SYSTEMS RESEARCH INSTITUTE
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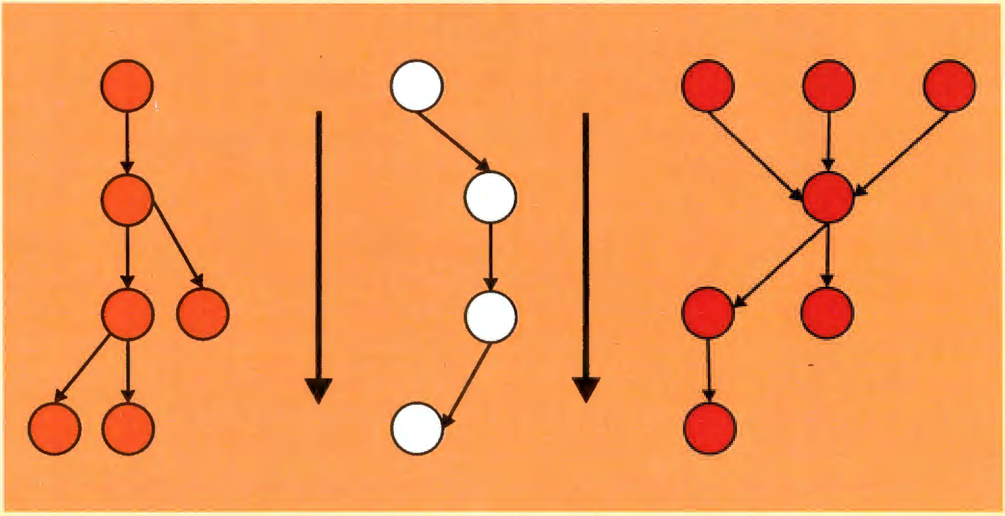
MULTICRITERIA ORDERING AND RANKING:
PARTIAL ORDERS, AMBIGUITIES AND APPLIED ISSUES



Jan W. Owsiński and Rainer Brüggemann
Editors

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This book is the outcome of the international workshop held in Warsaw in October 2008 within the premises of the Systems Research Institute. All papers were refereed and underwent appropriate modification in order to appear in the volume. The views contained in the papers are, however, not necessarily those officially held by the respective institutions involved, especially the Systems Research Institute of the Polish Academy of Sciences.

ISBN 83-894-7521-9
EAN 9788389475213

Technical editing and typesetting:
Jan W.Owsiński, Anna Gostyńska, Aneta M.Pielak

Theoretical Developments

Multicriteria Ordering and Ranking: Partial Orders, Ambiguities and Applied Issues
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Discriminating Fuzzy Preference Relations Based on Heuristic Possibilistic Clustering

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The paper deals in a preliminary way with the problem of the selection of a unique fuzzy preference relation from the set of fuzzy preference relations. A direct algorithm of possibilistic clustering is the basis of the method of fuzzy preference relation discriminating. A method of decision-making based on a fuzzy preference relation is described and basic concepts of the heuristic method of possibilistic clustering are considered. An illustrative example is given and some preliminary conclusions are made,

Keywords: fuzzy preference relation, alternative, fuzzy tolerance, clustering, allotment

1. Introduction

The first subsection of this introduction provides some preliminary remarks. A method of decision-making with a fuzzy preference relation is presented in the second subsection.

1.1. Fuzzy orderings and group decision theory

Fuzzy orderings were introduced by Zadeh (1971). Fuzzy orderings have been applied effectively in information processing, decision-making and control. In general, a fuzzy ordering is a transitive or negatively transitive fuzzy binary relation. Fuzzy preference relations can be considered as a case of fuzzy orderings. The fact was demonstrated by Ovchinnikov (1991).

A fuzzy subset of most appropriate alternatives corresponds to a fuzzy preference relation and a problem of choosing of a unique alternative is investigated by different researchers. However, a set of fuzzy preference relations can be obtained from several decision makers. A fuzzy subset of most appropriate alternatives corresponds to each fuzzy preference relation in this case and a unique alternative must be selected from the set of fuzzy subsets. So, a unique fuzzy preference relation must be obtained and the construction of appropriate decision is a fundamental problem in group decision theory.

A few approaches can be used for the purpose. In the first place, the construction of an appropriate consensus measure is very useful approach, as considered by Fedrizzi et al. (1999). A similar approach was developed by Kuzmin (1982). On the other hand, a unique fuzzy preference relation can be selected from a set of fuzzy preference relations which were proposed by members of a team. The problem was outlined by Nojiri (1979).

