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FUZZY TOURNAMENTS AND THEIR SOLUTION CONCEPTS IN GROUP DECISION MAKING

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Abstract. Starting from individual (or collective) fuzzy tournaments as preference representations, we define new solution concepts in group decision making which are mostly counterparts of those derived for nonfuzzy tournaments as, e.g., sets of undominated options, uncovered sets, and the Banks set; nonfuzzy and fuzzy majorities are considered.

Keywords: group decision making, social choice, tournament, fuzzy tournament, solution concept, fuzzy majority.

1. INTRODUCTION

Suppose we have a finite set of alternatives (options) $X = \{x_1,\dots,x_n\}$ and a finite set of individuals $N = \{1,\dots,m\}$. Each individual k provides his or her (connected and transitive) preference relation on X, P_k. The problem is to find a solution which is a subset $S \subseteq X$ of alternatives which are 'most acceptable' by N as a whole. S should satisfy some plausible criteria whose diversity leads to numerous solution concepts. Unfortunately, there is no 'ideal' solution [cf. Nurmi (1987, 1988a) for an analysis].

In the derivation of a solution there are two basic approaches:

- a direct one, i.e. $(P_1,\dots,P_m) \longrightarrow$ solution
- an indirect one, i.e. $(P_1,\dots,P_m) \longrightarrow P \longrightarrow$ solution

that is, directly from the indivudual preference relations, or by constructing first a collective preference relation, P.

A fruitful point of departure is to start with individual or collective tournaments, i.e. complete and assymetric relations on X; see Section 2 for many interesting tournament - based solutions.

In practice the human preferences are seldom clear - cut, hence fuzzy preference relations are often advocated. A *fuzzy preference relation* of individual k, R_k, is a fuzzy set in X × W whose membership function is $\mu_{R_k}: X \times X \longrightarrow [0, 1]$; $\mu_{R_k}(x_i, x_j)$ denotes the degree (strength) of preference of alternative x_i over x_j as felt by individual k, from 0 for full preference of x_i over x_j to 1 for full preference of x_j over x_i through all intermediate values (0.5 for indifference). If card X is small, R_k may be represented in matrix form $R_k = [r^k_{ij}]$; $r^k_{ij} = \mu_{R_k}(x_i, x_j)$. In general, it is assumed that $r^k_{ii} = 0$, and $r^k_{ij} + r^k_{ji} = 1$.

The fuzzy preference relations have been widely used - see, e.g., Blin (1974), Blin and Whinston (1973), Bezdek, Spillman and Spillman (1977, 1978, 1979), Kacprzyk (1984, 1985a, c, 1986a, 1987), Kacprzyk and Fedrizzi (1986, 1988), Kuzmin (1982), Kuzmin and Ovchinnikov (1980a, b), Montero and Tejada (1986), Nurmi (1981), Tanino (1984, 1988), etc. An extensive account of this topic is Kacprzyk and Roubens (1988). From the viewpoint of this paper, i.e. the derivation of a solution, Kacprzyk (1984, 1985a, c, 1986a) and Nurmi (1981) are the most relavant.

Here we start from fuzzy individual and collective tournaments, and propose some counterparts of the solution concepts proposed in the case of nonfuzzy tournaments.

1. SOLUTIONS IN THE CASE OF NONFUZZY TOURNAMENTS

A *tournament* P (or P_k if it concerns individual k) is a complete (either $x_i P x_j$ or $x_j P x_i$) and assymetric ($x_i P x_j \Rightarrow \neg x_j P x_i$) relation on X. It may be interpreted as a (strict) preference relation on the set of alternatives. If each individual has a complete, transitive and asymmetric preference relation on X and the number of individuals is odd, then a tournament may be constructed by pairwise comparisons of alternatives in the particular preference orders (corresponding to preference relations), i.e. in that tournament $x_i P x_j$ iff the number of individuals preferring x_i to x_j is larger than of those with the opposite preference.

Let us now briefly review some basic *solution concepts*:

The Condorcet winner. This solution concept comprises that alternative which is preferred by a majority of individuals. Unfortunately, it need not exist [see, e.g., Fishburn (1974) for this and related issues].

