Polska Akademia Nauk • Instytut Badań Systemowych

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Książka jubileuszowa
z okazji
70-lecia urodzin

PROFESORA KAZIMIERZA MAŃCZAKA

pod redakcją
Jakuba Gutenbauma

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# OPTIMAL CONTROL OF THE MULTIDIMENSIONAL LINEAR SYSTEMS WITH POLYNOMIAL PERFORMANCE INDEX[1]

***Henryk Górecki***

*Institute of Automatics, University of Mining and Metallurgy*
*Al.Mickiewicza 30, 30-059 Cracow, Poland*
*Tf/fax +48+12 6341568,< e-mail: head@ia.agh.edu.pl>*
*Life Member of IEEE*

***Abstract:*** *A stable non stationary linear system is considered. A quality criterion is represented by a functional in the integral form of the sum of positive polynomiaes with respect to the state variables and squared control. It is proved that the optimal control is an irrational function of polynomial depending on the state variables. The minimum value of performance index can have two forms: the hyperelliptic integral or polynomial depending on the initial state.*

***Keywords:*** *Optimal control, dynamic programming, polynomial performance index.*

## 1. Introduction

In the paper the problem of optimal control of the multidimensional nonstationary linear systems is considered. The criterion of optimality is to be assumed the integral of the polynomial of the state variables and squared control. The work is the generalization of the linear-quadratic problem. In the industrial applications the dependence of the functional of the state variables is determined by the technological process. This dependence may be well approximated by the polynomials of degrees higher than second.

Linear quadratic problem was considered in many reports (Bellman 1957, Górecki 2001, Kwakernaak and Sivan 1972). The problem of the

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parametric optimization represented by the positive polynomial as the function of the state variables was considered in (Górecki 1994, 1995, 1996). The optimal control of the scalar stationary linear system with polynomial performance index was considered in (Kalman 1960).

## 2. Problem formulation

Let us consider the system described by the linear differential equation:

$$\overset{\circ}{x}(t) = A(t)x(t) + B(t)u(t) \quad t \in [t_0, t_f] \tag{2.1}$$

where $t$ is time, $x(t)$ is a real $n$-dimensional state vector, and $u(t)$ is a real $k$-dimensional control vector.
The nonstationary matrices $A(t)$ and $B(t)$ have appropriate dimensions. The terminal time $t_f$ is established and the initial state $x(t_0)$ is given.
We assume that:

$$\int_0^{t_f} u^T(t)u(t)dt < \infty \tag{2.2}$$

Consider also the criterion:

$$\begin{aligned} J(u) = &\sum_{k=1}^{m} \left[x^{2k}(t_f)\right]^T P_k(t_f)x^{2k}(t_f) + \\ &+ \int_{t_0}^{t_f} \{ \sum_{k=1}^{m} \left[x^{2k}(t)\right]^T Q_k(t)x^{2k}(t) + u^T(t)R(t)\}dt \end{aligned} \tag{2.3}$$

where $P_k(t_f)$, $Q_k(t)$ and are the nonnegative symmetric matrices and $R(t)$ is positive–definite symmetric matrix for $t_0 \le t \le t_f$ The problem is of determination of control $u^0(t)$, $t_0 \le t \le t_f$, for which the criterion (2.3) is minimal. This control $u^0(t)$ is called optimal and problem is called the deterministic optimal regulator problem.

**Remark 1.** If all matrices occurring in this formulation of the optimal regulator problem are constant, we call it the stationary linear optimal regulator problem.

## 3. Solution of the regulator problem

We solve the deterministic regulator problem using the dynamic programming method.
Application of this method to the system described by the relation (2.1), (2.3) gives the well-known Bellman's equation:

$$-\frac{\partial s}{\partial t_0}=\min_{u(t_0)}\left\{\sum_{k=1}^{m}\left[x^{2k}(t_0)\right]^T Q_k(t_0)x^{2k}(t_0)+u^T(t_0)R(t_0)u(t_0)+\right. \tag{3.1}$$
$$\left.+\,grad^T_{x(t_0)}S[A(t_0)x(t_0)+B(t_0)u(t_0)]\right\}$$

with the final condition:

$$S\cdot\left[x(t_0),t_f\right]=-\sum_{k=1}^{m}\left[x^{2k}(t_f)P_k(t_f)x^{2k}(t_f)\right] \tag{3.2}$$

