New Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume II: Applications

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Dedicated to Professor Beloslav Riečan on his 75th anniversary

Effective solution of the Kemeny median. The case of ties in group ranking

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Abstract

In the paper a heuristic procedure of determining the Kemeny median is presented. It is assumed that all the alternatives are compared and ties may occur in experts' opinions as well as in group rankings. The loss matrix is applied to derive the solution. Lower bound of a distance of a given ranking from experts' opinions is evaluated for the case of ties in group ranking. It is also shown that the procedure of determining the Kemeny median can be simplified by means of the analysis of the loss matrix.

Keywords: pairwise comparison matrix, group ranking, ties in group ranking, the Kemeny median method.

1 Introduction

The problem of determining the Kemeny median is generally NP-hard and there are many methods developed which simplify its solution. In the paper [1] one of such methods was presented. For the case of no ties in group ranking Litvak's theorem, formulating necessary and sufficient condition for a preference order to be a median, was used.

In the present paper this approach is generalized for the case of ties in the median and a loss matrix R is applied to determine the Kemeny median.

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2 Definitions

In this section some necessary definitions are given. The detailed description of notions considered is provided in [1].
Given a set of alternatives $\mathcal{O} = \{O_1, \ldots, O_n\}$ and a set of experts $\mathcal{K} = \{1, \ldots, K\}$. The experts are expected to rank the set of alternatives according to an adopted criterion (set of criteria). It is assumed that all the alternatives from the set $\mathcal{O}$ are compared and tied alternatives can occur in experts' opinions as well as in group rankings. We also assume that all experts' opinions considered reveal true preferences and are of the same importance. The result of pairwise comparisons of alternatives may be as follows:

$O_i \overset{k}{\succ} O_j$, if the $k-th$ expert regards alternative O_i better than O_j,
$O_i \overset{k}{\approx} O_j$, if the $k-th$ expert regards alternatives O_i and O_j equally important,
$O_i \overset{k}{\prec} O_j$, if the $k-th$ expert regards alternative O_j better than O_i.

The following notation for a preference order can be used:

$$P = \{O_{i_1}, O_{i_2}, \ldots, (O_{i_s}, O_{i_t}), \ldots, O_{i_{n-1}}, O_{i_n}\} \tag{1}$$

It denotes that the alternative O_{i_j} is better than $O_{i_{j+1}}$ and tied alternatives (O_{i_s}, O_{i_t}) are given in brackets.
The $k-th$ expert opinion can be formulated in the form of a pairwise comparisons matrix A^k [2]:

$$A^k = [a^k_{ij}], \text{ where } a^k_{ij} = \begin{cases} 1 & \text{if } O_i \overset{k}{\succ} O_j \\ 0 & \text{if } O_i \overset{k}{\approx} O_j \\ \text{-1} & \text{if } O_i \overset{k}{\prec} O_j \end{cases} \tag{2}$$

In general it is assumed that $a^k_{ii} = 0, i = 1, \ldots, n$. Moreover it is assumed that experts opinions are given in the form of preference orders and P^k denotes the $k-th$ expert opinion.

Definition 1 *The distance between a pair of alternatives O_i, O_j in a given preference order P and a pair of the same alternatives in a ranking P^k is as follows*

$$d_{ij}(P, P^k) = \left| a_{ij} - a^k_{ij} \right| \tag{3}$$

Definition 2 *The distance between two preference orders P and P^k is as follows*

$$d(P,P^k)=\sum_{i=1}^{n-1}\sum_{j>i}^{n}d_{ij}(P,P^k)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i}^{n}\left|a_{ij}-a_{ij}^k\right| \tag{4}$$

Assume that the alternative O_i precedes O_j in the preference order P, hence $a_{ij}=1$. The r_{ij}^k coefficient is defined as follows

$$r_{ij}^k=d_{ij}(P,P^k)=\left|a_{ij}^k-a_{ij}\right|=\left|a_{ij}^k-1\right|,k=1,\ldots,K. \tag{5}$$

It may take the following values

$$r_{ij}^k=\left|a_{ij}^k-1\right|=\begin{cases}0 & \text{if } a_{ij}^k=1\\ 1 & \text{if } a_{ij}^k=0\\ 2 & \text{if } a_{ij}^k=-1\end{cases} \tag{6}$$

The sum of r_{ij}^k is denoted as r_{ij}

$$r_{ij}=\sum_{k=1}^{K}r_{ij}^k \tag{7}$$

$R=[r_{ij}]$ is called the loss matrix. It follows from (6) that the values of its elements depend only on preference orders given by experts.

