ANALIZA SYSTEMOWA I ZARZĄDZANIE

Książka jubileuszowa
z okazji
50-lecia pracy naukowej

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ISBN 83-85847-34-0

Druk: "ARGRAF" Agencja Poligraficzno-Wydawnicza, Warszawa
Skład: Barbara Kotuszewska

TOWARD THE UNIFIED GEOMETRIC THEORY OF CONTROL (UGTU)

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1 Introduction

I had a great honour and happiness to be the editor of the Russian version of the remarkable book by Roman Kulikowski "Optimal and Adaptive Processes in the Control Systems" translated into Russian by D.I.Iordanski and published by the "Nauka" Publishing House in Moscow in 1967.

It was a remarkable and pioneering book since in that period of 1950s-1960s the transition to the language of functional analysis was taking place in the theory of control.

Professor Roman Kulikowski was probably one of the first in the world who begun to present and use this mathematical language for control problems with clearness and masterness. It is important that Roman Kulikowski begun to apply functional analysis methods to the nonlinear systems for optimal and adaptive control including applications to the social and human systems.

Nowadays we can observe the new shift to the language of cybernetics with the wide use of the modern geometry and topology. This language promises to be a language of the most general theoretical cybernetics or control theory.

The storming developments in the modern technology require complex distributed parameter regulators. We mention just some of them: stabilizing and control systems for hot and cool plasma, control for coherent fields in lasers for control on quantum and microlevels, theory of composite materials and constructions with complex rheological properties, creation of intellectually active media etc. On the other hand, in spite of about 40 years of development and progress in the theory of distributed parameter systems (DPS) we have nowadays no sufficiently powerful theory and even rather language for solving the above mentioned problems. The UGTU-program [1-5] was born to overcome those disadvantages. This programme is a consequence of the natural developments in the geometric theory of phase portraits for differential inclusions [6,7].

The geometrical trend in control is quite coherent with the common trends of modern sciences, in particular, mechanics and theoretical physics (see, e.g., [8,9]).

To begin with we formulate several new results in the Optimal Control Theory for DPS presented in the general and compact geometrical forms.

2 Some new results in optimal control theory for DPS

2.1 Initial definitions

Consider the so-called (m,n)-CDS, where CDS means Controlled Differential Systems. For more details see [2,4] where the series of so-called proper (or eigen) structures of (m,n)-CDS are considered.

So, (m,n)-CDS means a control plant described by the general systems of PDE in normal matrix form.

$$\frac{\partial Q}{\partial x} = \bar{f}\ (x, Q, u),\ \frac{\partial Q^\beta}{\partial x^\alpha} = \bar{f}^\beta_\alpha\ (x, Q, u), \tag{1}$$

where $\alpha = 1, ..., m$; $\beta = 1, ..., n$; $x = (x^\alpha) \in \bar{D}$; $\bar{D}$ is open region in R^m; $Q = (Q^\beta) \in R^n$; u is control; $u \in U(x, Q)$; U is nonempty closed set in R^r possibly depending on (x, Q); f^β_α are given functions of their arguments. In our further considerations we assume that all functional operations have respective classical or "generalized" sense.

The pair $(Q(x), u(x))$ is called admissible iff (1) becomes identity for $\forall x \in D$ when $Q = Q(x)$ and $u = u(x) \in U(x, Q(x))$. By changing the independent variables $x = x(\xi)$ with positive Jacobian we transform (1) into the "autonomous" form

$$\frac{\partial y}{\partial \xi} = f(y, u),\ \frac{\partial y^\gamma}{\partial \xi^\alpha} = f^\gamma_\alpha(y, u), \tag{2}$$

where $\gamma = (\alpha, m + \beta) = 1, ..., m, m + 1, ...m + n$; $\xi = (\xi^\alpha) \subset \bar{D}$; $y = (x, Q) = (x^\alpha, Q^\beta) = (y^\gamma) \in D \times R^n = D_1$.

Functions $u = u(x)$ and $u = u(\xi)$ are called control; functions $Q = Q(x)$ and $y = y(\xi)$ are called the global state of (m,n)-CDS or simply: state. The point $y = (x, Q)$ is called imaging point or local state of (m,n)-CDS.

2.2 Geometrical representations

Consider $y = y(\xi), \xi \in \bar{D}$, $y \in \bar{D} \times R^n = D_1$, as a parametric equation for m-dimensional surface S in R^{m+n} in some Cartesian (rectangular and rectilinear) system of coordinates. Then we identify S with the state of (m,n)-CDS and point $y \in S$ is a local state. So, we call S admissible. Further, we shall consider only admissible $y = y(\xi)$, $u = u(\xi)$ and S, and so the word "admissible" will be omited.

