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UNCERTAINTY IN THE ANALYTIC HIERARCHY PROCESS

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ABSTRACT

The paper is concerned with the so-called analytic hierarchy process proposed by Th. L. Saaty. This approach is based on the pairwise comparison of alternatives, made by experts. Results of such a comparison are used to determine the relative importance of alternatives under examination. Numerous papers have been published presenting modifications of Saaty's method, mostly concerned with different ways of dealing with uncertainty in experts' opinions. In papers making use of statistical analysis for solving this problem the assumptions that are introduced are, in general, very strong, e.g. the independence of comparisons, known distributions of comparison errors. In the paper an attempt is made at solving the problem considered under weaker assumptions. A proposed method is based on analogy between decisions resulting from statistical tests and comparisons made by experts. It is an extension of the method, worked out by L. Klukowski, of clustering parameters of a piecewise linear trend.

Keywords: pairwise comparison, alternatives ordering, comparison random errors.

1. INTRODUCTION

The approach called analytic hierarchy process was proposed by Th. L. Saaty (1980) as a method of ranking alternatives. This method relies on pairwise comparisons and using it there is no need of applying the notion of utility function. The method consists in determining the principal right eigenvector of the comparison matrix. Many papers have been published presenting modifications of the Saaty method, mostly concerned with different ways of dealing with uncertainty in experts' opinions. Recently, Saaty and Vargas (1987) applied the concept of interval judgement to analytic hierarchy process. Krovak (1987) used various versions of the least squares method for ranking alternatives and compared results with those obtained when using Saaty's eigenvector method. Some elements of statistical analysis were used in both papers. However, assumptions adopted for the purpose of this analysis as well as in other papers concerning the considered problem, seem to be too strong, e.g. the independence of comparisons, known distributions of experts' errors. Moreover, there are some difficulties in determining properties of results obtained by these methods.

It should be mentioned that a detailed statistical analysis of pairwise comparison procedures is given in the book by David (1963).

This paper presents an attempt at solving the problem of ranking alternatives in the case of uncertainty in experts' opinions, but under weaker assumptions. The basis of a proposed method can be stated as follows: a family of random variables corresponding to the set of feasible orderings of alternatives is defined, i.e. every variable is a function of comparisons made and corresponds to an ordering considered. It is proved under weak assumptions related to the probabilities of comparison errors that the random variable defined, corresponding to the proper ordering of alternatives, assumes the minimal expected value on this family and the probability of the event that the value of this random variable is less than that of any other variable of this family is greater than some desired threshold.

The theorem proved makes it possible to suggest that the problem under consideration can be reduced to the search for an ordering of the set of alternatives corresponding to the random variable assuming the minimal value for given comparisons.

It should be mentioned that the presented method, being an extention of one worked out by Klukowski (1986) for clustering parameters of a piecewise linear trend, is based on analogy between decisions resulting from statistical tests and comparisons made by experts.

2. FORMULATION OF THE PROBLEM

Given a finite set of elements $\mathfrak{X}=\{x_1,\ldots,x_m\}$. It is assumed that there exists, but is not known apriori, the preference relation defined on this set. This relation makes it possible to partition the set $\mathfrak{X}$ into ordered, non-overlapping, non-empty subsets $\mathfrak{X}_1^*,\ldots,\mathfrak{X}_n^*$, $n \le m$. To each of the subsets belong equivalent elements only (if they exist).