If in a given ranking P alternatives O_i and O_j are tied, then $a_{ij}=0$. In this case

$$\left|a_{ij}^k-a_{ij}\right|=\left|a_{ij}^k\right|=e_{ij}^k,k=1,\ldots,K. \tag{8}$$

This coefficient takes the following values

$$e_{ij}^k=\left|a_{ij}^k\right|=\begin{cases}1 & \text{if } a_{ij}^k=1 \text{ or } a_{ij}^k=-1\\ 0 & \text{if } a_{ij}^k=0\ .\end{cases} \tag{9}$$

hence

$$e_{ij}=\sum_{k=1}^{K}e_{ij}^k=\sum_{k}\left|a_{ij}^k\right|_{O_i\overset{k}{\succ}O_j}+\sum_{k}\left|a_{ij}^k\right|_{O_i\overset{k}{\approx}O_j}+\sum_{k}\left|a_{ij}^k\right|_{O_i\overset{k}{\prec}O_j} \tag{10}$$

Given

l_{ij}	-	number of experts who prefer O_i to O_j,
l_{ji}	-	number of experts who prefer O_j to O_i,
m_{ij}	-	number of experts in whose opinions O_i and O_j are tied

we have

$$l_{ij} + m_{ij} + l_{ji} = K \tag{11}$$

hence

$$e_{ij} = 1 \cdot l_{ij} + 0 \cdot m_{ij} + 1 \cdot l_{ji} = K - m_{ij} \tag{12}$$

Taking into account Definition 2 one can decompose the set of indices (i, j) into three groups

$$\begin{aligned} I_P^1 &= \left\{(i,j) : O_i \succ^P O_j\right\}, && \text{i.e. } a_{ij} = 1 \text{ for } (i,j) \in I_P^1 \\ I_P^{1*} &= \left\{(i,j) : O_j \succ^P O_i\right\}, && \text{i.e. } a_{ij} = -1 \text{ for } (i,j) \in I_P^{1*} \\ I_P^0 &= \left\{(i,j), j > i : O_i \approx^P O_j\right\}, && \text{i.e. } a_{ij} = 0 \text{ for } (i,j) \in I_P^0 \end{aligned} \tag{13}$$

Making use of (13) formulae (4) can be rewritten as follows

$$\begin{aligned} d(P, P^k) &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| a_{ij} - a_{ij}^k \right| \\ &= \frac{1}{2} \left(\sum_{(i,j) \in I_P^1} \left| a_{ij}^k - 1 \right| + \sum_{(i,j) \in I_P^{1*}} \left| a_{ij}^k + 1 \right| \right) + \sum_{(i,j) \in I_P^0} \left| a_{ij}^k \right| \end{aligned} \tag{14}$$

We have

$$\begin{aligned} &\sum_{(i,j) \in I_P^1} \left| a_{ij}^k - 1 \right| + \sum_{(j,i) \in I_P^1} \left| a_{ji}^k + 1 \right| = \\ &\sum_{(i,j) \in I_P^1} \left| a_{ij}^k - 1 \right| + \sum_{(i,j) \in I_P^1} \left| -a_{ij}^k + 1 \right| = 2 \sum_{(i,j) \in I_P^1} \left| a_{ij}^k - 1 \right| \end{aligned} \tag{15}$$

Taking into account (15) the distance of a given ranking P from the set of experts' opinions $P^{(k)}$ can be written as follows

$$\begin{aligned} d(P, P^{(k)}) = \sum_{k=1}^{K} d(P, P^k) &= \sum_{(i,j) \in I_P^1} \sum_{k=1}^{K} \left| a_{ij}^k - 1 \right| + \sum_{(i,j) \in I_P^0} \sum_{k=1}^{K} \left| a_{ij}^k \right| \\ &= \sum_{(i,j) \in I_P^1} r_{ij} + \sum_{(i,j) \in I_P^0} e_{ij} \end{aligned} \tag{16}$$

It follows from (16) that for a given preference order P the distance $d(P, P^{(k)})$ is the sum of two components.

