ANALIZA SYSTEMOWA I ZARZĄDZANIE

Książka jubileuszowa
z okazji
50-lecia pracy naukowej

ROMANA KULIKOWSKIEGO

ISBN 83-85847-34-0

Druk: "ARGRAF" Agencja Poligraficzno-Wydawnicza, Warszawa
Skład: Barbara Kotuszewska

POSITIVE 2nd CONTINUOUS-DISCRETE LINEAR SYSTEMS

Tadeusz Kaczorek

Warsaw Technical University
Faculty of Electrical Engineering Institute of Control and Industrial Electronics

A new class of positive 2D continuous-discrete linear models is introduced. Sufficient conditions for 2D continuous-discrete linear model to be a positive system are established. The solution to the positive 2D continuous-discrete model is derived. Necessary and sufficient conditions for the local reachability of the positive 2D system are established. The minimum energy control problem for positive 2D continuous-discrete systems is formulated and solved.

Keywords: Positive, continuous-discrete, reachability, energy control.

1. Introduction

The most popular models of two-dimensional (2D) systems are the models introduced by Roesser [28], Fornasini and Marchesini [4,5]. The reachability and controllability and the minimum energy control of 2D linear systems have been considered in many papers and books [20-24]. The minimum energy control problem for the classical 2D Roesser model was formulated and solved by Klamka [24] and next the method was extended for 2D linear systems with variable coefficients [20] and other type of 2D models [23]. A positive linear system is a dynamical system in which the input, state, and output space are spaces over the nonnegative real numbers. The positive linear systems have been used in biomathematics, economics, chemometrics and other research areas [1,2,26]. Positive realisations of a given transfer function have been considered in many papers [1,3,6,7-9]. The reachability, observability and realisability of continuous-time positive systems have been discussed in [27]. The realisation problem of a given positive impulse response function has been considered in [1,6] and the realisation of an nth-order linear difference equation has been treated in [25]. A procedure for finding a positive realisation problem of a given improper transfer matrix has been presented in [8]. 2D continuous-discrete linear systems have been considered in [10-18].

In this paper a new class of positive 2D continuous-discrete linear systems will be introduced. Sufficient conditions for the local reachability will be established and the minimum energy control problem will be solved of the class of positive 2D systems.

2. Preliminaries

Let Z_+ be the set of nonnegative integers and R the field of real numbers. Denote by $R^{n\times m}$ the set of $n\times m$ matrices with entries from R, by $R_+^{n\times m}$ the set of $n\times m$ matrices with nonnegative entries and $R^n := R^{n\times 1}$.

Consider the 2D continuous-discrete linear system

$$\dot{x}(t,k+1)=A_0x(t,k)+A_1\,\dot{x}(t,k)+A_2x(t,k+1)+Bu(t,k)$$

$$t\in R, k\in Z_+ \tag{1a}$$

$$y(t,k)=Cx(t,k)+Du(t,k) \tag{1b}$$

where $\dot{x}(t,k)=\dfrac{\partial x(t,k)}{\partial t}, x(t,k)\in R_+^n$ is state vector, $u(t,k)\in R^m$ is the input vector, $y(t,k)\in R^p$ is the output vector and $A_i\in R^{n\times n}$, $i=0,1,2$,

$$B\in R^{n\times m}, C\in R^{p\times n}, D\in R^{p\times n}$$

Definition 1. The system (1) is called positive if for any boundary conditions

$$x(t,0)\in R_+^n, \dot{x}(t,0)\in R_+^n, t\ge 0$$

and

$$x(0,k)\in R_+^n, k\ge 1 \tag{2}$$

and all input vectors $u(t,k)\in R_+^m, t\ge 0, k\in Z_+$ the state and output vectors satisfy the condition $x(t,k)\in R_+^n, y(t,k)\in R_+^p$ for $t\ge 0, k\in Z_+$

Definition 2. A matrix $A \in R^{n \times n}$ is called a Metzler matrix if all its off-diagonal entries are nonnegative.

