$$\begin{aligned} \mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ = \max_{u_0, \ldots, u_{N-1}} [\mu_C^0(u_0) \wedge \mu_{G^1}(x_1) \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)] = \\ = \max_{u_0, \ldots, u_{N-1}} \bigwedge_{t=0}^{N-1} [\mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1})]. \end{aligned} \tag{8}$$

Often, also here, if not otherwise specified, at each stage t fuzzy constraints are given, $\mu_{C^0}(u_0), \ldots, \mu_{C^{N-1}}(u_{N-1})$, and a fuzzy goal, $\mu_{G^N}(x_N)$, is only imposed on the final state x_N. Then

$$\begin{aligned} \mu_D(u_0, \ldots, u_{N-1} \mid x_0) = \\ = \mu_C^0(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N) \end{aligned} \tag{9}$$

and the problem is to find $u_0^*, \ldots, u_{N-1}^*$ such that

$$\begin{aligned} \mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ = \max_{u_0, \ldots, u_{N-1}} [\mu_C^0(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)]. \end{aligned} \tag{10}$$

This general problem formulation may be a starting point for numerous extensions. A convenient classification of problem classes is here with respect to (Kacprzyk 1983b, 1983c, 1997, 2001):

- the type of the termination time: fixed and specified in advance, fuzzy, implicitly given (by entering a termination set of states), and infinite,
- the type of the dynamic system: deterministic, stochastic, and fuzzy,

and in virtually all cases a dynamic-programming-type algorithm can be devised.

3. Fuzzy dynamic programming for multistage decision making with a fixed and specified termination time

This case is basic and will be discussed in more detail.

3.1. The case of a deterministic dynamic system

A deterministic system is described by its state transition equation (6), i.e. $x_{t+1} = f(x_t, u_t); x_t, x_{t+1} \in X = \{s_1, \ldots, s_n\}, u_t \in U = \{c_1, \ldots, c_m\}; t = 0, 1, \ldots, N-1; x_0 \in X$ is the initial state, and $N < \infty$ is a fixed and specified termination time (planning horizon).

The fuzzy constraints are $\mu_{C^0}(u_0), \ldots, \mu_{C^{N-1}}(u_{N-1})$ and the fuzzy goal is $\mu_{G^N}(x_N)$. The fuzzy decision is

$$\begin{aligned} &\mu_D(u_0, \ldots, u_{N-1} \mid x_0) = \\ &\quad = \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N) \end{aligned} \tag{11}$$

and the problem is to find $u_0^*, \ldots, u_{N-1}^*$ such that

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ &\quad = \max_{u_0, \ldots, u_{N-1}} [\mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)] \end{aligned} \tag{12}$$

It is easy to see that the structure of (12) makes the use of dynamic programming scheme possible, and the set of recurrence equations is

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \max_{u_{N-i}} [\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1})] \\ x_{N-i+1} = f(x_{N-i}, u_{N-i}); i = 1, \ldots, N \end{cases} \tag{13}$$

where $\mu_{G^{N-i}}(.)$ is a fuzzy goal at $t = N - i$ induced by a fuzzy goal at $t = N - i + 1$.

An optimal sequence of decisions sought, $u_0^*, \ldots, u_{N-1}^*$, is given by the successive maximizing values of u_{N-i} in (13). The solution, u_t^*, is usually given by an optimal policy $a_t^* : X \longrightarrow U$, such that $u_t^* = a_t^*(x_t), t = 0, 1, \ldots, N-1$, i.e. relating an optimal decision to the current state.

3.2. The case of a stochastic dynamic system

The stochastic system is assumed to be a Markov chain described by a conditional probability

$p(x_{t+1} \mid x_t, u_t)$

such that $x_t, x_{t+1} \in X = \{s_1, \ldots, s_n\}$, $u_t \in U = \{c_1, \ldots, c_m\}$, $x_0 \in X$ is an initial state, $t = 0, 1, \ldots, N-1$, and $N < \infty$ is a fixed and specified termination time.