If there are no tied alternatives in P then the distance

$$d(P, P^{(k)}) = \sum_{(i,j) \in I_P^1} r_{ij} \tag{17}$$

Definition 3 [3] *The Kemeny median is a preference order* $\widetilde{P}$ *such that*

$$d(\widetilde{P}, P^{(k)}) = \min_P d(P, P^{(k)}) \tag{18}$$

In the classic definition of the Kemeny median [2, 3] it is assumed there are no tied alternatives in the median. The admissibility of ties in the median extends this definition.

Definition 4 [3] *Condorcet winner is an alternative which precedes all the others in the opinions of the plurality of experts* $(K_W > K/2)$[1)].

Definition 5 [3] *A set of preference orders has the Condorcet property if for every subset of alternatives there exists the Condorcet winner.*

Litvak [3] showed that for the case of no ties in the median the following theorems hold.

Theorem 1 [3] *If a given set of preference orders* $P^{(k)}$ *has the Condorcet property, then the Kemeny median is a preference order* $\widetilde{P} = \{O_{i_1}, O_{i_2}, \ldots, O_{i_{n-1}}, O_{i_n}\}$ *of subsequent Condorcet winners. For this preference order the distance* $d(\widetilde{P}, P^{(k)})$ *is equal to the lower bound of the distance - denoted as* H *- from the set of preference orders* $P^{(k)}$.

The lower bound H of the distance (17) is given by Litvak [3]

$$H = \sum_{i=1}^{n-1} \sum_{j>i}^{n} \min(r_{ij}, r_{ji}) \tag{19}$$

[1)] Some authors formulate this condition as $(K_W \geqslant K/2)$

It is important to estimate the minimum value of the distance i.e. its lower bound. It is evident that in the case of ties it depends on the class of rankings considered. From (16) we have

$$\min d(P, P^{(k)}) = \sum_{(i,j)\in I_P^1} \min(r_{ij}, r_{ji}) + \sum_{(i,j)\in I_P^0} e_{ij} \tag{20}$$

Let's denote the modified I_P^0 set as I_P^{0+}.

$$I_P^{0+} = \{(i,j), j > i : e_{ij} \leqslant \min(r_{ij}, r_{ji})\} \tag{21}$$

The modified I_P^{1+} is as follows

$$I_P^{1+} = I_P^1 \setminus I_P^{0+} \tag{22}$$

Formulae (20) may be rewritten as follows

$$\min d(P, P^{(k)}) = \sum_{(i,j)\in I_P^{1+}} \min(r_{ij}, r_{ji}) + \sum_{(i,j)\in I_P^{0+}} e_{ij} \tag{23}$$

Let's denote the lower bound of the distance for the case of ties as $\widetilde{H}$.

$$\widetilde{H} = \sum_{(i,j)\in I_P^{1+}} \min(r_{ij}, r_{ji}) + \sum_{(i,j)\in I_P^{0+}} e_{ij} \tag{24}$$

We have

$$\begin{aligned} \widetilde{H} &= \sum_{i=1}^{n}\sum_{j>i}^{n} \min(r_{ij}, r_{ji}) - \sum_{(i,j)\in I_P^{0+}} \min(r_{ij}, r_{ji} + \sum_{(i,j)\in I_P^{0+}} e_{ij} \\ &= H - \sum_{(i,j)\in I_P^{0+}} [\min(r_{ij}, r_{ji}) - e_{ij}] \end{aligned} \tag{25}$$

We introduce ties in the group ranking in order to decrease the value of $\widetilde{H}$ with respect to H[1)].
From (21) we have

$$[\min(r_{ij}, r_{ji}) - e_{ij}] \geqslant 0, \forall (i,j) \in I_P^{0+} \tag{26}$$

hence $\widetilde{H} \leqslant H$.
It follows from (24) that for the case of ties in a given ranking P the lower bound $\widetilde{H}$ depends on the cardinality of I_P^{0+}.
This relationship is important for checking whether a ranking considered is a median, specially for the case of ties in group ranking or when the loss matrix does not have the Condorcet property.

[1)] If the assumption $e_{ij} \leqslant min(r_{ij}, r_{ji})$ imposed on the set I_P^{0+} is not satisfied it can happen that $\widetilde{H} > H$. Hence this assumption seems to be justified.

Example 1.