The main goal of the present paper is to consider the method of choosing a subset of most appropriate weak fuzzy preference relations from the set of all weak fuzzy preference relations. A direct algorithm of possibilistic clustering constitutes the basis of the method. For this purpose, a short consideration of a method of the construction of a set of non-dominated alternatives based on a fuzzy preference relation is presented and basic concepts of a heuristic method of possibilistic clustering are considered. A method of data preprocessing is described and a general plan of the procedure of determining a unique weak fuzzy preference relation is proposed. An illustrative example is presented and results obtained from the proposed method are compared with the results obtained from the Orlovsky's method of decision-making. Some conclusions are formulated.

1.2. Fuzzy preference relations and non-dominated alternatives

An effective method of decision-making based on a fuzzy preference relation was proposed by Orlovsky (1978). A unique alternative can be selected from the set of alternatives using this method. Moreover, the method is used in the proposed approach. That is why the method must be outlined in the first place.

Let $X = \{x_1, \ldots, x_n\}$ be a finite set of alternatives and $R : X \times X \to [0,1]$ some binary fuzzy relation on X with $\mu_R(x_i, x_j), \forall x_i, x_j \in X$ being its membership function. The weak fuzzy preference relation is the fuzzy binary relation which possesses the reflexivity property

$$\mu_R(x_i, x_i) = 1, \forall x_i \in X \,. \qquad (1)$$

Let us consider the strong fuzzy preference relation concept. The strong fuzzy preference relation is used for the construction of a fuzzy set of non-dominated alternatives. The strong fuzzy preference relation P, which corresponds to the weak fuzzy preference relation R, is the binary fuzzy relation on X with the membership function $\mu_P(x_i,x_j), \forall x_i,x_j \in X$ defined as

$$\mu_P(x_i,x_j)=\begin{cases}\mu_R(x_i,x_j)-\mu_R(x_j,x_i), & if \quad \mu_R(x_i,x_j)>\mu_R(x_j,x_i)\\ 0, & if \quad \mu_R(x_i,x_j)\le\mu_R(x_j,x_i)\end{cases}. \tag{2}$$

The fuzzy set $\widetilde{R}$ of non-dominated alternatives is defined as follows:

$$\mu_{\widetilde{R}}(x_i)=1-\sup_{x_j\in X}\mu_P(x_j,x_i), \forall x_i\in X. \tag{3}$$

So, alternatives $x_i \in X$ with maximal values of the membership function $\max\limits_{x_i\in X}\mu_{\widetilde{R}}(x_i)$ are most appropriate alternatives.

2. Outline of the approach

The basic concepts of the heuristic method of possibilistic clustering, based on the allotment concept are considered first. The problem of data preprocessing is then considered in the second subsection. The problem of selection of the unique fuzzy preference relation is discussed in the third subsection.

2.1. Basic ideas of the clustering method

An outline for a new heuristic method of fuzzy clustering was presented by Viattchenin (2004), where concepts of fuzzy α-cluster and allotment among fuzzy α-clusters were introduced and a basic version of direct fuzzy clustering algorithm was described. The basic version of the algorithm is called D-AFC(c)-algorithm and this algorithm requires that the number c of fuzzy α-clusters be fixed. The allotment of elements of the set of classified objects among fuzzy clusters can be considered as a special case of possibilistic partition. That is why the D-AFC(c)-algorithm can be considered as a direct algorithm of possibilistic clustering. The fact was demonstrated by Viattchenin (2007).

Let us remind the basic concepts of the fuzzy clustering method based on the concept of allotment among fuzzy clusters, which was proposed by Viattchenin (2004). The notion of fuzzy tolerance constituted the basis for the concept of fuzzy

α-cluster. That is why definition of fuzzy tolerance must be considered in the first place.

Let $X = \{x_1,...,x_n\}$ be the initial set of elements and $T : X \times X \to [0,1]$ some binary fuzzy relation on $X = \{x_1,...,x_n\}$ with $\mu_T(x_i, x_j) \in [0,1], \forall x_i, x_j \in X$ being its membership function.

Definition 2.1. *Fuzzy tolerance is the fuzzy binary intransitive relation which possesses the symmetricity property*

$$\mu_T(x_i, x_j) = \mu_T(x_j, x_i), \forall x_i, x_j \in X, \tag{4}$$

and the reflexivity property (1).

The notions of powerful fuzzy tolerance, feeble fuzzy tolerance and strict feeble fuzzy tolerance were considered by Viattchenin (2004), as well. In this context the classical fuzzy tolerance in the sense of Definition 2.1 was called usual fuzzy tolerance and this kind of fuzzy tolerance was denoted by T_2.