For given S and any given $y \in S$ we also consider the tangent space $T_yS \subset T_yR^{m+n}$ and degree m of the tangent spaces $\hat{m}\, T_yS \subset \hat{m}\, T_yR^{m+n}$.

Then it becomes easy to understand that (2) is equivalent to m-vector equation

$$v = f(y,u),\ v \in \hat{m}\, T_yR^{m+n}, \tag{3}$$

where v is so-called m-vector which can be presented as an ordered set of "usual" vectors $v_\alpha \in T_yR^{m+n}$, $v = (v_\alpha) = (v_1, ..., v_m)$.

Associate differential inclusion from (3)

$$v \in I(y),\ I(y) = f\left(y, u(y)\right) \subset T_yD_1 \subset T_yR^{m+n}, \tag{4}$$

where $I(y)$ is called indicatrix on (m,n)-CDS.

So, S is an admissible iff it exists $v \in T_yS \subset I(y)$ for $\forall y \in S$.

2.3 Metric eigenstructures of (m,n)-CDS

We introduce important structures associated with any given (m,n)-CDS. (For other structures see [2,4]). First of all we introduce the field of cones of admissible "directions" for m-vectors v:

$$K(y) = \{\lambda_v : v \in I(y), \lambda > 0\},\ y \in D_1. \tag{5}$$

Then we define

$$\|v\| = L_0(y,v) = \left\{\inf \frac{1}{\lambda} : \lambda > 0, \lambda_v \in I(y)\right\},\ y \in D_1,\ v \in K(y). \tag{6}$$

We call function L_0 local eigen m-metric (m-measure) of (m,n)-CDS. Define also unit ball and unit sphere

$$L_0(y,v) \le 1 \Longleftrightarrow v \in I(y) \text{ and} \tag{7}$$

$$L_0(y,v) = 1 \Longleftrightarrow v \in \partial I(y). \tag{8}$$

For $m = 1$ function L_0 is gauge or Minkovski function, which gives the value of the square of surface element $v \in T_yS$ or norm $\|v\|$.

In Euclidean case $\|v\|$ means the usual volume of a parallelepiped spanned over v_α, $\alpha = 1, ..., m$.

Surface S is called measurable iff $\forall v \in T_yS$ is measurable in the sense of (6). Thus, we substitute admissibility by measurability. This is important.

Use L_0 for defining of square $\mu(S)$ for S :

$$\mu(S) = \int_S L_0(y,v)dy \tag{9}$$

S is summable iff $\mu(S)$ is defined by (9) (S is measurable) and exists.

2.4 Eigenoptimum of (m,n)-CDS

Denote $\{S\}$ as a set of all summable S with fixed $\partial S = \Gamma$. Then $\bar{S}$ is eigenoptimum of (m,n)-CDS iff by definition

$$\mu(\bar{S}) = \{\min \mu(S) : \in \{S\}\}. \tag{10}$$

In this conditions $\mu\left(\bar{S}\right)$ becomes the function of $\Gamma = \partial S$. Denote it by $\bar{\mu}\left(\Gamma\right)$.

3 The main results

3.1 Multidimensional Principle of Maximum (MPM)

Multidimentional Principle of Maximum (MPM) as a necessary condition of eigen optimality for (m,n)-CDS. MPM was formulated in noninvariant form in [2]. Here we formulate it in terms of differential forms:

Theorem 1 *MPM Theorem. If $\bar{S}$ is eigenoptimum then there exists closed differential m-form P such that the MPM condition is completed:*

$$\{\max P(y,v) \ : \ v \in I(y)\} = P(y,\bar{v}) = c \text{ for } A\,\forall y \in \bar{S},\ c > 0. \tag{11}$$

Here $\bar{v} \in T_y\ \bar{S}$, $\|\bar{v}\| = 1$ and $A\forall$ means "for almost any".

3.2 Extended Multidimensional Principle of Maximum (EMPM) as a sufficient condition of eigen optimality for (m,n)-CDS.

Theorem 2 *EMPM Theorem. Let $\bar{S}$ be summable (in the sense of (9)). Then $\bar{S}$ is eigenoptimum (in the sense of (10)) if the EMPM condition is completed: there exists an exact differential m−form P such that*

$$\{\max P(y,v) \ : \ v \in I(y),\ y \in D_1\} = P(y,\bar{v}) = 1 \text{ for } A\forall y \in \bar{S}\,. \tag{12}$$

Remarks.

1. For rather general conditions for L_0 the proof of MPM Theorem is rather difficult and not comleted. But EMPM Theorem can be proved easily.

2. It is not easy to see that the problem of optimal control for (1) with functional
$$\int_D f^0(x,Q,u)dx \Longrightarrow \min \tag{13}$$
can be transformed to the problem of eigenoptimum for other (m,n)-CDS.