Given preference orders of five alternatives presented by five experts.

$$\begin{array}{ll} P^1: & (O_2,O_5),O_4,(O_1,O_3) \\ P^2: & (O_2,O_3),O_1,O_5,O_4 \\ P^3: & O_1,O_4,(O_3,O_5),O_2 \\ P^4: & O_4,O_3,(O_1,O_2,O_5) \\ P^5: & O_3,O_5,O_4,O_1,O_2 \end{array} \tag{27}$$

The loss matrix R and the E matrix are as follows

$$\begin{array}{c|c|c|c|c|c} R & O_1 & O_2 & O_3 & O_4 & O_5 \\ \hline O_1 & 0 & \boxed{5} & 7 & 6 & \boxed{5} \\ \hline O_2 & \boxed{5} & 0 & 7 & 6 & \boxed{6} \\ \hline O_3 & 3 & 3 & 0 & 6 & 3 \\ \hline O_4 & 4 & 4 & 4 & 0 & 6 \\ \hline O_5 & \boxed{5} & \boxed{4} & 7 & 4 & 0 \end{array} \qquad \begin{array}{c|c|c|c|c|c} E & O_1 & O_2 & O_3 & O_4 & O_5 \\ \hline O_1 & 0 & \boxed{4} & 4 & 5 & \boxed{4} \\ \hline O_2 & \boxed{4} & 0 & 4 & 5 & \boxed{3} \\ \hline O_3 & 4 & 4 & 0 & 5 & 4 \\ \hline O_4 & 5 & 5 & 5 & 0 & 5 \\ \hline O_5 & \boxed{4} & \boxed{3} & 4 & 5 & 0 \end{array} \tag{28}$$

For the case of no ties in the group ranking the lower bound H (19) determined for the set of preference orders (27) is

$$H = \sum_{i=1}^{n-1} \sum_{j>i}^{n} min(r_{ij}, r_{ji}) = 39 \tag{29}$$

If ties can occur in the group ranking then the set I_P^{0+} is to be determined. The condition (21) is met for $(i,j) \in I_P^{0+} = \{(1,2),(1,5),(2,5)\}$. Corresponding elements of R and E matrices are given in frames.
The lower bound $\widetilde{H}$ (25) determined for ties in group ranking is

$$\widetilde{H} = H - \sum_{(i,j)\in I_P^{0+}} [\min(r_{ij}, r_{ji}) - e_{ij}] = 39 - (1+1+1) = 36 \tag{30}$$

3 Examples of determining the Kemeny median

In some cases determining the Kemeny median may be significantly simplified even if the loss matrix does not possess the Condorcet property. Some examples will illustrate the procedure proposed.

The process of determining the Kemeny median begins with verifying whether the set of rankings considered has the Condorcet property.

1. If it is true, then the Kemeny median is a ranking $\widetilde{P} = \{O_{i_1}, O_{i_2}, \ldots, O_{i_{n-1}}, O_{i_n}\}$ that consists of Condorcet winners for subsequent subsets of alternatives. For this ranking the distance $d(\widetilde{P}, P^{(k)})$ is equal to the lower bound of the distance from the set of rankings $P^{(k)}$.

2. If this is not the case, then one assumes that an alternative close to the Condorcet winner is taken as this winner and removed from the set of alternatives. Next the Condorcet winner (or an alternative close to) for the $(n-1)$ elements set is to be determined and to be removed from the set of alternatives. This procedure is repeated until an empty set remains. If there is more then one alternative close to the Condorcet winner the procedure is repeated for all the sequences of alternatives and all the possible rankings that may constitute the Kemeny median are determined.

3. It should be checked which ranking is the closest one (in the sense of distance (16)) to the set of experts' rankings.
   If the distance (16) is equal to the lower bound, then according to the Theorem 1 the ranking (rankings) considered constitutes (constitute) the Kemeny median.
   Otherwise one has to determine the difference between the distance analyzed and the corresponding (for the case of ties or no ties) lower bound of the distance Δd. If this difference is equal to the minimum value Δd_{min} then the ranking (rankings) considered constitute the Kemeny median. For the case of no ties in group ranking the minimum value of the difference of distance between rankings is equal to $\Delta d_{min} = 2$.

Example 2.