The kind of the fuzzy tolerance imposed determines the nature of the revealed structure of data. So, the notions of powerful fuzzy tolerance, feeble fuzzy tolerance and strict feeble fuzzy tolerance must be considered, as well.

Definition 2.2. *The feeble fuzzy tolerance is the fuzzy binary intransitive relation which possesses the symmetricity property (4) and the feeble reflexivity property*

$$\mu_T(x_i, x_j) \leq \mu_T(x_i, x_i), \forall x_i, x_j \in X. \tag{5}$$

This kind of fuzzy tolerance is denoted by T_1.

Definition 2.3. *The strict feeble fuzzy tolerance is the feeble fuzzy tolerance with strict inequality in (5):*

$$\mu_T(x_i, x_j) < \mu_T(x_i, x_i), \forall x_i, x_j \in X. \tag{6}$$

This kind of fuzzy tolerance is denoted by T_0.

Definition 2.4. *The powerful fuzzy tolerance is the fuzzy binary intransitive relation which possesses the symmetricity property (4) and the powerful reflexivity* *property. The powerful reflexivity property is defined as the condition of reflexivity (1) together with the condition*

$$\mu_T(x_i, x_j) < 1, \forall x_i, x_j \in X, x_i \neq x_j. \quad (7)$$

This kind of fuzzy tolerance is denoted by T_3.

Notable that fuzzy tolerances T_1 and T_0 are subnormal binary fuzzy relations if the condition

$$\mu_T(x_i, x_i) < 1, \forall x_i \in X \quad (8)$$

is met. However, the essence of the method here considered does not depend on the kind of fuzzy tolerance. That is why the method herein is described for any fuzzy tolerance T.

Let us consider the general definition of fuzzy cluster, the concept of the fuzzy cluster's typical point and the concept of the fuzzy allotment of objects. The number c of fuzzy clusters can be equal to the number of objects, n. This is taken into account in further considerations.

Let $X = \{x_1, ..., x_n\}$ be the initial set of objects. Let T be a fuzzy tolerance on X and α be α-level value of T, $\alpha \in (0,1]$. Columns or lines of the fuzzy tolerance matrix are fuzzy sets $\{A^1, ..., A^n\}$. Let $\{A^1, ..., A^n\}$ be fuzzy sets on X, which are generated by some fuzzy tolerance T.

Definition 2.5. *The α-level fuzzy set $A^l_{(\alpha)} = \{(x_i, \mu_{A^l}(x_i)) \mid \mu_{A^l}(x_i) \geq \alpha\}$ is fuzzy α-cluster. So, $A^l_{(\alpha)} \subseteq A^l, \alpha \in (0,1], A^l \in \{A^1, \ldots, A^n\}$ and μ_{li} is the membership degree of the element $x_i \in X$ for some fuzzy cluster $A^l_{(\alpha)}, \alpha \in (0,1], l \in \{1, \ldots, n\}$. Value of α is the tolerance threshold of fuzzy clusters elements.*

The membership degree of the element $x_i \in X$ for some fuzzy cluster $A^l_{(\alpha)}, \alpha \in (0,1], l \in \{1, \ldots, n\}$ can be defined as a

$$\mu_{li} = \begin{cases} \mu_{A^l}(x_i), & x_i \in A_\alpha^l \\ 0, & otherwise \end{cases}, \tag{9}$$

where an α-level $A_\alpha^l = \{x_i \in X \mid \mu_{A^l}(x_i) \geq \alpha\}$ of a fuzzy set A^l is the support of the fuzzy cluster $A_{(\alpha)}^l$. So, the α-level A_α^l of a fuzzy set A^l is a crisp set and condition $A_\alpha^l = Supp(A_{(\alpha)}^l)$ is met for each fuzzy cluster $A_{(\alpha)}^l$.

Membership degree can be interpreted as a *degree of typicality* of an element with respect to a fuzzy cluster. The value of a membership function of each element of the fuzzy cluster in the sense of Definition 2.5 is the degree of similarity of the object to some typical object of fuzzy cluster.

In other words, if columns or lines of a fuzzy tolerance T matrix are fuzzy sets $\{A^1,...,A^n\}$ on X then fuzzy clusters $\{A_{(\alpha)}^1,...,A_{(\alpha)}^n\}$ are fuzzy subsets of fuzzy sets $\{A^1,...,A^n\}$ for some value $\alpha, \alpha \in (0,1]$. The value zero for a fuzzy set membership function means that an element does not belong to a fuzzy set. That is why values of tolerance threshold α are considered in the interval $(0,1]$.