There are the following two problem formulations:

- due to Bellman and Zadeh (1970): find an optimal sequence of decisions $u_0^*, \ldots, u_{N-1}^*$ to maximize the probability of attainment of the fuzzy goal subject to the fuzzy constraints, i.e.

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ &= \max_{u_0, \ldots, u_{N-1}} [\mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \\ &\quad \wedge E\mu_{G^N}(x_N)] \qquad (14) \end{aligned}$$

where the fuzzy goal is viewed as a fuzzy event in X whose (non-fuzzy) probability is (Zadeh 1968)

$$E\mu_{G^N}(x_N) = \sum_{x_N \in X} p(x_N \mid x_{N-1}, u_{N-1}) \cdot \mu_{G^N}(x_N) \qquad (15)$$

- due to Kacprzyk and Staniewski (1980): find an optimal sequence of decisions $u_0^*, \ldots, u_{N-1}^*$ to maximize the expectation of the fuzzy decision's membership function, i.e.

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ &= \max_{u_0, \ldots, u_{N-1}} E[\mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \\ &\quad \wedge \mu_{G^N}(x_N)] \qquad (16) \end{aligned}$$

and these formulations are clearly not equivalent.

3.2.1. Bellman and Zadeh's approach

Since the structure of (14) is similar to that of (12), we obtain the set of fuzzy dynamic programming recurrence equations as

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \max_{u_{N-i}}[\mu_{C^{N-i}}(u_{N-i}) \wedge E\mu_{G^{N-i+1}}(x_{N-i+1})] \\ E\mu_{G^{N-i+1}}(x_{N-i+1}) = \sum_{x_{N-i+1}\in X} p(x_{N-i+1} \mid x_{N-1}, u_{N-1}) \times \\ \qquad \times\, \mu_{G^{N-i+1}}(x_{N-i+1}); \qquad i = 1, \ldots, N \end{cases} \tag{17}$$

and we consecutively obtain u^*_{N-i} or, in fact, optimal policies a^*_{N-i} such that $u^*_{N-i} = a^*_{N-i}(x_{N-i})$,$i = 1, \ldots, N$.

3.2.2. Kacprzyk and Staniewski's approach

To solve problem (16), we first introduce a sequence of functions $h_i : X \times \times_{j=1}^{i} U \longrightarrow [0,1]$ and $g_j : X \times \times_{l=1}^{j+1} U \longrightarrow [0,1]$; $i = 0, 1, \ldots, N$; $j = 1, \ldots, N-1$; such that

$$\begin{cases} h_N(x_N, u_0, \ldots, u_{N-1}) = \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \\ \qquad \wedge\, \mu_D(u_0, \ldots, u_{N-1} \mid x_0) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ g_k(x_k, u_0, \ldots, u_k) = \sum_{i=1}^{n} h_{k+1}(s_i, u_0, \ldots, u_k) p(s_i \mid x_k, u_k) \\ h_k(x_k, u_0, \ldots, u_{k-1}) = \max_{u_k} g_k(x_k, u_0, \ldots, u_k) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h_0(x_0) = \max_{u_0} g_0(x_0, u_0). \end{cases} \tag{18}$$

Basically, if the consecutive decisions and states are $u_0, \ldots, u_j$ and $x_0, \ldots, x_j$, respectively, then g_j is the expected value of $\mu_D(. \mid x_0)$ provided that the next decisions are optimal, i.e. $u^*_{j+1}, \ldots, u^*_{N-1}$.

As shown in Kacprzyk and Staniewski (1980), Kacprzyk (1983b, 19983c, 1997, 2001) there exist functions $w_k : X \times \times_{i=1}^{k} U \longrightarrow U$ such that $h_k(x_k, u_0, \ldots, u_{k-1}) = g_k(x_k, u_0, \ldots, u_{k-1}, w_k(x_k, u_0, \ldots, u_{k-1}))$.