It is easy to show [19] that $e^{At} \in R_+^{n \times n}$ for $t \geq 0$ if and only if $A \in R^{n \times n}$ is a Metzler matrix.

Theorem 1. The system (1) is positive if

$$A_0 \in R_+^{n \times n}, A_1 \in R_+^{n \times n}, A := A_0 + A_1 A_2 \in R_+^{n \times n}, B \in R_+^{n \times m}, C \in R_+^{p \times n}, D \in R_+^{p \times m}$$

and

A_2 is a Metzler matrix.

Proof. The proof will be accomplished by induction. The equation (1a) may be written in the form

$$\dot{x}(t,k+1) = A_2 x(t,k+1) + F(t,k) \tag{3a}$$

where

$$F(t,k) := A_0 x(t,k) + A_1 \dot{x}(t,k) + Bu(t,k) \tag{3b}$$

From (3) for $k = 0$ we have

$$\dot{x}(t,1) = A_2 x(t,1) + F(t,0) \tag{4a}$$

where

$$F(t,0) := A_0 x(t,0) + A_1 \dot{x}(t,0) + Bu(t,0) \tag{4b}$$

is known for given boundary conditions (2).

From (4b) it follows that if (i) and (2) hold then $F(t,0) \in R_+^n$. The solution of (4a) is given by

$$x(t,1) = e^{A_2 t} x(0,1) + \int_0^t e^{A_2(t-\tau)} F_1(\tau,0) d\tau \ , \ t \geq 0 \tag{5}$$

Note that $x(t,1) \in R_+^n$ since A_2 is a Metzler matrix and $e^{A_2 t} \in R_+^{n \times n}$ for $t \geq 0$.

From (1b) for $k = 1$ we have $y(t,1) = Cx(t,1) + Du(t,1) \in R_+^p$.

Using (3b) for $k = 1$ and (5) we obtain

$$F(t,1) = A_0 x(t,1) + A_1 \dot{x}(t,1) + Bu(t,1) =$$

$$(A_0 + A_1 A_2) e^{A_2 t} x(0,1) + (A_0 + A_1 A_2) \int_0^t e^{A_2 (t-\tau)} \times F(\tau,0) d\tau + A_1 F(t,0) + Bu(t,1) =$$

$$A e^{A_2 t} x(0,1) + A \int_0^t e^{A_2 (t-\tau)} F(\tau,0) d\tau + A_1 F(t,0) + Bu(t,1) \in R_+^n$$

since $A \in R_+^{n \times n}, e^{A_2 t} \in R_+^{n \times n}$, $A_1 \in R_+^{n \times n}$ and $F(t,0) \in R_+^n$.

Assuming that $x(t,i) \in R_+^n$ and $F(t,i-1) \in R_+^n$ for $t \geq 0$ and $i \geq 1$ we shall show that

$x(t,i+1) \in R_+^n$ for $t \geq 0$ and $i \geq 1$. From (3) for $k = i$ we get

$$\dot{x}(t,i+1) = A_2 x(t,i+1) + F(t,i) \tag{6a}$$

where

$$F(t,i) = A_0 x(t,i) + A_1 \dot{x}(t,i) + Bu(t,i) = \tag{6b}$$

$$A e^{A_2 t} x(0,i) + A \int_0^t e^{A_2 (t-\tau)} F_i(\tau,i-1) d\tau + A_1 F(t,i-1) + Bu(t,i) \in R_+^n$$

since $A \in R_+^{n \times n}, e^{A_2 t} \in R_+^{n \times n}$, $A_1 \in R_+^{n \times n}$ and $F(t,i-1) \in R_+^n$.

The solution of (6a) is given by

$$x(t,i+1) = e^{A_2 t} x(0,i+1) + \int_0^t e^{A_2 (t-\tau)} F(\tau,i) d\tau, t \geq 0 \tag{7}$$

From (7) it follows that $x(t,i+1) \in R_+^n$ if the condition i) is satisfied. From (1b) for $k = i+1$ we have $y(t,i+1) = Cx(t,i+1) + Du(t,i+1) \in R_+^p$. □

Remark. The system (1) is positive only if A_2 is a Metzler matrix.