Definition 2.6. *If T is a fuzzy tolerance on X, where X is the set of elements, and $\{A_{(\alpha)}^1,...,A_{(\alpha)}^n\}$ is the family of fuzzy clusters for some $\alpha \in (0,1]$, then the point $\tau_e^l \in A_\alpha^l$, for which*

$$\tau_e^l = \arg\max_{x_i} \mu_{li}, \forall x_i \in A_\alpha^l \tag{10}$$

is called typical point of the fuzzy cluster $A_{(\alpha)}^l$, $\alpha \in (0,1]$, $l \in \{1,\ldots,n\}$.

Obviously, a typical point of a fuzzy cluster does not depend on the value of tolerance threshold. Moreover, a fuzzy cluster can have several typical points. That is why symbol e is the index of the typical point.

Definition 2.7. *Let $R_z^\alpha(X) = \{A_{(\alpha)}^l \mid l = \overline{1,c}, 2 \leq c \leq n, \alpha \in (0,1]\}$ be a family of fuzzy clusters for some value of tolerance threshold $\alpha, \alpha \in (0,1]$, which are* *generated by some fuzzy tolerance* T *on the initial set of elements* $X = \{x_1,...,x_n\}$. *If condition*

$$\sum_{l=1}^{c} \mu_{li} > 0, \forall x_i \in X \tag{11}$$

is met for all fuzzy clusters $A^l_{(\alpha)}, l = \overline{1,c}, c \le n$, *then the family is the allotment of elements of the set* $X = \{x_1,...,x_n\}$ *among fuzzy clusters* $\{A^l_{(\alpha)} \mid l = \overline{1,c}, 2 \le c \le n\}$ *for some value of the tolerance threshold* $\alpha, \alpha \in (0,1]$.

It should be noted that several allotments $R^{\alpha}_{z}(X)$ can exist for some tolerance threshold $\alpha, \alpha \in (0,1]$. That is why symbol z is the index of an allotment.

The condition (11) requires that every object $x_i, i = 1,\ldots,n$, be assigned to at least one fuzzy cluster $A^l_{(\alpha)}, l = 1,\ldots,c, c \le n$ with the membership degree higher than zero. The condition $2 \le c \le n$ requires that the number of fuzzy clusters in $R^{\alpha}_{z}(X)$ be at least two. Otherwise, the unique fuzzy cluster will contain all objects, possibly with different positive membership degrees.

The concept of allotment is the central point of the method. But the next concept introduced should be paid attention to, as well.

Definition 2.8. *Allotment* $R^{\alpha}_{I}(X) = \{A^l_{(\alpha)} \mid l = \overline{1,n}, \alpha \in (0,1]\}$ *of the set of objects among* n *fuzzy clusters for some tolerance threshold* $\alpha, \alpha \in (0,1]$ *is the initial allotment of the set* $X = \{x_1,...,x_n\}$.

In other words, if initial data are represented by a matrix of some fuzzy T, then lines or columns of the matrix are fuzzy sets $A^l \subseteq X, l = \overline{1,n}$ and level fuzzy sets $A^l_{(\alpha)}, l = \overline{1,n}, \alpha \in (0,1]$ are fuzzy clusters. These fuzzy clusters constitute an initial allotment for some tolerance threshold and they can be considered as clustering components.

Thus, the problem of fuzzy cluster analysis can be defined in general as the problem of discovering the unique allotment $R^{*}(X)$, resulting from the classification process, which corresponds to either most natural allocation of objects among fuzzy clusters or to the researcher's opinion about classification. In the first case, the number of fuzzy clusters c is not fixed. In the second case, the researcher's opinion determines the kind of the allotment sought and the number of fuzzy clusters c can be fixed.

If some allotment $R_z^{\alpha}(X) = \{A_{(\alpha)}^l \mid l = \overline{1,c}, c \le n, \alpha \in (0,1]\}$ corresponds to the formulation of a concrete problem, then this allotment is an adequate allotment. In particular, if conditions

$$\bigcup_{l=1}^{c} A_{\alpha}^{l} = X, \tag{12}$$

and

$$card(A_{\alpha}^{l} \cap A_{\alpha}^{m}) = 0, \forall A_{(\alpha)}^{l}, A_{(\alpha)}^{m}, l \ne m, \alpha \in (0,1] \tag{13}$$

are met for all the fuzzy clusters $A_{(\alpha)}^l, l = \overline{1,c}$ of some allotment $R_z^{\alpha}(X) = \{A_{(\alpha)}^l \mid l = \overline{1,c}, c \le n, \alpha \in (0,1]\}$, then the allotment is the allotment among fully separate fuzzy clusters.