Then, due to Kacprzyk and Staniewski (1980), an optimal policy sought, a^*_t, $t = 0, 1, \ldots, N-1$, is given by

$$\begin{cases} a^*_0 = w_0(x_0) \\ a^*_1(x_0, x_1) = w_1(x_1, a^*_0(x_0)) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^*_{N-1}(x_0, \ldots, x_{N-1}) = \\ \qquad = w_{N-1}(x_{N-1}, b^*_{N-2}(x_0, \ldots, x_{N-2}, \ldots, b^*_0(x_0)) \ldots) \end{cases} \tag{19}$$

and it depends now not only on the current state but also on the trajectory. The solution of this formulation is evidently more difficult than of that due to Bellman and Zadeh.

3.3. The case of a fuzzy dynamic system

Now the system is fuzzy and is described by a fuzzy state transitions equation

$$X_{t+1} = F(X_t, U_t), \qquad t = 0, 1, \ldots \tag{20}$$

where X_t, X_{t+1} are fuzzy states at time (stage) t and $t+1$, and U_t is a fuzzy decision at t, characterized by $\mu_{X_t}(x_t), \mu_{X_{t+1}}(x_{t+1})$, and $\mu_{U_t}(u_t)$, respectively; (20) is equivalent to a conditioned fuzzy set $\mu_{X_{t+1}}(x_{t+1} \mid x_t, u_t)$ (Kacprzyk 1983b, 1983c, 1997, 2001).

Baldwin and Pilsworth (1982) proposed here a dynamic programming scheme. First, for each $t = 0, 1, \ldots, N-1$ a fuzzy relation $\mu_{R^t}(u_t, x_{t+1}) = \mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1})$ is constructed. The degree to which U_t and X_{t+1} satisfy C^t and G^{t+1} is

$$\begin{aligned} &\mu_T(u_t, \mu_{R^t}(u_t, x_{t+1}), x_{t+1}) = \\ &\quad = \max_{u_t}[(\mu_{U_t}(u_t) \wedge \mu_{C^t}(u_t)) \wedge \max_{x_{t+1}}(\mu_{X_{t+1}}(x_{t+1}) \wedge \\ &\qquad \wedge \mu_{G^{t+1}}(x_{t+1}))]. \end{aligned} \tag{21}$$

Then, they obtain the following set of recurrence equations (Kacprzyk 1983b, 1983c, 1997, 2001)

$$\begin{cases} \mu_{\overline{G}^N}(X_N) = \max_{x_N}[\mu_{X_N}(x_N) \wedge \mu_{G^N}(x_N)] \\ \mu_{\overline{G}^{N-i}}(X_{N-i}) = \max_{U_{N-i}[(\max_{u_{N-i}}}(\mu_{U_{N-i}}(u_{N-i}) \wedge \\ \qquad \wedge \mu_{C^{N-i}}(u_{N-i})) \wedge \mu_{\overline{G}^{N-i+1}}(X_{N-i+1})] \\ \mu_{X_{N-i+1}}(x_{N-i+1}) = \max_{x_{N-i}[\max_{u_{N-i}}}(\mu_{U_{N-i}}(u_{N-i}) \wedge \\ \qquad \wedge \mu_{X_{N-i+1}}(x_{N-i+1} \mid x_{N-i}, u_{N-i})] \wedge \mu_{X_{N-i}}(x_{N-i})) \\ i = 1, \ldots, N-1. \end{cases} \tag{22}$$

In principle, this may be solved, though a prohibitive difficulty is that $\mu_{\overline{G}^{N-i}}(X_{N-i})$ must be specified for all the possible X_{N-i}'s, and the maximization is to proceed over all the possible U_{N-i}'s. As the number of both of them may be very high (theoretically infinite), Baldwin and Pilsworth (1982) predefine some (sufficiently small) number of reference (standard) fuzzy states and fuzzy decisions. Then, they redefine their problem formulation in terms of the reference fuzzy states and fuzzy decisions to finally make (22) solvable.

Experience with Baldwin and Pilsworth's (1982) approach is often discouraging (Zimmermann, 1987) and an earlier and simple branch-and-bound approach by Kacprzyk (1979) may be a better choice.

4. Fuzzy dynamic programming for multistage decision making with a fuzzy termination time

Very often it may be more adequate to assume a fuzzy termination time as *more or less 5 years, a couple of days, ...* . This idea appeared in Fung and Fu (1977) and Kacprzyk (1977).