Given the set of five rankings of four alternatives presented by five experts.

$$\begin{array}{ll} P^1: & O_1, (O_3, O_4), O_2 \\ P^2: & O_4, (O_1, O_3), O_2 \\ P^3: & O_1, O_3, (O_2, O_4) \\ P^4: & O_4, O_1, (O_2, O_3) \\ P^5: & (O_1, O_2), O_3, O_4 \end{array} \tag{31}$$

There are tied alternatives in the preference orders (31). It is generally accepted that in such a case every tied alternative receives $\frac{1}{2}$ of the expert's vote. Hence the outranking matrix and the loss matrix are as follows

$$L = \begin{array}{c|c|c|c|c} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & 0 & 4.5 & 4.5 & 3 \\ \hline O_2 & 0.5 & 0 & 1.5 & 1.5 \\ \hline O_3 & 0.5 & 3.5 & 0 & 2.5 \\ \hline O_4 & 2 & 3.5 & 2.5 & 0 \end{array} \qquad R = \begin{array}{c|c|c|c|c} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & 0 & 1 & 1 & 4 \\ \hline O_2 & 9 & 0 & 7 & 7 \\ \hline O_3 & 9 & 3 & 0 & \boxed{5} \\ \hline O_4 & 6 & 3 & \boxed{5} & 0 \end{array} \tag{32}$$

For the case of no ties in the group ranking the lower bound of the distance H is equal to 17.
In this example the alternative O_1 is the Condorcet winner. After removing it from the set of alternatives the outranking matrix becomes

$$\begin{array}{c|c|c|c} & O_2 & O_3 & O_4 \\ \hline O_2 & 0 & 1.5 & 1.5 \\ \hline O_3 & 3.5 & 0 & 2.5 \\ \hline O_4 & 3.5 & 2.5 & 0 \end{array} \tag{33}$$

There is no Condorcet winner in this matrix but one choose an alternative close to it. It follows from (33) that alternatives O_3 or O_4 can to be taken into account.
For the case of no ties, group rankings consisting of the subsequent Condorcet winners are of the form

$$O_1, O_3, O_4, O_2 \tag{34}$$

and

$$O_1, O_4, O_3, O_2 \tag{35}$$

The distance of the preference order (34) from the set (31) may be determined from the loss matrix R

$$R = \begin{array}{c|c|c|c|c||c} & O_1 & O_3 & O_4 & O_2 & \sum \\ \hline O_1 & 0 & 1 & 4 & 1 & 6 \\ \hline O_3 & & 0 & 5 & 3 & 8 \\ \hline O_4 & & & 0 & 3 & 3 \\ \hline O_2 & & & & 0 & \mathbf{17} \end{array} \tag{36}$$

Similarly, the distance of the ranking (35) is equal to 17 and is equal to its lower bound. Hence both rankings are medians.
For the case of ties in group ranking the matrix E, the set I_P^{0+} and the lower bound of the distance $\widetilde{H}$ are to be determined.

From (21) we have the $I_P^{0+} = \{(3,4)\}$. The E matrix is of the form

$$E = \begin{array}{c|c|c|c|c} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & 0 & 4 & 4 & 5 \\ \hline O_2 & 4 & 0 & 4 & 4 \\ \hline O_3 & 4 & 4 & 0 & \boxed{4} \\ \hline O_4 & 3 & 4 & \boxed{4} & 0 \end{array} \tag{37}$$

If one assumes that in a given preference order P alternatives O_3 and O_4 are tied then the lower bound of the distance is (25)

$$\widetilde{H} = H - [\min(r_{3,4}, r_{4,3}) - e_{3,4}] = 17 - 1 = 16 \tag{38}$$

For the preference order $O_1, (O_3, O_4), O_2$ the distance (23) from the set of preference orders (31) is equal to 16 and is equal to the lower bound. Hence - for the case of ties - it is the median .

Example 3.

For the set of rankings from Example 1 the outranking matrix L is as follows

L	O_1	O_2	O_3	O_4	O_5
O_1	0	2.5	1.5	2	2.5
O_2	2.5	0	1.5	2	2
O_3	3.5	3.5	0	2	3.5
O_4	3	3	3	0	2
O_5	2.5	3	1.5	3	0

(39)

There is no Condorcet winner however, alternatives O_3 and O_4 are close to it. After removing O_3 from the matrix (39) one gets

L	O_1	O_2	O_4	O_5
O_1	0	2.5	2	2.5
O_2	2.5	0	2	2
O_4	3	3	0	2
O_5	2.5	3	3	0

(40)

In the matrix (40) alternative O_5 is close to the Condorcet winner. After removing it from this matrix one obtains

L	O_1	O_2	O_4
O_1	0	2.5	2
O_2	2.5	0	2
O_4	3	3	0

(41)