The preference relation can be characterized by the function T defined as follows

$$T:\quad \mathfrak{X}\times\mathfrak{X}\rightarrow D,\quad D=\{0,\pm 1,\ldots,\pm(n-1)\} \tag{1}$$

$$T(x_i;x_j)\ =\ d \Longleftrightarrow x_i \in \mathfrak{X}_k^*,\ x_j \in \mathfrak{X}_l^*,\ l-k=d \tag{2}$$

$$T(x_i,x_j)\ =\ -T(x_j,x_i)\ \text{for}\ T(x_i,x_j)\neq 0\ . \tag{3}$$

The preference relation we are looking for (or in other words a partition of the set $\mathfrak{X}$), is to be determined on the basis of pairwise comparisons made by an expert (or experts). It is assumed that the following conditions hold:

A1. Comparisons are to be made for every pair of elements $x_i, x_j \in \mathfrak{X}$

A2. The results of comparison can be as follows: (i) elements x_i, $x_j \varepsilon \mathfrak{X}$ are equivalent; in other words the expert's opinion is that $T(x_i,x_j)=0$; (ii) the element x_i is preferred to x_j; in other words the expert's opinion is that $T(x_i,x_j)<0$; (iii) the element x_j is preferred to x_i; in other words the expert's opinion is that $T(x_i,x_j)>0$.

A3. Errors, random in nature, can occur in comparisons made by expert(s). For every pair of elements, the probability that the comparison made by each of the experts is correct is greater than $\frac{1}{2}$.

The result of comparison made by the k-th expert $(k=1,\ldots,N)$ is described by the function $g^{(k)}: \mathfrak{X} \times \mathfrak{X} \rightarrow \{-1,0,1\}$. If the expert's opinion is such that x_i is preferred to x_j, than $g^{(k)}(x_i,x_j)=-1$. In the opposite case, $g^{(k)}(x_i,x_j)=1$. If these two elements are considered to be equivalent, then $g^{(k)}(x_i,x_j)=0$.

Hence it follows that comparisons are to be made only for pairs $x_i,x_j \varepsilon \mathfrak{X}, j>i$, $i=1,\ldots,m-1$. The total number of necessary comparisons is equal to $\frac{m}{2}(m-1)$.

Under the above assumptions it can be said that a given comparison is correct if one of the following conditions is satisfied:

$$
\begin{aligned}
g^{(k)}(x_i,x_j) &= -1 \text{ and } T(x_i,x_j)<0 \\
g^{(k)}(x_i,x_j) &= 0 \text{ and } T(x_i,x_j)=0 \\
g^{(k)}(x_i,x_j) &= 1 \text{ and } T(x_i,x_j)>0 \; .
\end{aligned}
\tag{4}
$$

For the sake of notation simplicity the probabilities of comparison errors are denoted as follows:

$$P[g^{(k)}(x_i,x_j) = -1/T(x_i,x_j) = 0] = \alpha^{(1)}_{ijk}$$
$$P[g^{(k)}(x_i,x_j) = 1/T(x_i,x_j) = 0] = \alpha^{(2)}_{ijk}$$
$$P[g^{(k)}(x_i,x_j) = 0/T(x_i,x_j) \leq -1] = \beta^{(1)}_{ijk} \quad ;(5)$$
$$P[g^{(k)}(x_i,x_j) = 1/T(x_i,x_j) \leq -1] = \beta^{(2)}_{ijk}$$
$$P[g^{(k)}(x_i,x_j) = 0/T(x_i,x_j) \geq 1] = \gamma^{(1)}_{ijk}$$
$$P[g^{(k)}(x_i,x_j) = -1/T(x_i,x_j) \geq 1] = \gamma^{(2)}_{ijk} \quad .$$

It follows from the assumption A3 that the following inequalities hold

$$\alpha_{ijk} = \alpha^{(1)}_{ijk} + \alpha^{(2)}_{ijk} < \frac{1}{2}$$
$$\beta_{ijk} = \beta^{(1)}_{ijk} + \beta^{(2)}_{ijk} < \frac{1}{2} \quad (6)$$
$$\gamma_{ijk} = \gamma^{(1)}_{ijk} + \gamma^{(2)}_{ijk} < \frac{1}{2} \quad .$$

To simplify the consideration it is assumed that N=1. The case when comparisons are made by a group of experts, i.e. N > 1, can be considered almost in the same way.