4 Energy tensor of (m,n)-CDS

Following consideration puts more of physical sense into the nature of (m,n)-CDS. There are no logical obstacles to consider L_0 as Lagrangian for a physical system. It is easy to show that by means of "homogeneisation" (see [2]) any Lagrangian $L(x, Q, \frac{\partial Q}{\partial x})$ can be transformed into the equivalent homogeneous Lagrangian of type L_0 and vice versa. Since L_0 is a positive homogeneous function of the first degree as a function of each v_α, $\alpha = 1, ..., m$, we have energy tensor $E_\alpha^{\alpha'}$ (by means of Euler identities).

$$E = \left(E_\alpha^{\alpha'}\right) = \left(P_\gamma^{\alpha'} \frac{\partial y^\gamma}{\partial \xi^\alpha} - \delta_\alpha^{\alpha'} L_0 \left(y, \frac{\partial y^\gamma}{\partial \xi^\alpha}\right)\right) \tag{14}$$

and

$$trE = E_\alpha^\alpha = P_\gamma^\alpha \frac{\partial y^\gamma}{\partial \xi^\alpha} - mL_0 \left(y, \frac{\partial y^\gamma}{\partial \xi^\alpha}\right) = 0,$$

where P_γ^α has sense of the flow components of some substance (e.g. a sort of multicomponent gas). In these terms the two theorems, *MPM* and *EMPM*, can be formulated shortly as follows 1) MPM means that along the optimum S the trace trE of energy tensor E as a function of $v = \frac{\partial y^\gamma}{\partial \xi^\alpha}$ assumes the maximum value equal to zero and 2) EMPM means that if the trace trE of energy tensor E assumes maximum value equal to zero on some S as function of y and $v = \frac{\partial y^\gamma}{\partial \xi^\alpha}$, then S is optimum.

Remark. All of the above mentioned items can be symmetrically reformulated in terms of $H_0(y, P)$, Hamiltonian of (m,n)-CDS, instead of $L_0(y, \frac{\partial y}{\partial \xi})$, on the basis of identities at each $i = 1, ..., m$

$$P_\gamma^{\alpha'} \frac{\partial y^\gamma}{\partial \xi^\alpha} = L_0 \left(y, \frac{\partial y^\gamma}{\partial \xi^\alpha}\right) = H_0(y, P_\gamma^\alpha),\ P_\gamma^\alpha = \frac{\partial L_0}{\partial \overset{Q}{D}\, \partial y^\gamma \partial \xi^\alpha}; \cdot \frac{\partial y^\gamma}{\partial \xi^\alpha} = \frac{\partial H_0}{\partial P_\gamma^\alpha} \tag{15}$$

with the gauging condition $L_0 = H_0 = 1$.

References

[1] Butkovskiy, A.G.: *Toward the Unified Geometric Theory of Differential Systems with Control.* Vortragsauszuge der Herbstschule, Stralsund, 13-18 Oktober, 1993. "Variationsrechnung, Optimal Prozesse und Anwendungen", Preprint-Reihe Mathematik, Greifswald, 1994, No 1, pp.5-8 (in English).

[2] Butkovskiy, A.G.: *Toward Geometric Theory of Control for Distributed Parameter Systems.* "Theory and Control Systems" (the former "Technical Cybernetics"), No 4, 1995 (in Russian; to appear in English version by "Sripta Technika").

[3] Butkovskiy, A.G.: *Geometrical Approach to the Controlled Differential Systems (CDS).* "Systems Science", vol. 22, No 2, 1996 (Technical Univ. of Wroclaw). In English.

[4] Butkovskiy, A.G.: *"Problems of Control and Informatics"*, Cybernetics and Structures. No 1-2, 1996 (in Russian, to appear in English version by "Sripta Technika").

[5] Butkovskiy, A.G.: *On the Way to Geometrization of Control.* "Theory and Control Systems". No 1, 1997 (The Former "Technical Cybernetics") (in Russian, to appear in English version by "Interperiodica Publishing", Nauka, Moscow).

[6] Butkovskiy, A.G.: *Phase Space Portraits for Control Dynamical Systems.* Kluwer Publishing, Netherlands, 1991 (book in English).

[7] Butkovskiy, A.G., Samoilenko, Yu.I.: *Control of Quantum Processes and Systems.* Kluwer Publishing, Netherlands, 1990 (book in English).

[8] von Westenholz, C.: *Differential Forms in Mathematical Physics.* North Holland, 1981.

[9] Atiyah, M.: *The Geometry and physics of knots.* Cambridge Univ. Press, Cambridge, New York, Sydney.

ISBN 83-85847-34-0