It follows from the matrix (41) that the following preference orders can be taken into account as the group ranking

$$O_3, O_5, O_4, O_1, O_2 \tag{42}$$

and

$$O_3, O_5, O_4, O_2, O_1 \tag{43}$$

Similarly, when the alternative O_4 is to be removed from (39) as the first, then O_3 as the next and finally (O_1 or O_5) one gets three preference orders

$$O_4, O_3, O_1, O_5, O_2 \tag{44}$$

$$O_4, O_3, O_5, O_1, O_2 \tag{45}$$

$$O_4, O_3, O_5, O_2, O_1 \tag{46}$$

The distances (16) of these rankings from the set (27) are all equal to 41. The lower bound of the distance (19) is $H = 39$. There is no ranking for which $d(P, P^{(k)}) = H$. Hence rankings (42) to (46) are medians because their distance from the lower bound is minimum and equal to 2.

For the case of ties the set $I_P^{0+} = \{(1,2),(1,5),(2,5)\}$ and the lower bound $\widetilde{H} = 36$. Let's consider the following preference orders

$$O_4, O_3, (O_1, O_2, O_5) \tag{47}$$

and

$$O_3, O_4, (O_1, O_2, O_5) \tag{48}$$

The distance (16) from the set of rankings (27) for both preference orders equals to 38. There is no ranking for which $d(P, P^{(k)}) = \widetilde{H}$. Hence rankings (47) and (48) are medians because its distance from the lower bound is minimum.

Example 4.

Given the set of rankings of eight alternatives presented by eleven experts.

$$\begin{array}{ll} P^1: & O_5, (O_7, O_8), (O_3, O_6), (O_1, O_2), O_4 \\ P^2: & O_5, O_4, O_2, O_8, O_7, O_1, O_6, O_3 \\ P^3: & O_3, O_7, O_2, O_5, (O_4, O_6, O_8), O_1 \\ P^4: & O_5, O_8, O_7, O_1, O_6, O_3, O_4, O_2 \\ P^5: & O_3, (O_4, O_5, O_8), (O_1, O_2), O_6, O_7 \\ P^6: & O_3, O_2, O_7, O_5, O_8, O_6, O_4, O_1 \\ P^7: & O_3, (O_5, O_7), O_1, (O_2, O_4, O_6, O_8) \\ P^8: & O_3, O_8, O_4, O_2, O_1, O_5, O_6, O_7 \\ P^9: & (O_1, O_4, O_8), O_5, (O_2, O_3, O_6, O_7) \\ P^{10}: & O_1, O_5, O_3, O_7, O_8, O_2, O_4, O_6 \\ P^{11}: & O_7, (O_4, O_8), (O_1, O_2), O_6, O_3, O_5 \end{array} \tag{49}$$

The corresponding outranking matrix L is as follows

L	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8
O_1	0	5.5	5	4.5	4	8	4	2.5
O_2	5.5	0	2.5	4.5	4	8	4.5	3.5
O_3	6	8.5	0	8	6	7	6.5	6
O_4	6.5	6.5	3	0	3.5	7	4	3.5
O_5	7	7	5	7.5	0	10	7.5	7.5
O_6	3	3	4	4	1	0	2.5	1
O_7	7	6.5	4.5	7	3.5	8.5	0	5.5
O_8	8.5	7.5	5	7.5	3.5	10	5.5	0

(50)

The loss matrix R and the E matrix are as follows

R	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8
O_1	0	$\boxed{11}$	12	13	14	6	14	17
O_2	$\boxed{11}$	0	17	13	14	6	13	15
O_3	10	5	0	6	10	8	9	10
O_4	9	9	16	0	15	8	14	$\boxed{15}$
O_5	8	8	12	7	0	2	7	7
O_6	16	16	14	14	20	0	17	20
O_7	8	9	13	8	15	5	0	$\boxed{11}$
O_8	5	7	12	$\boxed{7}$	15	2	$\boxed{11}$	0