3. DEFINITIONS AND NOTIONS

For a given partition $\mathfrak{X}_1,\ldots,\mathfrak{X}_r$ of the set $\mathfrak{X}$, being a feasible solution to the formulated problem, the following notation is introduced:

$L(\mathfrak{X})$ - the set of all the pairs of indices <i,j> satisfying the conditions

$$1 \leq i,j \leq m;\ j > i \quad (7)$$

(m - the number of elements of the set $\mathfrak{X}$);

$I(\mathfrak{X}_1,\ldots,\mathfrak{X}_r)$ – the subset of $L(\mathfrak{X})$ such that

$$I(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \{<i,j> : x_i, x_j \varepsilon \mathfrak{X}_q, 1 \le q \le r\} \quad (8)$$

$J(\mathfrak{X}_1,\ldots,\mathfrak{X}_r)$ – the subset of $L(\mathfrak{X})$ such that

$$J(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \{<i,j> : x_i \varepsilon \mathfrak{X}_k, x_j \varepsilon \mathfrak{X}_l ; l-k<0 ; l,k=1,\ldots,r\} \quad (9)$$

$K(\mathfrak{X}_1,\ldots,\mathfrak{X}_r)$ – the subset of $L(\mathfrak{X})$ such that

$$K(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \{<i,j> : x_i \varepsilon \mathfrak{X}_k, x_j \varepsilon \mathfrak{X}_l ; l-k>0 ; l,k=1,\ldots,r\} . \quad (10)$$

It follows from the definitions given above that

$$I(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \cup J(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \cup K(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = L(\mathfrak{X}) . \quad (11)$$

It should be emphasized that the sets I,J,K result from a given partition; the set L depends upon the number of elements of the set $\mathfrak{X}$ only.

The following random variables (W_1, W_2, W_3) are defined:

$$W_1(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \sum_{I(\mathfrak{X}_1,\ldots,\mathfrak{X}_r)} U_{ij}(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \quad (12)$$

where

$$U_{ij}(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \begin{cases} 0 \text{ if } g(x_i,x_j)=0 \text{ for } <i,j> \varepsilon\, I(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \\ 1 \text{ if } g(x_i,x_j)\neq 0 \text{ for } <i,j> \varepsilon\, I(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \end{cases} \quad (13)$$

$$W_2(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \sum_{J(\mathfrak{X}_1,\ldots,\mathfrak{X}_r)} V_{ij}(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \quad (14)$$

where

$$V_{ij}(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \begin{cases} 0 \text{ if } g(x_i,x_j) = -1 \text{ for } <i,j> \varepsilon\, J(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \\ 1 \text{ if } g(x_i,x_j) \ge 0 \text{ for } <i,j> \varepsilon\, J(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \end{cases} \quad (15)$$

$$W_3(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \sum_{K(\mathfrak{X}_1,\ldots,\mathfrak{X}_r)} Z_{ij}(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \quad (16)$$

where

$$Z_{ij}(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \begin{cases} 0 \text{ if } g(x_i,x_j) = 1 \text{ for } <i,j> \varepsilon\, K(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) \\ 1 \text{ if } g(x_i,x_j) \le 0 \text{ for } <i,j> \varepsilon\, K(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) . \end{cases} \quad (17)$$

and

$$W(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) = \sum_{i=1}^{3} W_i(\mathfrak{X}_1,\ldots,\mathfrak{X}_r) . \quad (18)$$

4. BASIC THEOREM

Let us denote by $\tilde{\mathfrak{X}}_1,\ldots,\tilde{\mathfrak{X}}_r$ any feasible solution to the problem, different from $\mathfrak{X}_1^*,\ldots,\mathfrak{X}_n^*$. To simplify the notation, the following symbols are used $I^* = I(\mathfrak{X}_1^*,\ldots,\mathfrak{X}_n^*),\ldots,U_{ij}^*=U_{ij}(\mathfrak{X}_1^*,\ldots,\mathfrak{X}_n^*),\ldots,$ $W^* = W(\mathfrak{X}_1^*,\ldots,\mathfrak{X}_n^*)$, $\tilde{I}=I(\tilde{\mathfrak{X}}_1,\ldots,\tilde{\mathfrak{X}}_r),\ldots,\tilde{U}_{ij} = U_{ij}(\tilde{\mathfrak{X}}_1,\ldots,\tilde{\mathfrak{X}}_r),\ldots,\tilde{W}=W(\tilde{\mathfrak{X}}_1,\ldots,\tilde{\mathfrak{X}}_r)$. For the defined variables W^* and $\tilde{W}$ the following theorem can be stated:

<u>Theorem</u>: If the maximal values of the probabilities (6), i.e. α_{ijmax}, $\langle i,j\rangle \in I^*$, β_{ijmax}, $\langle i,j\rangle \in J^*$, γ_{ijmax}, $\langle i,j\rangle \in K^*$, are such that the following inequalities hold,

$$\alpha_{ijmax},\ \beta_{ijmax},\ \gamma_{ijmax} < \delta\ ,\ \delta \in (0,\tfrac{1}{2}) \quad (19)$$

then

$$E(W^* - \tilde{W}) < 0 \quad (20)$$

and

$$P(W^* < \tilde{W}) > 1 - 2\delta\ . \quad (21)$$

<u>Proof</u> of inequality (20).

It follows from relations (12), (13), (14), (15), (17) and (18) that W^* can be written in the form:

$$W^* = \sum_{I^*} U_{ij}^* + \sum_{J^*} V_{ij}^* + \sum_{K^*} Z_{ij}^* = \sum_{I^*\cap\tilde{I}} U_{ij}^* + \sum_{I^*-\tilde{I}} U_{ij}^* + \sum_{J^*\cap\tilde{J}} V_{ij}^* + \sum_{J^*-\tilde{J}} V_{ij}^* +$$

$$\sum_{K^*\cap\tilde{K}} Z_{ij}^* + \sum_{K^*-\tilde{K}} Z_{ij}^* \quad . \quad (22)$$

It is evident that the variable $\tilde{W}$ can be written in the same form.

Hence

$$W^* - \tilde{W} = \sum_{I^*-\tilde{I}} U^*_{ij} + \sum_{J^*-\tilde{J}} V^*_{ij} + \sum_{K^*-\tilde{K}} Z^*_{ij} - \sum_{\tilde{I}-I^*} \tilde{U}_{ij} - \sum_{\tilde{J}-J^*} \tilde{V}_{ij} - \sum_{\tilde{K}-K^*} \tilde{Z}_{ij} \qquad (23)$$

It can be shown that

$$I^* - \tilde{I} = [I^* \cap (\tilde{J}-J^*)] \cup [I^* \cap (\tilde{K}-K^*)] \qquad (24a)$$

$$J^* - \tilde{J} = [J^* \cap (\tilde{I}-I^*)] \cup [J^* \cap (\tilde{K}-K^*)] \qquad (24b)$$

$$K^* - \tilde{K} = [K^* \cap (\tilde{I}-I^*)] \cup [K^* \cap (\tilde{J}-J^*)] . \qquad (24c)$$

Moreover, it can be easily proved that sets, which are the intersection of subsets constituting the right-hand side of equalities (24) are empty, e.g.

$$[I^* \cap (\tilde{J}-J^*)] \cap [I^* \cap (\tilde{K}-K^*)] = \{\emptyset\} \qquad (25)$$

The subsets $\tilde{I}-I^*$, $\tilde{J}-J^*$ and $\tilde{K}-K^*$ can be written in the same form. Fig.1 gives the evidence that all the relations mentioned hold.

I^*			J^*			K^*		
$\tilde{I}$	$\tilde{J}$	$\tilde{K}$	$\tilde{I}$	$\tilde{J}$	$\tilde{K}$	$\tilde{I}$	$\tilde{J}$	$\tilde{K}$

$\tilde{I}$			$\tilde{J}$			$\tilde{K}$		
I^*	J^*	K^*	I^*	J^*	K^*	I^*	J^*	K^*

Fig.1.