Fuzzy clusters in the sense of Definition 2.5 can have a non-empty intersection, as shown in Viattchenin (2007). If the intersection area of any pair of different fuzzy cluster is an empty set, then the condition (13) is met and fuzzy clusters are called fully separate fuzzy clusters. Otherwise, fuzzy clusters are called particularly separate fuzzy clusters and $w = \{0, \ldots, n\}$ is the maximum number of elements in the intersection area of different fuzzy clusters. Obviously, for $w = 0$ fuzzy clusters are fully separate fuzzy clusters. The conditions (12) and (13) can be generalized for a case of particularly separate fuzzy clusters. Conditions

$$\sum_{l=1}^{c} card(A_{\alpha}^{l}) \ge card(X), \forall A_{(\alpha)}^{l} \in R_z^{\alpha}(X), \alpha \in (0,1], card(R_z^{\alpha}(X)) = c, \tag{14}$$

and

$$card(A_{\alpha}^{l} \cap A_{\alpha}^{m}) \le w, \forall A_{(\alpha)}^{l}, A_{(\alpha)}^{m}, l \ne m, \alpha \in (0,1], \tag{15}$$

are generalizations of conditions (12) and (13). Obviously, if $w = 0$ in conditions (14) and (15), then conditions (12) and (13) are met.

The adequate allotment $R_z^\alpha(X)$ for some value of tolerance threshold $\alpha, \alpha \in (0,1]$ is a family of fuzzy clusters which are elements of the initial allotment $R_I^\alpha(X)$ for the value of α, and the family of fuzzy clusters should satisfy the conditions (14) and (15). Hence, construction of adequate allotments $R_z^\alpha(X) = \{A_{(\alpha)}^l \mid l = \overline{1,c}, c \le n\}$ for every α constitutes a trivial problem of combinatorics.

Several adequate allotments can exist. Thus, the problem consists in the selection of the unique adequate allotment $R^*(X)$ from the set B of adequate allotments, $B = \{R_z^\alpha(X)\}$, which is the class of possible solutions of the concrete classification problem and $B = \{R_z^\alpha(X)\}$ depends on the parameters of the classification problem. The selection of the unique adequate allotment $R^*(X)$ from the set $B = \{R_z^\alpha(X)\}$ of adequate allotments must be made on the basis of evaluation of allotments. The criterion

$$F_1(R_z^\alpha(X),\alpha) = \sum_{l=1}^{c} \frac{1}{n_l} \sum_{i=1}^{n_l} \mu_{li} - \alpha \cdot c, \tag{16}$$

where c is the number of fuzzy clusters in the allotment $R_z^\alpha(X)$ and $n_l = card(A_\alpha^l)$, $A_{(\alpha)}^l \in R_z^\alpha(X)$ is the number of elements in the support of the fuzzy cluster $A_{(\alpha)}^l$, can be used for evaluation of allotments. The criterion

$$F_2(R_z^\alpha(X),\alpha) = \sum_{l=1}^{c} \sum_{i=1}^{n_l} (\mu_{li} - \alpha), \tag{17}$$

can also be used for evaluation of allotments. Both criteria were considered by Viattchenin (2007).

The maxima of criteria (10) or (11) correspond to the best allotments of objects among c fuzzy clusters. So, the classification problem can be characterized formally as determination of a solution $R^{*}(X)$ satisfying

$$R^{*}(X) = \arg \max_{R_{z}^{\alpha}(X) \in B} F(R_{z}^{\alpha}(X), \alpha), \tag{18}$$

where $B = \{R_{z}^{\alpha}(X)\}$ is the set of adequate allotments corresponding to the formulation of a concrete classification problem and criteria (16) and (17) are denoted by $F(R_{z}^{\alpha}(X), \alpha)$.

The condition (18) must be met for the some unique allotment $R_{z}^{\alpha}(X) \in B(c)$. Otherwise, the number c of fuzzy clusters in the allotment sought $R^{*}(X)$ is suboptimal. Identification of c, the number of fuzzy clusters, can be considered as the aim of classification. The clustering procedure is considered, for instance, by Viattchenin (2007).

2.2. A technique of data preprocessing

Let $X = \{x_1, \ldots, x_n\}$ be a finite set of alternatives and $\{R^1, \ldots, R^g\}$ a finite set of weak fuzzy preference relations on X. After application of distance

$$d(R^k, R^f) = \sum_{(x_i, x_j)} \left| \mu_{R^k}(x_i, x_j) - \mu_{R^f}(x_i, x_j) \right|, k, f = 1, \ldots, g, \tag{19}$$

to the set $\{R^1, \ldots, R^g\}$ of weak fuzzy preference relations a matrix of pair-wise dissimilarity coefficients $D_{g \times g} = [d(R^k, R^f)]$, $k, f = 1, \ldots, g$ is obtained. The distance (19) was proposed by Kuzmin (1982). The matrix $D_{g \times g} = [d(R^k, R^f)]$ can be normalized as follows:

$$\mu_I(R^k, R^f) = \frac{d(R^k, R^f)}{\max_{k,f} d(R^k, R^f)}, \tag{20}$$

for all $k,f=1,\ldots,g$. Actually, the matrix of normalized pair-wise dissimilarity coefficients $I=[\mu_I(R^k,R^f)]$, $k,f=1,\ldots,g$ is the matrix of fuzzy intolerance relation on the set of weak fuzzy preference relations $\{R^1,\ldots,R^g\}$. A matrix of fuzzy tolerance $T=[\mu_T(R^k,R^f)]$, $k,f=1,\ldots,g$ is obtained after application of the complement operation

$$\mu_T(R^k,R^f)=1-\mu_I(R^k,R^f), \forall k,f=1,\ldots,g, \tag{21}$$

to the matrix of fuzzy intolerance $I=[\mu_I(R^k,R^f)]$. The D-AFC(c)-algorithm can be applied directly to the matrix of fuzzy tolerance $T=[\mu_T(R^k,R^f)]$.

2.3. A methodology of choosing a unique fuzzy preference relation

The D-AFC(c)-algorithm can be applied to solve the problem of selection of appropriate weak fuzzy preference relation. The basic idea of the approach is that weak fuzzy preference relations can be classified and a typical point of each fuzzy cluster can be considered as an element of a subset of appropriate weak fuzzy preference relations. So, a methodology of solving the problem of selection of most appropriate alternative can be described as follows:

1. The matrix of fuzzy tolerance correlation must be constructed for weak fuzzy preference relations;
2. The D-AFC(c)-algorithm can be applied directly to the matrix of the fuzzy tolerance relation for a given number c of classes;
3. Typical points of fuzzy clusters of the received allotment $R^*(X)$ can be selected as elements of a subset of appropriate weak fuzzy preference relations;
4. Elements of the subset of appropriate weak fuzzy preference relations must be ordered on the basis of the evaluation and appropriate weak fuzzy preference relation must be selected.

3. An illustrative example

We provide, first, the description of the data set. The results of processing of these data by the D-AFC(c)-algorithm are presented in the second subsection and the results are compared with the results obtained from application of the Orlovsky's method to weak fuzzy preference relations.

3.1. The data

Let $X = \{x_1, x_2, x_3, x_4\}$ be the finite set of alternatives under consideration. Let $\{R^1, R^2, R^3, R^4, R^5, R^6\}$ be a set of weak fuzzy preference relations on X. The set of relations is presented below.

R^1	x_1	x_2	x_3	x_4
x_1	1.0	0.2	0.3	0.1
x_2	0.5	1.0	0.2	0.6
x_3	0.1	0.6	1.0	0.3
x_4	0.6	0.1	0.5	1.0

R^2	x_1	x_2	x_3	x_4
x_1	1.0	0.3	0.4	0.2
x_2	0.6	1.0	0.3	0.7
x_3	0.2	0.7	1.0	0.4
x_4	0.7	0.2	0.6	1.0

R^3	x_1	x_2	x_3	x_4
x_1	1.0	0.3	0.4	0.2
x_2	0.5	1.0	0.3	0.7
x_3	0.1	0.6	1.0	0.4
x_4	0.6	0.1	0.5	1.0

R^4	x_1	x_2	x_3	x_4
x_1	1.0	0.4	0.1	0.3
x_2	0.2	1.0	0.4	0.4
x_3	0.3	0.4	1.0	0.5
x_4	0.4	0.3	0.3	1.0

R^5	x_1	x_2	x_3	x_4
x_1	1.0	0.5	0.2	0.4
x_2	0.1	1.0	0.5	0.5
x_3	0.2	0.3	1.0	0.6
x_4	0.3	0.2	0.2	1.0

R^6	x_1	x_2	x_3	x_4
x_1	1.0	0.4	0.1	0.3
x_2	0.3	1.0	0.4	0.4
x_3	0.4	0.5	1.0	0.5
x_4	0.5	0.4	0.4	1.0

So, six weak fuzzy preference relations form the set of objects for classification.

3.2. The results of classification

The matrix of the fuzzy tolerance relation was constructed for six weak fuzzy preference relations. By executing the D-AFC(c)-algorithm for two classes, we obtain the following: the first class is formed by 3 elements, and the second class is composed of 3 elements too. Membership functions of two classes are presented in Fig. 1.