$$
\begin{array}{c|c|c|c|c|c|c|c|c}
E & O_1 & O_2 & O_3 & O_4 & O_5 & O_6 & O_7 & O_8 \\
\hline
O_1 & 0 & \boxed{8} & 11 & 10 & 11 & 11 & 11 & 10 \\
\hline
O_2 & \boxed{8} & 0 & 10 & 10 & 11 & 9 & 10 & 10 \\
\hline
O_3 & 11 & 10 & 0 & 11 & 11 & 9 & 10 & 11 \\
\hline
O_4 & 10 & 10 & 11 & 0 & 10 & 9 & 11 & \boxed{6} \\
\hline
O_5 & 11 & 11 & 11 & 10 & 0 & 11 & 10 & 10 \\
\hline
O_6 & 11 & 9 & 9 & 9 & 11 & 0 & 10 & 0 \\
\hline
O_7 & 11 & 10 & 10 & 11 & 10 & 10 & 0 & \boxed{10} \\
\hline
O_8 & 10 & 10 & 11 & \boxed{6} & 10 & 9 & \boxed{10} & 0
\end{array}
\tag{51}
$$

For the case of no ties in the group ranking the lower bound of the distance is $H = 208$. It follows from the L matrix that alternatives O_3 are O_5 the first and the second Condorcet winner respectively and O_7 and O_8 are close to the Condorcet winner. After eliminating these alternatives from the L matrix it can be shown that O_4 is Condorcet winner and O_1 and O_2 are close to the Condorcet winner while O_6 is the Condorcet loser. The preference orders to be considered are of the form

$$O_3, O_5, O_7, O_8, O_4, O_1, O_2, O_6 \tag{52}$$

$$O_3, O_5, O_7, O_8, O_4, O_2, O_1, O_6 \tag{53}$$

$$O_3, O_5, O_8, O_7, O_4, O_1, O_2, O_6 \tag{54}$$

$$O_3, O_5, O_8, O_7, O_4, O_1, O_1, O_6 \tag{55}$$

The distance from the set (49) for all these preference orders is equal to 208, i.e. to its lower bound, hence all the rankings considered are medians.

For the case of ties in the group ranking we have (21) that the set I_P^{0+} consists of three pairs $\{(1,2),(4,8),(7,8)\}$ and the lower bound (25) of the distance $\widetilde{H}$ is equal to 203. Preference orders to be taken into account are as follows (values of the distance d are also given)

$$O_3, O_5, (O_7, O_8), O_4, O_1, O_2, O_6, \qquad d = 207 \tag{56}$$

$$O_3, O_5, (O_7, O_8), O_4, O_2, O_1, O_6, \qquad d = 207 \tag{57}$$

$$O_3, O_5, O_7, O_8, O_4, (O_1, O_2), O_6, \qquad d = 205 \tag{58}$$

$$O_3, O_5, O_8, O_7, O_4, (O_1, O_2), O_6, \qquad d = 205 \tag{59}$$

$$O_3, O_5, (O_7, O_8), O_4, (O_1, O_2), O_6, \qquad d = 204 \tag{60}$$

$$O_3, O_5, O_7, (O_4, O_8), (O_1, O_2), O_6, \qquad d = 204 \tag{61}$$

There is no ranking for which $d(P, P^{(k)}) = \widetilde{H}$. Hence the rankings (60) and (61) are medians because their distance from the lower bound is minimum.

4 Conclusions

Taking into account ties in the group ranking is essential for practical applications of the Kemeny median method. The notion of lower bound of the distance H is extended for the case of ties $\widetilde{H}$.

An important topic element of the procedure proposed is evaluation of the distance of solutions obtained. In some cases, even when the loss matrix does not possess the Condorcet property, the approach presented - according to the elimination of Condorcet winners or losers - enables efficient search for median (e.g. brute search over the limited set of alternatives).

Together with the method presented in ([1]) the procedure considered in the paper provides an heuristic tool for determining the Kemeny median without the necessity of application of sophisticated numerical procedures of integer programming.

References

[1] Bury H., Wagner D. (2011) Efektywne metody wyznaczania oceny grupowej (Effective solution of group ranking), in Polish, referat na konferencję "Modelowanie Preferencji a Ryzyko'11", Ustroń, 3-5 kwietnia 2011.

[2] Kemeny J.G., Snell L.J., (1962), Preference Ranking: An Axiomatic Approach. In J.G. Kemeny and L.J. Snell, Mathematical Models in the Social Sciences, New York, Ginn.

[3] Litvak B.G., (1982), Ekspertnaja informacija. Mietody połuczienija i analiza (in Russian), Radio i Swjaz, Moskwa.

The papers presented in this Volume 2 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.

It may be viewed as a result of fruitful discussions held during the Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) organized in Warsaw on September 30, 2011 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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