Using relations (24) and (25) it is possible to show that

$$W^* - \tilde{W} = \sum_{I^* \cap (\tilde{J}-J^*)} (U^*_{ij} - \tilde{V}_{ij}) + \sum_{I^* \cap (\tilde{K}-K^*)} (U^*_{ij} - \tilde{Z}_{ij}) + \sum_{J^* \cap (\tilde{I}-I^*)} (V^*_{ij} - \tilde{U}_{ij})$$

$$+ \sum_{J^* \cap (\tilde{K}-K^*)} (V^*_{ij} - \tilde{Z}_{ij}) + \sum_{K^* \cap (\tilde{I}-I^*)} (Z^*_{ij} - \tilde{U}_{ij}) + \sum_{K^* \cap (\tilde{J}-J^*)} (Z^*_{ij} - \tilde{V}_{ij}) \qquad (26)$$

To simplify expression (26), the following notation is introduced

$$\frac{Q_{ij}^{(1)} = U_{ij}^{*} - \tilde{V}_{ij}; \quad Q_{ij}^{(2)} = U_{ij}^{*} - \tilde{Z}_{ij} \;;}{Q_{ij}^{(5)} = Z_{ij}^{*} - \tilde{U}_{ij}; \quad Q_{ij}^{(6)} = Z_{ij}^{*} - \tilde{V}_{ij}.} \tag{27}$$

and

$$\frac{S_1 = I^{*} \wedge (\tilde{J}-J^{*}) \qquad S_2 = I^{*} \wedge (\tilde{K}-K^{*})}{S_5 = K^{*} \wedge (\tilde{I}-I^{*}) \qquad S_6 = K^{*} \wedge (\tilde{J}-J^{*})} \;. \tag{28}$$

Hence this expression can be written as follows

$$W^{*} - W = \sum_{l=1}^{6} \sum_{S_l} Q_{ij}^{(l)} \;. \tag{29}$$

To determine the distributions of the random variables involved one has to notice that $Q_{ij}^{(l)}$ $(l=1,\dots,6)$ can assume only the values $\{-1,0,+1\}$. Moreover, it follows from relations (13), (15), (17) and (29) that

$$P(Q_{ij}^{(1)} = -1) = P[(U_{ij}^{*} = 0) \wedge (\tilde{V}_{ij} = 1)] = P\{[g(x_i,x_j)=0 /\ \langle i,j\rangle \varepsilon\ S_1] \wedge \wedge [g(x_i,x_j) \geq 0 / \langle i,j\rangle \varepsilon\ S_1]\} = P[g(x_i,x_j)=0/\langle i,j\rangle \varepsilon\ S_1] = 1 - \alpha_{ij} > \frac{1}{2} \;. \tag{30}$$

Analogously, it is possible to show that

$$P[Q_{ij}^{(1)} = 0] = \alpha_{ij}^{(2)} \tag{31}$$

$$P[Q_{ij}^{(1)} = 1] = \alpha_{ij}^{(1)} \tag{32}$$

Taking into account relations (6) and (19), we have

$$E[Q_{ij}^{(1)}] = -1\cdot(1-\alpha_{ij}) + 0 \cdot \alpha_{ij}^{(2)} + 1 \cdot \alpha_{ij}^{(2)} < 0 . \quad (33)$$

It can be proved in the same manner that each of the variables $Q_{ij}^{(l)}$ $(l=1,\ldots,6)$ assumes the value -1 with the probability greater than $\frac{1}{2}$. Therefore we have

$$E[Q_{ij}^{(l)}] < 0 \quad l=1,\ldots,6; \ \langle i,j\rangle \varepsilon\ S_l \ . \quad (34)$$

Partitions $\tilde{\mathfrak{X}}_1,\ldots,\tilde{\mathfrak{X}}_r$ are different from $\mathfrak{X}_1^*,\ldots,\mathfrak{X}_n^*$, hence at least one of sets S_l, $(l=1,\ldots,6)$ is not empty. The expected value operator is additive, therefore

$$E(W^* - \tilde{W}) = \sum_{l=1}^{6} \sum_{S_l} E[Q_{ij}^{(l)}] < 0 . \quad (35)$$

<u>Proof</u> of inequality (21).