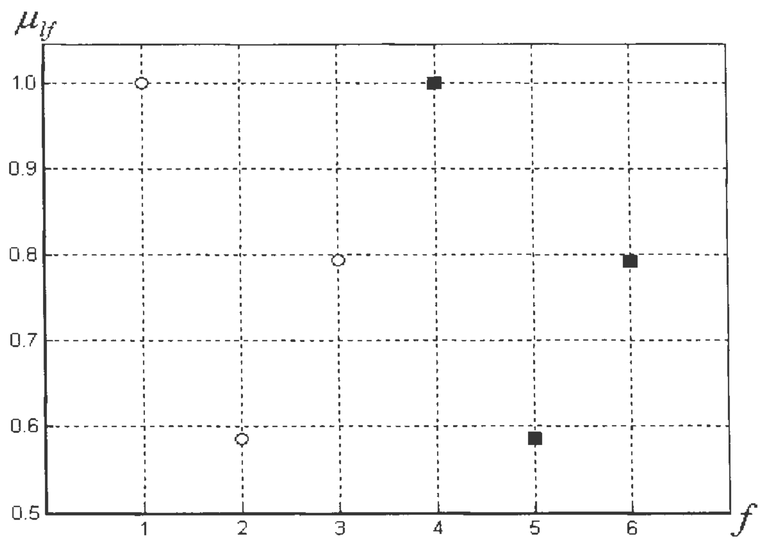

Figure 1. Membership functions of two classes of the allotment

Membership values of the first class are represented in Fig. 1 by ○, membership values of the second class are represented in Fig. 1 by ■.

The value of the membership function of the fuzzy cluster which corresponds to the first class is maximal for the first object and is equal one. So, the first object is the typical point of the fuzzy cluster which corresponds to the first class. The membership value of the fourth object is equal one for the fuzzy cluster which corresponds to the second class. Thus, the fourth object is the typical point of the fuzzy cluster which corresponds to the second class. That is why the set $\{R^1, R^4\}$ is the subset of most appropriate weak fuzzy preference relations. Obviously, the analysis of the set of weak fuzzy preference relations is a simple problem.

A modified linear index of fuzziness can be used for the evaluation of the results of classification. A linear index of fuzziness was defined by Kaufmann (1975) and the index can be modified for binary fuzzy relations as follows

$$\ell(R)=\frac{2}{n^{2}}\sum_{(x_i,x_j)}|\mu_R(x_i,x_j)-\mu_{\underline{R}}(x_i,x_j)|,\ \forall(x_i,x_j)\in X\times X\,, \tag{22}$$

where $n=card(X)$ is the number of alternatives and the crisp binary relation $\underline{R}$ nearest to the fuzzy preference relation R The membership function $\mu_{\underline{R}}(x_i,x_j)$ of the crisp relation $\underline{R}$ can be defined as

$$\mu_{\underline{R}}(x_i,x_j)=\begin{cases}0, & \mu_R(x_i,x_j)\le 0.5\\ 1, & \mu_R(x_i,x_j)>0.5\end{cases},\ \forall(x_i,x_j)\in R\,. \tag{28}$$

A unique fuzzy preference relation R can be selected if the condition $\min\limits_{f}\ell(R^f),\ f=1,2,\ldots,$ is met, where fuzzy preference relations R^f are typical points of fuzzy clusters obtained from the D-AFC(c)-algorithm. For example, $\ell(R^1)=0.425$ and $\ell(R^4)=0.500$. So, the fuzzy preference relation R^1 can be selected as a most appropriate weak fuzzy preference relation from the set $\{R^1,R^2,R^3,R^4,R^5,R^6\}$ of weak fuzzy preference relations on X. A fuzzy set $\widetilde{R}^1$ of non-dominated alternatives can be constructed as follows:

$$\widetilde{R}^1=\big((x_1,0.5),(x_2,0.6),(x_3,0.8),(x_4,0.5)\big). \tag{29}$$

So, the alternative x_3 can be selected as a most appropriate alternative from the set $X=\{x_1,x_2,x_3,x_4\}$ and the membership degree $\mu_{\widetilde{R}^1}(x_3)=0.8$ is a grade of the acceptance of the alternative x_3.

However, the method must be compared with another effective method. Let us consider the results obtained from the Orlovsky's method. The method was applied for each weak fuzzy preference relation from the set $\{R^1,R^2,R^3,R^4,R^5,R^6\}$. So, six strong fuzzy preference relations P^f, $f=1,\ldots,6$, and six fuzzy sets of non-dominated alternatives $\tilde{R}^f, f = 1,\ldots,6$, were constructed. Fuzzy sets of on-dominated alternatives $\tilde{R}^f, f = 1,\ldots,6$ are presented in Table 1.