Let $R = \{0, 1, \ldots, K-1, K, K+1, \ldots, N\}$ be the set of decision making stages. At each $t \in R$ we have a fuzzy constraint $\mu_{C^t}(u_t)$, and a fuzzy goal $\mu_{G^v}(x_v)$, $v \in R$, is imposed on the final state. The fuzzy termination time is given by $\mu_T(v), v \in R$, which can be viewed as a degree of how preferable v is as the termination time.

The fuzzy decision is now (Kacprzyk 1977, 1978b, c)

$$\begin{aligned} \mu_D(u_0, \ldots, u_{v-1} \mid x_0) \;&=\; \mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{v-1}}(u_{v-1}) \wedge \\ &\wedge \mu_T(v) \cdot \mu_{G^v}(x_v) \end{aligned} \tag{23}$$

and we seek an optimal termination time v^* and an optimal sequence of decisions $u_0^*, \ldots, u_{v^*-1}^*$ such that

$$\begin{aligned} &\mu_D(u_0^*, \ldots u_{v^*-1}^* \mid x_0) = \\ &= \max_{v, u_0, \ldots, u_{v-1}} [\mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{v-1}}(u_{v-1}) \wedge \\ &\quad \wedge \mu_T(v) \cdot \mu_{G^v}(x_v)]. \end{aligned} \tag{24}$$

Problem (24) may be solved using Kacprzyk's (1977, 1978b) and Stein's (1980) approaches which will now be briefly presented below.

In Kacprzyk's (1977, 1978c) formulation the set of possible termination times is $\{v \in R : \mu_T(v) > 0\} = \{K, K+1, \ldots, N\} \subseteq R$, hence an optimal sequence of decisions is $u_0^*, \ldots, u_{K-2}^*, u_{K-1}^*, \ldots, u_{v^*-1}^*$.

The part $u_{K-1}^*, u_K^*, \ldots, u_{v^*-1}^*$ is determined by solving

$$\begin{cases} \mu_{G^{v-i}}(x_{v-i}, v) = \max_{v_{v-i}}[\mu_{C^{v-i}}(u_{v-i}) \wedge \mu_{G^{v-i+1}}(x_{v-i+1}, v)] \\ x_{v-i+1} = f(x_{v-i}, u_{v-i}) \\ i = 1, \ldots, v-i+1; v = K, K+1, \ldots, N-1 \end{cases} \tag{25}$$

where $\mu_{G^v}(x_v, v) = \mu_T(v)\mu_{G^v}(x_v)$.

An optimal termination time v^* is then found by the maximizing v in

$$\mu_{G^{K-1}}(x_{K-1}) = \max_v \mu_{G^{K-1}}(x_{K-1}, v). \tag{26}$$

The part $u_0^*, \ldots, u_{K-2}^*$ is then determined by solving

$$\begin{cases} \mu_{G^{K-i-1}}(x_{K-i-1}) = \max_{u_{K-i-1}}[\mu_{C^{K-i-1}}(u_{K-i-1}) \wedge \mu_{G^{K-i}}(x_{K-i})] \\ x_{K-i} = f(x_{K-i-1}, u_{K-i-1}); i = 1, \ldots, K-1 \end{cases} \tag{27}$$

Stein (1980) presented a computationally more efficient dynamic programming approach whose idea is as follows. At $t = N-i, i \in \{1, \ldots, N-1\}$, we can either stop and attain $\mu_{\overline{G}^{N-i}}(x_{N-i}) = \mu_T(N-i)\,\mu_{G^{N-i}}(x_{N-i})$, or apply u_{N-i} and attain $\mu_{C^{N-i}}(u_{N-i}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1})$. The better alternative should evidently be chosen, and this is repeated for $t = N-i-1, N-i-2, \ldots, 0$. The set of recurrence equations is therefore

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \mu_{\overline{G}_{N-i}}(x_{N-i}) \wedge \\ \qquad \wedge \max_{u_{N-i}}[\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1})] \\ x_{N-i+1} = f(x_{N-i}, u_{N-i}); i = 1, 2, \ldots, N \end{cases} \tag{28}$$

and an optimal termination time is such a $t = N-i$ at which the terminating decision, $u_{v^*-1}^*$, occurs, i.e., when

$$\mu_{G^{N-i}}(x_{N-i}) > \max_{u_{N-i}}[\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1})]. \tag{29}$$

Using a similar reasining, we can formulate and solve the case with a fuzzy termination time and a stochastic system (Kacprzyk 1978b, c, 1983b, 1983c, 1997, 2001, and Stein 1980).