Making use of the variables (27), (28), inequality (21) can be written in the form

$$P[\sum_{l=1}^{6} \sum_{S_l} Q_{ij}^{(l)} < 0] > 1 - 2\delta . \quad (36)$$

Introducing the new variables

$$Q'^{(l)}_{ij} = Q_{ij}^{(l)} + 1 \quad (37)$$

and taking into account that

$$P[\sum_{l=1}^{6} \sum_{S_l} Q_{ij}^{(l)} < 0] = 1 - P[\sum_{l=1}^{6} \sum_{S_l} Q_{ij}^{(l)} \geq 0] \quad (38)$$

inequality (36) can be written as follows

$$P[\sum_{l=1}^{6} \sum_{S_l} Q'^{(l)}_{ij} \geq \nu] < 2\delta \tag{39}$$

where

$$\nu = \text{card}\ (\bigcup_{l=1}^{6} S_l) \quad . \tag{40}$$

Each of the variables $Q'^{(l)}_{ij}$ $(l=1,\ldots,6)$ assumes non-negative values only, i.e. 0,1,2; hence making use of the Tchebysheff inequality we have

$$P[\sum_{l=1}^{6} \sum_{S_l} Q'^{(l)}_{ij} \geq \nu] \leq \frac{E[\sum_{l=1}^{6} \sum_{S_l} Q'^{(l)}_{ij}]}{\nu} \quad . \tag{41}$$

We have that

$$E[Q'^{(l)}_{ij}] = 0 \cdot P[Q'^{(l)}_{ij} = 0] + 1 \cdot P[Q'^{(l)}_{ij} = 1] + 2 \cdot P[Q'^{(l)}_{ij} = 2], l=1,\ldots,6 \quad . \tag{42}$$

Making use of inequality (19) and applying the same reasoning as used to derive relations (31) and (32), one obtains

$$P[Q'^{(l)}_{ij} = 1] + P[Q'^{(l)}_{ij} = 2] < \delta; l=1,\ldots,6 \quad . \tag{43}$$

Therefore

$$P[Q'^{(l)}_{ij} = 1] + 2P[Q'^{(l)}_{ij} = 2] < 2\delta; l=1,\ldots,6 \quad . \tag{44}$$

Substituting inequality (43) into relation (42) we have

$$E[Q'^{(l)}_{ij}] < 2\delta \ ; \ l=1,\ldots,6 \tag{45}$$

The average value operator is additive; hence

$$E[\sum_{l=1}^{6} \sum_{S_l} Q'^{(l)}_{ij}] = \sum_{l=1}^{6} \sum_{S_l} E[Q'^{(l)}_{ij}] < 2\nu\delta \tag{46}$$

Taking into account inequality (41) one can show that

$$P[\sum_{l=1}^{6} \sum_{S_l} Q'^{(1)}_{ij} \geq \nu] < 2\delta \ . \qquad (47)$$

It follows from definitions (37) and (40) that

$$P[\sum_{l=1}^{6} \sum_{S_l} Q'^{(1)}_{ij} \geq \nu] = P[\sum_{l=1}^{6} \sum_{S_l} Q^{(1)}_{ij} \geq 0] \ . \qquad (48)$$

Combining relations (38) and (48) with inequality (47) one obtains

$$P[\sum_{l=1}^{6} \sum_{S_l} Q^{(1)}_{ij} < 0] = P(W^{*} < \tilde{W}) > 1 - 2\delta \ . \qquad (49)$$