Proof. Note that $e^{A_2 t} \in R_+^{n \times n}$ if and only if A_2 is a Metzler matrix. From (7) it follows that $x(t,i) \in R_+^n$ for $t \geq 0$ and $i \in Z_+$ only if A_2 is a Metzler matrix. □

3. Solution of the positive system

Theorem 2. The solution $x(t,k)$ of the equation (1a) with boundary conditions (2) has the form

$$x(t,k) = e^{A_2 t}x(0,k) + \int_0^t e^{A_2(t-\tau)}F(t,k-1)d\tau\,,$$

$$t \geq 0, k \in Z_+ \tag{8}$$

where $F(t,k)$ is given by

$$F(t,k) = P_t^k[A_0 x(t,0) + A_1\dot{x}(t,0) + Bu(t,0] + \\ + \sum_{i=0}^{k-1} P_t^{k-i-1}[Ae^{A_2 t}x(0,i+1) + Bu(t,i+1)] \tag{9}$$

and P_t is an operator defined by

$$P_t F(t) := A\int_0^t e^{A_2(t-\tau)}F(\tau)d\tau + A_1 F(t) \tag{10}$$

Proof. The solution (8) follows from (7) for $i+1=k$.

Using the operator P_t we may write the difference equation (6b) in the form

$$F(t,i) = P_t F + Ae^{A_2 t}x(0,i) + Bu(t,i)$$

and its solution is given by (9) for $i = k-1$. □

4. Reachability

Definition 3. The positive system (1) is called locally reachable in the rectangle (point)

$$D_{hr} := \{(t,k) \in R_+ \times Z_+ ; 0 \leq t \leq h, 0 \leq k \leq r\} \tag{11}$$

if for zero boundary conditions (2) and every vector $x_f \in R_+^n$ there exists $u(t,k) \in R_+^m$ for $0 \leq t \leq h, 0 \leq k < r$ such that $x(h,r) = x_f$.

Definition 4. The positive system (1) is called reachable if for zero boundary conditions (2) and every vector $x_f \in R_+^n$ there exists and pair (h,r) such that the system is locally reachable in the rectangle (11).

Definition 5. A set of all nonnegative linear combinations of columns of the matrix $A \in R^{n\times m}$ is called the positive image of A and it is denoted by $\mathrm{Im}_+ A$, i.e.

$$\mathrm{Im}_+ A := \{y \in R_+^n : y = Ax, for\ all\ \ x \in R_+^m\} \tag{12}$$

Theorem 3. The positive system (1) is locally reachable in the rectangle (11) if and only if

$$\mathrm{Im}_+ R_{hr} = R_+^n \tag{13}$$

where

$$R_{hr} = [R_0, R_1, \dots, R_{r-1}], \tag{14}$$

$$R_i := \int_0^h e^{A_2(h-\tau)} P_t^{r-i-1} B d\tau \qquad i = 0,1,\dots,r-1$$

and the operator P_t is defined by (10).

Proof. Let us assume that

$$u(t,k) := u_k \quad \text{for} \quad 0 \le t \le h,\ 0 \le k < r \tag{15}$$

where u_k is independent of t.

Using (8) and (9) for $t = h$ and $k = r$ and zero boundary conditions (2) we obtain

$$x_f = x(h,r) = R_{hr} u_{0r}, \qquad u_{0r} := [u_0 \quad u_1 \cdots u_{r-1}]^T \tag{16}$$

where T denotes the transposition.

From (16) and (12) it follows that for every $x_f \in R_+^n$ there exists a sequence $u_i \in R_+^m$, $i = 0,1,\dots,r-1$ if and only if (13) holds f.

A matrix is called the generalised permutation matrix if in each its row and its column only one entry is positive and the remaining entries are zero.

Theorem 4.