A dynamic programming scheme for the case of a fuzzy termination time and a fuzzy system can be obtained by first fixing some (finite and relatively small) number of reference fuzzy states (and possibly decisions), and obtaining an auxiliary approximate system whose state transitions are of a deterministic system type (Section 3.3 or Kacprzyk 1983b and Kacprzyk and Staniewski 1982). Then, Kacprzyk's or Stein's approach can be employed. In many cases, however, a simple Kacprzyk's (1983b) branch-and-bound algorithm is a better choice.

This concludes in fact our analysis of fuzzy dynamic programming. Now, for completeness, we will sketch the two other problem classes: with an implicit and infinite termination time for which some iterative algorithms are available.

5. Multistage decision making with an implicitly specified termination time

The process terminates now when the state enters for the first time the termination set of states $W = \{s_{p+1}, s_{p+2}, \ldots, s_n\} \subset X$. The problem is to determine an optimal sequence of decisions $u_0^*, \ldots, u_{\overline{N}-1}^*$ such that

$$\begin{aligned}\mu_D(u_0^*, \ldots, u_{\overline{N}-1}^* \mid x_0) &= \\ &= \max_{u_0, \ldots, u_{\overline{N}-1}} [\mu_C(u_0 \mid x_0) \wedge \ldots \wedge \mu_C(u_{\overline{N}-1} \mid x_{\overline{N}-1}) \wedge \\ &\quad \wedge \mu_{G^{\overline{N}}}(x_{\overline{N}})]\end{aligned} \tag{30}$$

where $x_0, \ldots, x_{\overline{N}-1} \in X \setminus W$, and $x_{\overline{N}} \in W$; we seek an optimal stationary strategy in fact.

The solution of (30) may proceed by using:

- Bellman and Zadeh's (1970) iterative approach,
- Komolov's *et et al.* (1979) graph-theoretic approach, and
- Kacprzyk's (1978a, b) branch-and-bound approach,

and the first one is somehow related to dynamic programming. For details we refer the reader to, e.g., Kacprzyk's (1983b, 1996) books.

6. Multistage decision making with an infinite termination time

This case may well represent problems of a low variability or meant to maintain some level of activity. Fuzzy multistage decision making problem with an infinite termination time was first formulated and solved by Kacprzyk and Staniewski (1982, 1983) – see also Kacprzyk's (1983b, 1983c, 1997, 2001) books.

For the deterministic dynamic system (6), the fuzzy decision is

$$\begin{aligned}
&\mu_D(u_0, u_1, \ldots \mid x_0) = \\
&= \mu_C(u_0 \mid x_o) \wedge \mu_G(x_1) \wedge \mu_C(u_1 \mid x_1) \wedge \mu_G(x_2) \wedge \ldots = \\
&= \lim_{N \to \infty} \bigwedge_{t=0}^{N} [\mu_C(u_t \mid x_t) \wedge \mu_G(x_{t+1})]. \qquad (31)
\end{aligned}$$

We seek an optimal stationary strategy $a^*_\infty = (a^*, a^*, \ldots)$ such that

$$\begin{aligned}
&\mu_D(a^*_\infty \mid x_0) = \max_{a_\infty} \mu_D(a_\infty \mid x_0) = \\
&= \max_{a_\infty} \lim_{N \to \infty} \bigwedge_{t=0}^{N} [\mu_C(a(x_t) \mid x_t) \wedge \mu_G(x_{t+1})]. \qquad (32)
\end{aligned}$$

As shown in Kacprzyk and Staniewski (1983), problem (32) may be solved in a finite number of steps by a policy iteration algorithm, with a step-by-step improvement of stationary policies.