We denote by $S\cdot[x(t_0),t_0]$ the minimum value of the functional $J(u)$ defined by (2.3). There are no constraints on control $u(t_0)$ besides (2.2). We can apply the well-known necessary condition to the right hand side of the equation (3.1):

$$grad_{u(t_0)}\left\{\sum_{k=1}^{m}\left[x^{2k}(t_0)\right]^T Q_k(t_0)x^{2k}(t_0)+u^T(t_0)R(t_0)u(t_0)+\right. \tag{3.3}$$
$$\left.+\,grad^T_{x(t_0)}S[A(t_0)x(t_0)+B(t_0)u(t_0)]\right\}=0$$

From equation (3.3) we obtain directly that optimal control:

$$u^0(t_0)=-\frac{1}{2}R^{-1}(t_0)B^T(t_0)\frac{\partial s}{\partial x(t_0)} \tag{3.4}$$

**Remark 2.** It is easy to observe that the relation (3.4) can be obtained directly from equation (3.1) if we complete its right hand-side to the full quadratic form with respect to control $u(t_0)$:

$$u^T(t_0)R(t_0)u(t_0)+\left(\frac{\partial s}{\partial x(t_0)}\right)^T B(t_0)u\ (t_0)=$$

$$=\left[u(t_0)+\frac{1}{2}R^{-1}(t_0)B^T(t_0)\frac{\partial s}{\partial x(t_0)}\right]^T R(t_0)\left[u(t_0)+\frac{1}{2}R^{-1}(t_0)B^T(t_0)\frac{\partial s}{\partial x(t_0)}\right]-$$

$$-\frac{1}{4}\left(\frac{\partial s}{\partial x(t_0)}\right)^T B(t_0)R^{-1}(t_0)B^T(t_0)\frac{\partial s}{\partial x(t_0)} \tag{3.5}$$

Equation (3.5) leads to the conclusion that the minimum of the expression (3.1) is obtained when $u(t_0)$ is determined by equation (3.4), because in this case we have:

$$u^T(t_0)R(t_0)u(t_0)+\left(\frac{\partial s}{\partial x(t_0)}\right)^T B(t_0)u(t_0)= =-\frac{1}{4}\left(\frac{\partial s}{\partial x(t_0)}\right)^T B(t_0)R^{-1}(t_0)B^T(t_0)\frac{\partial s}{\partial x(t_0)} \tag{3.6}$$

Substitution of (3.6) into (3.1) gives:

$$-\frac{\partial s}{\partial x(t_0)}=\sum_{k=1}^{m}\left[x^{2k}(t_0)\right]^T Q_k(t_0)x^{2k}(t_0)+\left[\frac{\partial s}{\partial x(t_0)}\right]^T A(t_0)x(t_0)- -\frac{1}{4}\left(\frac{\partial s}{\partial x(t_0)}\right)^T B(t_0)R^{-1}(t_0)B^T(t_0)\frac{\partial s}{\partial x(t_0)} \tag{3.7}$$

Taking into account that the equations (3.4) and (3.7) are true for arbitrary $t_0$ and arbitrary $x(t_0)$ we can write that they are true for arbitrary $t$ and arbitrary optimal state $x^{o}(t)$:

$$u^0(t)=-\frac{1}{2}R^{-1}(t)B^T(t)\frac{\partial s}{\partial x^o(t)} \tag{3.8}$$

$$-\frac{\partial S}{\partial t}=\sum_{k=1}^{m}\left[x^{o2k}(t)\right]^T Q_k(t)\left[x^{o2k}(t)\right]+\left[\frac{\partial S}{\partial x^0(t)}\right]^T A(t)x^0(t)-$$
$$-\frac{1}{4}\left(\frac{\partial S}{\partial x^0(t)}\right)^T B(t)R^{-1}(t)B^T(t)\frac{\partial S}{\partial x^0(t)} \tag{3.9}$$

In general the solution of this equation has a polynomial form of the optimal state and is in itself a difficult problem. We illustrate this on a very simple example.

