SYSTEMS RESEARCH INSTITUTE
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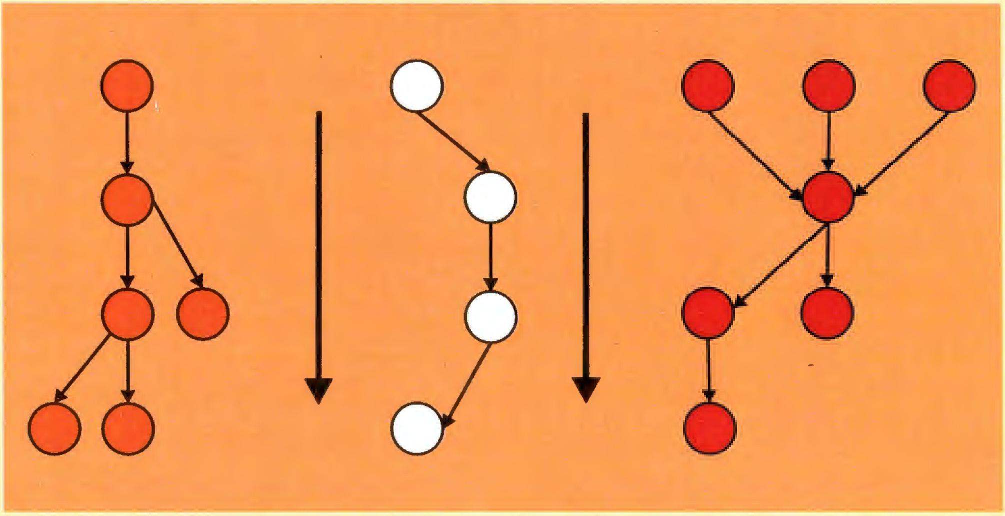
MULTICRITERIA ORDERING AND RANKING:
PARTIAL ORDERS, AMBIGUITIES AND APPLIED ISSUES



Jan W. Owsiński and Rainer Brüggemann
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This book is the outcome of the international workshop held in Warsaw in October 2008 within the premises of the Systems Research Institute. All papers were refereed and underwent appropriate modification in order to appear in the volume. The views contained in the papers are, however, not necessarily those officially held by the respective institutions involved, especially the Systems Research Institute of the Polish Academy of Sciences.

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Theoretical Developments

An Upper Bound on the Number of Rankings Satisfying Order Preferences

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We provide an upper bound on the number of rankings satisfying given preferences in the form of (partial) order. An application of this bound to ranking problems is briefly presented.

Keywords: order preferences, number of rankings, upper bounds

1. Introduction

The underlying task of any decision problem is to provide a linear order (below: *ranking*) of objects on the base of some preferences. Here we assume that preferences have the form of a partial order (below: *order*).

A ranking in a set is *consistent* with an order in that set if the order is a subset of the ranking.

We are concerned with the following question:

given order $\succ$ in finite set G, how many rankings of G consistent with $\succ$ do exist?

We provide a result which establish an upper bound on the number of rankings consistent with given order.

In Section 2 we present the result (Lemma 1 and Lemma 2), whereas in Section 3 we outline its application in the context of object ranking problems.

2. An upper bound

Lemma 2 (and Lemma 1 as a special case) provides an upper bound on the number of rankings of G consistent with a given order in G.

The number of rankings of G consistent with order $\succ$ in G is clearly less than or equal to the number of rankings of G consistent with a subset of $\succ$. Lemma 1 provides the number of rankings of G for the case a subset of $\succ$ is a ranking.

Given finite set G, $|G| = g$ and $|H| = h$, where $H \subseteq G$. Assume that a ranking of H is given.

Lemma 1. *The number of rankings of G when a ranking of H is given is equal to* $\frac{g!}{h!}$.

Proof In $g!$ rankings (permutations) of g elements of G there are $h!$ rankings (permutations) of h elements of H.

If in $g!$ rankings of G each ranking of H is replaced by a given ranking, then the number of rankings of G is reduced $h!$ times. Hence, the number of rankings of G when a ranking of H is given is equal to $\frac{g!}{h!}$. □

By the same token as in the case of Lemma 1, the number of rankings of G consistent with order $\succ$ in G is clearly less than or equal to the number of rankings of G consistent with disjoint subsets of $\succ$. Lemma 2 provides the number of rankings of G for the case each disjoint subset of $\succ$ is a ranking.

Let $H^1, ..., H^t$ be disjoint subsets of G, i.e. $H^i \cap H^j = \emptyset$, $i, j = 1, ..., t$, $i \neq j$, and $|H^i| = h^i$, $i = 1, 2, ..., t$. Assume that rankings of H^i, $i = 1, ..., t$, are given.

Lemma 2. *The number of rankings of G when rankings of* $H^1, ..., H^t$ *are given is equal to* $\frac{g!}{h^1! \cdot ... \cdot h^t!}$.

Proof For $t = 1$ the proof follows from Lemma 1.

Assume $t = 2$. Because ranking of H^1 is given, by Lemma 1 the number of rankings of G is equal to $\frac{g!}{h^1!}$.