If

$$P_h := \int_0^h e^{A_2\tau} BB^T e^{A_2^T} d\tau \in R_+^{n\times n} \tag{17}$$

is a generalised permutation matrix then the positive model (1) is reachable in the rectangle D_{h1} $(t = h, k = 1)$ and the desired input vector is given by

$$u(t,0) := B^T e^{A_2^T (h-t)} P_h^{-1} x_f \text{ for } 0 \le t \le h \tag{18}$$

Proof. It is well-known [2] that $P_h^{-1} \in R_+^{n\times n}$ if and only if $P_h \in R_+^{n\times n}$ is a generalised permutation matrix. Therefore, if the assumption of Theorem 4 is satisfied then $u(t,0) \in R_+^m$.

From (8) and (9) for $t = h, k = 1$ for zero boundary conditions (2) we obtain

$$x(h,1) = \int_0^h e^{A_2(h-\tau)} Bu(\tau,0) d\tau \tag{19}$$

Substitution of (18) into (19) yields

$$x(h,1) = \int_0^h e^{A_2(h-\tau)} BB^T e^{A_2^T(h-\tau)} d\tau P_h^{-1} x_f = \int_0^h e^{A_2\tau} BB^T e^{A_2^T \tau} d\tau P_h^{-1} x_f = x_f \qquad \square$$

5. Minimum energy control

Consider the positive system (1) and the performance index

$$I(u) := \sum_{k=0}^{r-1} u_k^T Q u_k \tag{20}$$

where Q is the $m \times m$ symmetric positive definite weighting matrix such that

$$Q^{-1} \in R_+^{m\times m} \tag{21}$$

and u_k is defined by (15).

The minimum energy control problem for the positive system (1) with zero boundary conditions (2) can be stated as follows. Given the matrices $A_k, k = 0,1,2$ and B of (1), the weighting matrix Q and the point (h,r), find a sequence $u_i \in R_+^m$ for $i = 0,1,...,r-1$ which transfer the system (1) from zero boundary conditions to the desired local state $x_f = x_{hr}$ and minimizes the performance index (19).

To solve the problem we define the matrix

$$W_Q(h,r) := R_{hr} Q_d R_{hr}^T \tag{22}$$

where R_{hr} is defined by (14) and $Q_d := diag\left[Q^{-1},...,Q^{-1}\right] \in R_+^{rm \times rm}$

Using (22) it is easy to show that for the positive system (1) the matrix (22) is nonsingular if and only if the matrix R_{hr} has full row rank.

Define the sequence of inputs

$$\hat{u}_{0r} := Q_d R_{hr}^T W_Q^{-1}(h,r) x_f \tag{23}$$

Note that $\hat{u}_{0r} \in R_+^{rm}$ for any $x_f \in R_+^n$ if and only if

$$W_Q^{-1}(h,r) \in R_+^{n \times n} \tag{24}$$

Theorem 5.

Let us assume that

i. the positive system (1) is reachable for zero boundary conditions at the point (h,r),

ii. and (24) hold,

iii. $\overline{u}_{0r}$ is any sequence of inputs which transfer the system (1) from zero boundary conditions to the desired local state $x_f = x_{hr}$.

Then the sequence of inputs (23) accomplishes the same task and

$$I\left(\hat{u}_{0r}\right) \le I\left(\overline{u}_{0r}\right) \tag{25}$$

Moreover, the minimum value of (20) is given by

$$I\left(\hat{u}_{0r}\right) = x_f^T W_Q^{-1}(h,k) x_f \tag{26}$$

Proof. First we shall show that the sequence of inputs (23) provides $x_{hr} = x_f$.