The Copeland winning set. This (and the next ones) is one of the so - called *Condorcet extensions* (solution concepts that reduce to the Condorcet winner if it exists). It is denoted by UC_C and consists of those alternatives that have the maximum number of 1's in their corresponding rows in the tournament matrix.

The uncovered set. Denoted by UC [Miller (1977, 1980)], this solution concept consists of those alternatives that are not covered by any other one where the *covering relation*, $C \subseteq X \times X$, is defined as: $x_i C x_j$ (x_i covers x_j) iff $x_j P x_k \Rightarrow x_i P x_k$, for all $x_i, x_j \in X$; $x_k \in X \setminus \{x_i, x_j\}$, i.e. x_i covers x_j if all the alternatives defeated by x_j are also defeated by x_i.

The Banks set. It consists of the endpoints of the *Banks chains* which are defined as follows: starting from x_i we find an x_j such that $x_j P x_i$, then an x_k such that $x_k P x_j$, etc. If no such an x_k exists, then x_j is the endpoint of the so - called *Banks chain*. For each $x_i \in X$ we obtain a Banks chain, and the set of their endpoints is called the *Banks set*.

For an analysis of various properties of the above solutions, and definitions of many other ones, see, e.g., Nurmi (1983, 1987, 1988a).

3. SOLUTIONS IN THE CASE OF FUZZY TOURNAMENTS

We will now define some new solution concepts, mainly counterparts of those from Section 2, starting from fuzzy tournaments, and proceeding first due to "$\{R_1,\ldots,R_m\}$ ---> R ---> solution concept", and then due to "$\{R_1,\ldots,R_m\}$ ---> solution concept".

3.1. Solutions derived from a collective fuzzy tournament

We assume that a collective fuzzy tournament, R, is obtained by pairwise comparisons, i.e.

$$R = [r_{ij}] = [\mathrm{card}\{k \in N:\ x_i P_k x_j\} / m] \tag{1}$$

A *strong covering relation*, $C_s \subseteq X \times X$, is defined as

$$x_i C_s x_j \Leftrightarrow r_{il} \geq r_{jl} \ \&\ r_{ij} \geq r_{ji} \tag{2}$$

for each i, j, l $\in \{1,\ldots,n\}$. Clearly, $x_i C_s x_j \Rightarrow x_i C x_j$. C_s is therefore a strengthening of C, hence the *set of C_s - uncovered elements*, UC_s, is a superset of that of C - uncovered ones, UC.

A *weak fuzzy covering relation*, $C_w \subseteq X \times X$, is defined as

$$x_i C_w x_j \Leftrightarrow r_{ij} \geq r_{ji} \ \&$$
$$\&\ \mathrm{card}\{x_q \in X:\ r_{iq} > r_{jq}\} \geq \mathrm{card}\{x_p \in X:\ r_{jp} > r_{ip}\} \tag{3}$$

for each x_i, $x_j \in X$.

Obviously, $x_i C_s x_j \Rightarrow x_i C_w x_j$. Thus, the *set of C_w - uncovered elements*, UC_w, is always a subset of UC_s.

As to some more important properties, a Condorcet winner (if it exists) C_w - covers all the other alternatives. It is impossible for a Copeland winning alternative to be C_s - covered, but it may be C_w - covered.

Example 1. Let X = {a, b, c, d, e}, m = 11, and the collective fuzzy preference relation (in fact, mR) be mR =

	a	b	c	d	e
a	–	6	7	5	3
b	5	–	9	7	5
c	4	2	–	6	7
d	6	4	5	–	9
e	8	6	4	2	–

Thus, $UC_c = X$, but $bC_v c$ which implies $UC_v \subset UC_c$. This is however not always the case as shown below.

Example 2. Let now $X = \{a, b, c, d, e, f\}$, $m = 5$, and

mR =

	a	b	c	d	e	f
a	–	4	3	2	1	3
b	1	–	4	3	2	3
c	2	1	–	3	3	2
d	3	2	2	–	4	4
e	4	3	2	1	–	3
f	2	2	3	1	2	–

Thus, $UC_c = \{a, b, d, e\}$ and $UC_v = X$, i.e. $UC_c \subset UC_v$. In fact, UC_v may even contain covered elements as shown below.