## 4. Example

Scalar case

We can write the dynamic equation of the system as:

$$\frac{dx}{dt}=a_1x+bu, \quad x(0)=x_0 \tag{4.1}$$

and taking into account the functional (2.3) we can write Bellman's equation (3.3) in the form:

$$\frac{d}{du}\left\{f(x)+\lambda u^2+\frac{d}{dx}\left[S(x)\cdot(a_1x+bu)\right]\right\}=0 \tag{4.2}$$

After differentiation we get:

$$2\lambda u^o+b\frac{dS}{dx}=0 \tag{4.3}$$

so the optimal control is equal to:

$$u^o=-\frac{b}{2\lambda}\frac{dS}{dx} \tag{4.4}$$

By putting control $u^o$ defined in (4.4) into equation (4.2) we obtain:

$$f(x)+a_1x\frac{dS}{dx}-\frac{b^2}{4\lambda}\left(\frac{dS}{dx}\right)^2=0 \tag{4.5}$$

After solving the equation (4.5) with respect to $\frac{dS}{dx}$, we obtain:

$$\frac{dS}{dx}=\frac{2\lambda a_1}{b^2}x\pm\sqrt{\left(\frac{2\lambda a_1}{b^2}x\right)^2+\frac{4\lambda}{b^2}f(x)} \tag{4.6}$$

Because of the fact that feedback control has to be negative, $\frac{dS}{dx}$ in (4.4) should be positive.

Therefore we take:

$$\frac{dS}{dx}=\frac{2\lambda a_1}{b^2}x+\sqrt{\left(\frac{2\lambda a_1}{b^2}x\right)^2+\frac{4\lambda}{b^2}f(x)} \tag{4.7}$$

The minimal value of $S$ is calculated with regard to the value $x(0)=x_0$, for $t=0$, and to the value $x(\infty)=0$ for $t=\infty$ because of assumed asymptotic stability of the closed system:

$$S=\int_{x_0}^{0}\left[\frac{2\lambda a_1}{b^2}x+\sqrt{\left(\frac{2\lambda a_1}{b^2}x\right)^2+\frac{4\lambda}{b^2}f(x)}\right]dx \tag{4.8}$$

If polynomial $f(x)$ is of higher than second degree then integral (4.8) cannot be described by elementary functions.

The function $f(x)$ is found using the condition:

$$\left(\frac{2\lambda a_1}{b^2}x\right)^2+\frac{4\lambda}{b^2}f(x)=\left(\gamma_0x^2+\gamma_1x+\gamma_2\right)^2 \tag{4.9}$$

to remove the root in the integrand in (4.8).

Therefore:

$$f(x)=\frac{b^2}{4\lambda}\left[\left(\gamma_0x^2+\gamma_1x+\gamma_2\right)^2-\left(\frac{2\lambda a_1}{b^2}x\right)^2\right] \tag{4.10}$$

Taking into consideration (4.10) in the expression in (4.8) we obtain the minimal value of the functional $J$ as:

$$S = -\int_0^{x_0} \left[ \frac{2\lambda a_1}{b^2} x + \gamma_0 x^2 + \gamma_1 x + \gamma_2 \right] dx \tag{4.11}$$

Finally, after integration we have

$$S = -\left[ \frac{1}{3}\gamma_0 x_0^3 + \frac{1}{2}\left( \gamma_1 + \frac{2\lambda}{b^2} a_1 \right) x_0^2 + \gamma_2 x_0 \right] \tag{4.12}$$

The optimal control is defined by equations (4.4), (4.7) and (4.10). Thus, at the end we get

$$u^o = -\frac{b}{2\lambda}\left[ \frac{2\lambda a_1}{b^2} x^o + \gamma_0 x^{o2} + \gamma_1 x^o + \gamma_2^o \right] \tag{4.13}$$

## 5. Conclusion

We can state that the optimal controller is not linear, as it was in the LQ problem.

Considering that the minimum value of the functional should be positive, the expression in the formula (4.12) has to satisfy:

$$\frac{1}{3}\gamma_0 x_0^3 + \frac{1}{2}\left( \gamma_1 + \frac{2\lambda}{b^2} a_1 \right) x_0^2 + \gamma_2 x_0 < 0 \tag{4.14}$$

which renders the domain of controllability.

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