Using (23) and (22) we obtain

$$x_{hr} = R(h,r)\hat{u}_{0r} = R_{hr}Q_d R_{hr}^T W_Q^{-1}(h,r)x_f = x_f$$

Since both $\overline{u}_{0r}$ and $\hat{u}_{0r}$ transfer the system from zero boundary conditions to x_f then

$$R_{hr}\overline{u}_{0r} = R_{hr}\hat{u}_{0r} \text{ and } R_{hr}\left[\overline{u}_{0r} - \hat{u}_{0r}\right] = 0 \qquad (27)$$

From (27) we have

$$\left[\overline{u}_{0r} - \hat{u}_{0r}\right]^T R_{hr}^T W_Q^{-1}(h,r)x_f = \left[\overline{u}_{0r} - \hat{u}_{0r}\right]^T Q_d^{-1}\hat{u}_{0r} = 0 \qquad (28)$$

Using (28) it is easy to show that

$$\overline{u}_{0r}^T Q_d^{-1}\overline{u}_{0r} = \hat{u}_{0r}^T Q_d^{-1}\hat{u}_{0r} + \left[\overline{u}_{0r} - \hat{u}_{0r}\right]^T Q_d^{-1}\left[\overline{u}_{0r} - \hat{u}_{0r}\right]$$

or

$$\sum_{i=0}^{r-1}\overline{u}_i^T Q\overline{u}_i = \sum_{i=0}^{r-1}\hat{u}_i^T Q\hat{u}_i + \sum_{i=0}^{r-1}\left[\overline{u}_i - \hat{u}_i\right]^T Q\left[\overline{u}_i - \hat{u}_i\right] \qquad (29)$$

The inequality (25) holds since the last term in (29) is always nonnegative.

To obtain the minimum value of (20) we substitute (23) into (20)

$$I(\hat{u}_{0r}) = \sum_{i=0}^{r-1}\hat{u}_i^T Q\hat{u}_i = \hat{u}_{0r}^T(0,r)Q_d^{-1}\hat{u}_{0r}(0,r) =$$

$$\left[Q_d R_{hr}^T W_Q^{-1}(h,r)x_f\right]^T Q_d^{-1}\left[Q_d R_{hr}^T W_Q^{-1}(h,r)x_f\right] =$$

$$= x_f^T W_Q^{-1}(h,r)R_{hr}Q_d R_{hr}^T W_Q^{-1}(h,r)x_f = x_f^T W_Q^{-1}(h,r)x_f$$

since by (22) $R_{hr}Q_d R_{hr}^T W_Q^{-1}(h,r) = I$. □

6. Conclusions

Sufficient conditions have been established for 2D continuous-discrete linear model (1) to be a positive one. It has been shown that the model (1) is positive only if A_2 is a Metzer matrix. The solution (8) to the model (1) has been derived. Necessary and sufficient conditions for the local reachability of the positive model (1) have been established. The minimum energy control problem for the positive (1) has been formulated and solved.