A policy iteration type algorithm was also proposed for the stochastic system (the most challenging case!) by Kacprzyk, Safteruk and Staniewski (1981), and for a fuzzy system by Kacprzyk and Staniewski (1982).

7. Applications of fuzzy dynamic programming: a short survey

These applications fall into two main categories:

- applications related to more general (standard) problems,
- applications related to more specific problems.

In the first group, we should first mention a pioneering work on a general resource allocation problem with fuzzy goals and constraints by Esogbue and Ramesh (1970).

Another problem class of a universal relevance is data clustering, and the application of fuzzy dynamic programming was proposed here by Esogbue (1986), and Esogbue and Bellman (1981). Moreover, there are here workd related to some mathematical programming problems. Hussein and Abo-Sinna (1993) apply a hybrid fuzzy dynamic programming approach to determine a set of efficient solutions of the multiobjective mathematical programming problem. Narshima Sastry, Tiwari, and Sastry (1993), on the other hand, solved a generalized multiple goal fuzzy control (optimization) problem by using fuzzy dynamic programming.

In the second group, we should mention, first, Esogbue's (1983) work on a general research and development (R&D) control problem in which some limited funds are to be optimally used to attain certain goals distributed over time. Needless to say that this problem may be a prototype for a much wider class.

Second, in a series of papers, Kacprzyk and Straszak (1981, 1982a, b, 1984) proposed a fuzzy dynamic programming model for determining socioeconomic regional development strategies which take into account limited resources, requirements on the effectiveness and stability of development, some objective and subjective aspects, etc.

Relevant medical applications have been proposed, and one should mention here, e.g., Esogbue and Elder's (1986) work on a fuzzy dynamic programming model of medical diagnosis and Esogbue's (1985) approach to intra-operative anesthesia adminstrations; in fact, Esogbue's (1983) work on R&D is much concerned with the cancer research appropriation process, too.

One of most fruitful research areas have been various aspects of energy systems. We should first of all mention Su and Hsu (1991) who applied fuzzy dynamic programming to the unit commitment of a power system, and practically implemented the results obtained for Taiwan's power system.

Huang, Lin and Huang (1992) presented a fuzzy dynamic programming model for the determination of an optimal schedule for the outage starting times of power generators for maintenance over the one year planning horizon for a Taiwanese power company.

Sugianto and Mielczarski (1995a, b, 1996) considered the optimization of a spare parts inventory in a power generation plant by balancing requirements on power availability and reduction of spare parts' costs.

Water resources systems were another relevant field of applications. Already in Esogbue and Bellman (1981) an application to the analysis of water resouces related data was presented. Esogbue and Ahipo (1982a, b) considered some effectiveness measures in water resources planning. Esogbue, Theologidu and Guo (1992) presented an application to flood control.

In chemical engineering Krasławski, Górak and Vogenpohl (1989) proposed an application of fuzzy dynamic programming to the determination of destillation sequence (sequence of destillation columns) for a mixture of some number of components. An important problem of determining a best itinerary for the transportation of hazardous waste loads ovex a given road structure was considered by Klein (1991) taking into account safety of the road segments and intersections, distance, driving difficulty, dangers related to population density, etc.

Yuan and Wu (1991) proposed an application of fuzzy dynamic programming to real-time control of a transportation system in a flexible manufacturing system, more particularly involving the so-called autonomous guided vehicles (AGVs) which are driverless, programmable vehicles that can move around the factory.

This concludes our brief analysis of main applications of fuzzy dynamic programming in widely perceived operations research related areas. For details on them, as well as on applications in other fields we refer the interested reader to Kacprzyk's book (1997).

8. Concluding remarks

We presented a short and readable survey of basic elements and applications of fuzzy dynamic programming. We hope this exposition will be useful for users from diverse areas who may be interested in applications of fuzzy dynamic programming as we witness an increase of interest in this topic in recent years. For a recent state of the art we refer the reders to two last Kacprzyk's books (1997, 2001).

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