Example 3. Let $X = \{a, b, c, d, e, f\}$, $m = 5$, and

mR =

	a	b	c	d	e	f
a	–	4	3	2	2	4
b	1	–	4	3	3	2
c	2	1	–	3	3	2
d	3	2	2	–	4	3
e	3	2	2	1	–	3
f	1	3	3	2	2	–

Hence, $UC_v = \{a, b, c, d\}$.

3.2. <u>Solutions derived from individual fuzzy tournaments</u>

First, we introduce the concept of an *α - dominance relation* on X, $D_\alpha \subseteq X \times X$, defined as

$$x_i D_\alpha x_j \Leftrightarrow r^k_{ij} \geq \alpha \qquad (4)$$

The *set of D_α-undominated alternatives*, UD_α, is obviously a plausible solution concept [equivalent to the set of α-consensus winners proposed by Nurmi (1981), and to the Pareto set for α = 0.5 - see, e.g. Nurmi (1987, 1988a)]. Moreover, $\alpha_1 < \alpha_2 < 0.5 \Rightarrow UD_{\alpha 1} \subseteq UD_{\alpha 2} \subseteq UD_{0.5}$. Hence, one can reduce the choice set to a managable size by reducing α.

UD_α is not based on any majority - related preference relation. More "majoritarian" solution concepts will be discussed below.

We define the concept of an *α - majority dominance* *relation* on X, $MD_\alpha \subseteq X \times X$, as

$$x_i MD_\alpha x_j \Leftrightarrow \frac{1}{n}\, \mathrm{card}\{k \in N:\ r^k_{ij} > \alpha\} \ge 0.5 \qquad (5)$$

Evidently, $\alpha_1 \ge \alpha_2 \Rightarrow [x_i MD_{\alpha 1} x_j \Rightarrow x_i MD_{\alpha 2} x_j,\ \forall\, x_i,\ x_j]$.

The *set of* α *- majority undominated options* is clearly a plausible solution concept.

Now, we can define the concept of an α *- covering relation* on X, $MC_\alpha \subseteq X \times X$, as

$$x_i MC_\alpha x_j \Leftrightarrow x_i MD_{0.5} x_j \ \&\ x_i MD_\alpha x_k \qquad (6)$$

for all $x_k \in X$ such that $x_j MD_\alpha x_k$.

The *set of* α *- uncovered options*, UMC_α, consists then of those alternatives which are not α - covered by any other one.

Notice that, first, $x_i C x_j \Leftrightarrow x_i MC_{0.5}$, and second, $\alpha_1 < \alpha_2 \Rightarrow UMC_{\alpha 1} \subseteq UMC_{\alpha 2}$. Thus, an obvious solution concept would be an $UMC_\alpha \neq \emptyset$ with a minimal α.

Example 4. Let X = {a, b, c}, and the individual fuzzy tournaments be

$$[r^1_{ij}] = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} - & .8 & .6 \\ .2 & - & .9 \\ .4 & .1 & - \end{bmatrix} \end{array} \quad [r^2_{ij}] = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} - & .2 & .3 \\ .8 & - & .8 \\ .7 & .2 & - \end{bmatrix} \end{array} \quad [r^3_{ij}] = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} - & .6 & .3 \\ .4 & - & .2 \\ .7 & .8 & - \end{bmatrix} \end{array}$$

Then: $aMC_{0.4}b$, $bMC_{0.4}c$, and $cMC_{0.2}a$, i.e. when $\alpha = 0.2$, all the alternatives are 0.2 - covered, hence $UMC_{0.2}$ is empty. When $\alpha = 0.3$, b is covered, i.e. $UMC_{0.3} = \{a, c\}$, and when $\alpha = 0.4$, $UMC_{0.4} = X$. Therefore, it seems that a choice set should be {a, c}.

The α - Banks set may be similarly constructed. For any $x_i \in X$ we find an x_j such that $x_j MD_\alpha x_i$, then an x_k such that $x_k MD_\alpha x_j$ and $x_k MD_\alpha x_i$, then an x_l such that $x_l MD_\alpha x_k$, $x_l MD_\alpha x_j$ and $x_l MD_\alpha x_i$, etc. Each such a sequence $x_i, x_j, x_k, x_l, \ldots$ forms an α *- Banks chain*, and the α *- Banks set* is the set of endpoints of those chains. Obviously, the 0.5 - Banks set and the Banks set are identical.