The result shows that the fuzzy set of non-dominated alternatives $\tilde{R}^6$ is the most appropriate fuzzy set in terms of essential positions. Then, obviously, alternative x_3 can be selected as the most appropriate alternative for fuzzy sets of non-dominated alternatives $\tilde{R}^1$, $\tilde{R}^2$, $\tilde{R}^4$, and $\tilde{R}^6$. The alternative x_1 can be selected as the most appropriate alternative for the fuzzy set of non-dominated alternatives $\tilde{R}^5$. Alternatives x_2 and x_3 can be selected as two most appropriate alternatives for the fuzzy set of non-dominated alternatives $\tilde{R}^3$.

Table 1. Fuzzy sets of non-dominated alternatives

Alternatives	Fuzzy sets of non-dominated alternatives					
	$\tilde{R}^1$	$\tilde{R}^2$	$\tilde{R}^3$	$\tilde{R}^4$	$\tilde{R}^5$	$\tilde{R}^6$
x_1	0.5	0.5	0.6	0.8	1.0	0.7
x_2	0.6	0.6	0.7	0.8	0.6	0.9
x_3	0.8	0.8	0.7	1.0	0.8	1.0
x_4	0.5	0.5	0.4	0.8	0.6	0.9

Let us consider values of the modified linear index of fuzziness (22) for all weak fuzzy preference relations. These values are presented in Table 2.

Table 2. Values of the $\ell(R^f)$ for weak fuzzy preference relations

Fuzzy relations	R^1	R^2	R^3	R^4	R^5	R^6
Values of the $\ell(R^f)$	0.4250	0.4625	0.4875	0.5000	0.4750	0.5750

So, the condition $\min_f \ell(R^f), f = 1,\ldots,6$ is met for the relation R^1 and the result corresponds to the result of the evaluation of weak fuzzy preference relations, which are typical points of fuzzy clusters. Moreover, the modified linear index of fuzziness (22) is seems to be appropriate index for the evaluation of the result of the classification.

4. Final remarks

4.1. Discussion

An appropriate weak fuzzy preference relation was selected on the basis of possibilistic clustering of weak fuzzy preference relations and a unique alternative was selected from the corresponding fuzzy set of non-dominated alternatives. The results were verified. The results obtained with the use of the Orlovsky's method show that the proposed method seems to be correct.

A linear index of fuzziness was modified for purposes of evaluation of the results of classification. Obviously, some other index can be proposed for this purpose. It should be noted that the values of the modified linear index of fuzziness (22) do not correspond to the results, which are presented in Table 1. For example, the fuzzy set $\widetilde{R}^1$ is equal to the fuzzy set $\widetilde{R}^2$, while $\ell(R^1) < \ell(R^2)$. So, equal fuzzy sets of non-dominated alternatives can be obtained from different weak fuzzy preference relations. Moreover, the classification results do not depend on values of the modified linear index of fuzziness (22). For example, the fuzzy relation R^4 is the typical point of the second class and membership function values are ordered as follows $\mu_{24} > \mu_{26} > \mu_{25}$ for elements of the second fuzzy cluster. However, values of the index (22) are ordered as follows $\ell(R^5) < \ell(R^4) < \ell(R^6)$ for fuzzy relations which are elements of corresponding class. These facts must be explained in further investigations.

In general, the proposed method of classification of weak fuzzy preference relations seems to be satisfactory for selecting a subset of weak fuzzy preference relations for the detailed analysis.

4.2. Conclusions

In conclusion it should be said that the concept of fuzzy cluster and allotment have an epistemological motivation. That is why the results of application of the heuristic possibilistic clustering method based on the allotment concept can be very well interpreted. Moreover, the possibilistic clustering method based on the allotment concept depends on the set of adequate allotments only. That is why the clustering results are stable. The results of application of the proposed algorithms to the experimental data set show that the algorithm is a precise and effective numerical procedure for solving the classification problem in group decision process.

Acknowledgements

I would like to thank the anonymous referees and for their constructive reviews and useful remarks. I am grateful to Dr Jan W. Owsinski and Professor Janusz Kacprzyk for their interest in my investigations and support. I also thank Mr Aliaksandr Damaratski for elaborating the experimental software.

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This book is a collection of papers, prepared in connection with the 8th International Workshop on partial orders, their theoretical and applied developments, which took place in Warsaw, at the Systems Research Institute, in October 2008. The papers deal with software developments (PYHASSE and other existing software), theoretical problems of ranking and ordering under various assumed analytic and decision-making-oriented conditions, as well as experimental studies and down-to-earth pragmatic questions.

ISBN 83-894-7521-9
EAN 9788389475213