References

1. Anderson, B.D.O. and M. Deistler, L. Farina and L. Benvenuti: *Nonnegative realization of a linear system with nonnegative impulse response*. IEEE Trans. on Circuits and Systems, vol. 43. No 2, 134-142, 1996.
2. Berman, A. and R.J. Plemmous: *Nonnegative Matrices in the Mathematical Science*. Academic Press, New York, 1979.
3. Farina, L.: *On the existence of a positive realization*. Systems and Control Letters, vol. 28, 219-226, 1996.
4. Fornasini, E. and G.Marchesini: *State space realization of two-dimensional filters*. IEEE Trans. Autom.Control, AC-21, 484-491, 1976.
5. Fornasini, E. and G.Marchesini: *Doubly indexed dynamical systems: State space models and structural properties*. Math. Syst. Theory 12, 1978.
6. Hof, J.M van den: *Realization of positive linear systems*. Linear Algebra and its Applications, vol. 256, 287-308, 1997.
7. Kaczorek, T.: *Realisation problem for discrete-time positive linear systems*. Appl. Math. and Comp. Sc., vol. 7, No 1, 117- 124, 1997.
8. Kaczorek, T.: *Positive realisations of improper transfer matrices of discrete-time linear systems*. Bull. Pol. Acad. Sci. Techn., vol. 45, No 1, 277-286, 1997.
9. Kaczorek, T.: *Positive stable realisations for linear systems*. Bull. Pol. Acad. Sci. Techn., vol. 45, No 4, 549-557, 1997.
10. Kaczorek, T.: *Singular 2-D continuous- discrete linear systems*. Bull. Pol. Acad. Techn. Sci., vol. 42, No 3, 417-422, 1994.
11. Kaczorek, T.: *2-D continuous-discrete linear systems*. Proc.IASTED. Intern. Conf. Modelling, Simulation and Identification, Sept. 12-16 Wakayama, Japan, 191-194, 1994.
12. Kaczorek, T.: *2-D continuous-discrete linear systems*. Proc.Tenth Intern. Conf. on Systems Engineering ICSE'94, vol. I, Coventry, 6-8 Sept., 550-557, 1994.
13. Kaczorek, T.: *Reachability and controllability of 2-D continuous-discrete linear systems*. First Intern. Symp. Mathematical Models in Automation and Robotics, Sept. 1-3, Międzyzdroje, 24-28, 1994.
14. Kaczorek, T.: *Generalized 2-D continuous- discrete linear systems with delays*. Applied Mathematics and Computer Science, vol. 5, No 3, 439-454, 1994.
15. Kaczorek, T:. *Local reachability and minimum energy control of 2-D continuous-discrete linear systems*. Proc. Tenth Intern. Conf. on Systems Engineering ICSE'94, vol. I, Coventry, 6-8 Sept., 558-565, 1994.
16. Kaczorek, T.: *Stabilization of singular 2-D continuous-discrete systems by state-feedback controllers*. IEEE Trans. on Automatic Control, vol. 41, No 7, 1007-1009, 1996.

17. Kaczorek, T.: *Singular two-dimensional continuous-discrete linear systems. Dynamics of continuous, discrete and impulsive systems.* An International Journal for Theory Systems Science and Applications, 193-204, 1996.

18. Kaczorek, T.: *Stabilization of singular 2-d continuous-discrete systems by output-feedback controllers.* Systems Analysis Modelling Simulation, vol. 00, 1-10, 1996.

19. Kaczorek, T.: *Positive linear systems and their relationship with electrical circuits.* XX Seminarium z Podstaw Elektrotechniki i Teorii Obwodów, SPETO'97, Gliwice-Ustroń, 21-24.05, 33-41, 1997.

20. Kaczorek, T. and J. Klamka: *Minimum energy control of 2-D linear systems with variable coefficients.* Int. J. Control, vol. 44, No. 3, 645-650, 1986.

21. Kaczorek, T. and J. Klamka: Minimum energy control for general model of 2-D linear systems. Int. J. Control, vol. 47, No. 5, 1555-1562, 1988.

22. Klamka, J.: *Minimum energy control of singular 2-D linear systems with variable coefficients.* Proc. IMACS Symp. Lille, vol. 2, 155-159, 1991.

23. Klamka, J. *Minimum energy control problem for general linear 2-D systems in Hilbert spaces.* Proc. IEEE Symp. Crete, 1993.

24. Klamka, J.: *Controllability of Dynamical Systems.* Kluwer Academic Publ., Dordrecht, 1991.

25. Maeda, H. and S. Kodama: *Positive realization of difference equations.* IEEE Trans. on Circuits and Systems, vol. CAS- 28, No. 1, 39-47, 1981.

26. Maeda, H., S. Kodama and F. Kajiya: *Compartmental system analysis: Realization of a class of linear systems with physical constraints.* IEEE Trans. on Circuits and Systems, vol. CAS-24, No.1, 8-14, 1977.

27. Ohta, Y., H. Maeda and S. Kodama: *Reachability, observability and reazability of continuous-time positive systems.* SIAM J. Control and Optimization, vol. 22, No.2, 171-180, 1984.

28. Roesser, P.R.: *A discrete state-space model for linear image processing.* IEEE Trans. Autom. Contr., vol. AC-20, No. 1, 1-10, 1975.

ISBN 83-85847-34-0