Clearly, the larger the value of α the more strongly the alternatives have to be preferred to the previous ones in the chain by a majority of individuals. One should therefore start from large values of a and proceed towards the smaller ones.

3.3. Remarks on fuzzy majorities

The above mentioned solution concepts were based on nonfuzzy majorities. Let us now consider the case of *fuzzy majority* [see Kacprzyk (1984, 1985b,c, 1987), and Kacprzyk and Nurmi (1988) for more detail].

First, we construct a *fuzzy majority relation* RM: X × X ---> [0, 1], labelled "is preferred to ... by *most* individuals", for instance as follows

$$v(x_i RMx_j) = 1 \quad \text{for } n_{ij} \geq 0.8$$
$$= 2n_{ij} - 0.6 \quad \text{for } 0.3 < n_{ij} < 0.8 \qquad (7)$$
$$= 0 \quad \text{for } n_{ij} \leq 0.3$$

where $100n_{ij}$ is the percentage of individuals preferring x_i over x_j, and v(.) is the intensity of preference by *most* individuals. And analogously for, e.g., "...preferred by *almost all* the individuals...".

Now, we define a (nonfuzzy) "*covers by a fuzzy a - majority*" *relation*, $M_a \subseteq X \times X$, as

$$x_i M_a x_j \iff v(x_i RMx_j) \geq a \ \&$$
$$\& \ [v(x_j RMx_k) \geq a \Rightarrow v\ (x_i RMx_k) \geq a, \ \forall x_k \in X] \qquad (8)$$

Obviously, $x_i M_a x_j \iff x_i Cx_j$.

The *fuzzy a - majority uncovered set*, $UM_a \subseteq X$, is then

$$UM_a = \{x_i \in X: \exists x_j \in X, x_j M_a x_i\} \qquad (9)$$

Evidently, UM_a is a nonfuzzy set. Notice also that UM_a depends crucially on the membership function (intensity) of RM given by (7), and hence not much can be generally said about the relation between UM_a and other solution concepts.

It may be sometimes viewed counterintuitive that UM_a is a nonfuzzy set, we can define the *fuzzy a - majority uncovered fuzzy set*, $UFM_a \subsetneq X$, as follows

$$UFM_a = f(n_a(x_1)/n)/x_1 + \ldots + f(n_a(x_n)/n)/x_n \qquad (10)$$

where $f: \{1/n, 2/n, \ldots, (n-a)/n\} \longrightarrow [0, 1]$ maps the sizes of relative majorities to degrees of membership in UFM_{α} as, e.g., (7) or even $f(x) = x$.

4. CONCLUDING REMARKS

In this paper we tried to develop some new solution concepts in group decision making starting from individual and collective fuzzy preference relations (tournaments). These solution concepts were meant to be counterparts of those developed by starting from nonfuzzy tournaments. The purpose was, first, to overcome some inherent difficulties related to the use of the conventional solution concepts as, e.g., too large choice sets, i.e. insufficient decisiveness, and second, to provide means to employ some more realistic and human - consistent representations of preferences.

And, indeed, several newly developed solution concepts are applicable in cases when their conventional counterparts do not exist or are not decisive enough. Unfortunately, there are also some plausible properties of certain non - fuzzy solution concepts that do not necessarily parallel those of their fuzzy counterparts. In particular this concerns the stategic properties of the uncovered set and the Banks set. Namely, the sophisticated voting outcomes in simple majority voting games are all in the uncovered set, and in particular comprise its subset - the Banks set. This cannot be said for the a - uncovered set, a - Banks set, fuzzy a - majority uncovered set, and fuzzy a - majority uncovered fuzzy set.

Basically, the fuzzy counterparts would rather be recommended in non- strategic voting contexts where instead of individuals we have equally important criteria

We hope that this introductory paper on fuzzy tournaments in group decision making would trigger further research in this interesting and important topic. This should provide a deeper analysis of properties of, and relations between the new and conventional solution concepts, and maybe even additional solution concepts.

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