If in $\frac{g!}{h^1!}$ rankings of G each ranking of H^2 is replaced by a given ranking, then the number of rankings of G is reduced $h^2!$ times. Hence, the number of rankings of G when rankings of H^1 and H^2 are given is equal to $\frac{g!}{h^1! \cdot h^2!}$.

The argument can be continued for any t. □

3. An application

The number of rankings consistent with given order $\succ$ is of practical interest in ranking problems. A ranking problem is understood here as a processes of constructing a ranking algorithm, which accounts for the the decision maker partial preferences with respect to the resulting rankings.

The decision maker can provide his partial preferences with respect to objects in a holistic manner or atomistic (parametric) manner.

In the holistic manner the decision maker defines his preferences with respect to objects directly.

For example, assume that in the set of five objects A, B, C, D, E, the decision maker prefers A to B and also prefers C to D. By assumption, those preferences are an order. By Lemma 2 the number of rankings consistent with these partial preferences is bounded by $\frac{5!}{2!2!} = 30$. By enumeration the number of consistent rankings is 30.

In an atomistic manner the decision maker defines his preferences with respect to objects indirectly via values of a set of $k \geq 2$ attributes.

Once attributes are selected, the dominance relation $\gg$ (i.e. $y \gg y' \Leftrightarrow y_i \geq y_i',\ i = 1, ..., k$, and $y_i > y_i'$ for some i) in the set of vectors of attribute values can provide some more rankings of subsets of objects, which can further constrain the number of consistent rankings. The assumption is that the dominance relation does not contradict the preferences defined with respect to objects.

To continue our example, assume that the five objects have two attributes with values

$$\begin{matrix} A & B & C & D & E \\ \begin{pmatrix} 5 \\ 4 \end{pmatrix} & \begin{pmatrix} 4 \\ 3 \end{pmatrix} & \begin{pmatrix} 7 \\ 3 \end{pmatrix} & \begin{pmatrix} 4 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{matrix}$$

If attributes are of the type "the more the better" A dominates B, C dominates D and D dominates E, and these relations provide rankings of disjoint subsets $\{A, B\}$ and $\{C, D, E\}$. By Lemma 2 the number of rankings consistent with these partial preferences is bounded by $\frac{5!}{2!3!} = 10$. By enumeration the number of consistent rankings is 2.

The decision maker can further provide his preferences with respect to vectors of attribute values under the assumption that those preferences contradict neither the preferences defined with respect to objects nor the dominance relation.

To continue our example, assume that after evaluating attributes the decision maker prefers A over C. With rankings of disjoint subsets $\{B\}$, $\{A, C, D, E\}$, by Lemma 2 the number of rankings consistent with these partial preferences is bounded by $\frac{5!}{1!4!} = 5$. By enumeration the number of consistent rankings is 1.

Given decision maker preferences in the form of an order, different disjoint sets with rankings produce different bounds on the number of rankings consistent with these preferences. In our example with rankings of disjoint subsets $\{A, B\}$, $\{C, D, E\}$, by Lemma 2 the number of rankings consistent with these partial preferences is bounded by $\frac{5!}{2!3!} = 10$. The rule for selecting disjoint subsets with rankings to get lowest bound is the greedy one: select the largest possible subset with ranking, then repeat the procedure with the remaining subset. Such a procedure ensures the greatest possible value of the denominator in the formula of Lemma 2, which is the consequence of the following lemma.

Lemma 3. *Let* $2 \leq h^1 \leq h^2$. *Then*

$$h^1! \cdot h^2! < (h^1 - 1)! \cdot (h^2 + 1)!$$

Proof By assumption we have

$$h^1! \cdot h^2! < h^1! \cdot h^2! \cdot \frac{h^2 + 1}{h^1} = (h^1 - 1)! \cdot (h^2 + 1)!$$

□

In another paper (Chmielewski, Kaliszewski 2008) we have proposed an interactive scheme to build ranking algorithms, in which the decision maker is encouraged and stimulated to provide his partial preferences with respect to objects directly and also indirectly via attributes, in the form of orders. The results presented here can be used to approximate the number of rankings, with a lower bounds obtained by an enumeration procedure. The gap between lower and upper bounds can be interpreted as a measure of arbitrariness when selecting a ranking algorithm with the partial preferences defined up to the current stage of the decision process.

References

Chmielewski M. (2008) Extrapolations of Partial Orders in the Problem of Rankings. PhD Thesis (in Polish), Systems Research Institute, Warsaw.

Chmielewski M. Kaliszewski I. (2008) Multiple Criteria Ranking Decision Support. Systems Research Institute Report RB/2/2008.

This book is a collection of papers, prepared in connection with the 8th International Workshop on partial orders, their theoretical and applied developments, which took place in Warsaw, at the Systems Research Institute, in October 2008. The papers deal with software developments (PYHASSE and other existing software), theoretical problems of ranking and ordering under various assumed analytic and decision-making-oriented conditions, as well as experimental studies and down-to-earth pragmatic questions.

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