5. PROPOSED ALGORITHM

The theorem proved makes it possible to suggest that a solution to the stated problem, i.e. determining a reasonable estimate of the partition of the set $\mathfrak{X}$ corresponding to the preference relation, can be found by solving the following optimization problem

$$\min w(\mathfrak{X}_1, \ldots, \mathfrak{X}_r)$$
$$\mathfrak{X}_1, \ldots, \mathfrak{X}_r \in G(\mathfrak{X}) \qquad (50)$$

where $w(\mathfrak{X}_1, \ldots, \mathfrak{X}_r)$ is the value of the random variable $W(\mathfrak{X}_1, \ldots, \mathfrak{X}_r)$ determined for given results of comparisons and $G(\mathfrak{X})$ is the set of all feasible partitions of the set $\mathfrak{X}$.

It is evident that a solution to this problem may not be unique.

Therefore the following algorithm for solving the problem under discussion can be proposed:

<u>Step 1.</u> One has to determine $g(x_i, x_j)$; $x_i, x_j \in \mathfrak{X}$, in such a way that the assumptions (6) are satisfied.

<u>Step 2.</u> Given results of comparisons $g(x_i, x_j)$; $x_i, x_j \in \mathfrak{X}$, one has to determine a partition $\hat{\mathfrak{X}}_1, \ldots, \hat{\mathfrak{X}}_{\hat{n}}$ such that the value of random variable $\hat{W}$ corresponding to this partition is minimal.

<u>Step 3.</u> If the result of Step 2 is not unique, one has to introduce another objective function, e.g. minimization of the number of subsets $\hat{n}$.

Using the Tchebysheff inequality one can construct a test to verify the correctness of an obtained solution.

The problem of determining a partition $\hat{\mathfrak{X}}_1, \ldots, \hat{\mathfrak{X}}_{\hat{n}}$ can be formulated in the form of 0-1 mathematical programming problem

$$\min \{ \sum_{I(\mathfrak{X}_1, \ldots, \mathfrak{X}_r)} |g(x_i, x_j)| + \sum_{J(\mathfrak{X}_1, \ldots, \mathfrak{X}_r)} h^{(1)}(x_i, x_j) + \sum_{K(\mathfrak{X}_1, \ldots, \mathfrak{X}_r)} h^{(2)}(x_i, x_j) \} \tag{51}$$

where

$$h^{(1)}(x_i, x_j) = \begin{cases} 0 \text{ if } g(x_i, x_j) = -1 \\ 1 \text{ otherwise} \end{cases}$$
$$h^{(2)}(x_i, x_j) = \begin{cases} 0 \text{ if } g(x_i, x_j) = 1 \\ 1 \text{ otherwise} \end{cases} \tag{52}$$

If a solution to the problem (51) - (52) results in a partition $\hat{\mathfrak{X}}_1, \ldots, \hat{\mathfrak{X}}_{\hat{n}}$, which is the same as $\mathfrak{X}_1^*, \ldots, \mathfrak{X}_n^*$ than

$$\hat{n} = n .$$

However, it should be emphasized that this equality can also hold in the case when these partitions are not the same (due to random errors of comparisons).

6. CONCLUDING REMARKS

It is possible to show that the theorem proved and algorithm proposed can also be applied in the case when the definitions of the random variables V_{ij} and Z_{ij} are generalized, e.g.

$$V_{ij}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)=\begin{cases}0 \text{ if } g(x_i,x_j)=-1 \text{ for } \langle i,j\rangle\varepsilon\, \mathfrak{J}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)\\ 1 \text{ if } g(x_i,x_j)=0 \text{ for } \langle i,j\rangle\varepsilon\, \mathfrak{J}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)\\ 2 \text{ if } g(x_i,x_j)=1 \text{ for } \langle i,j\rangle\varepsilon\, \mathfrak{J}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)\end{cases} \tag{53}$$

$$Z_{ij}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)=\begin{cases}0 \text{ if } g(x_i,x_j)=1 \text{ for } \langle i,j\rangle\varepsilon\, \mathfrak{K}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)\\ 1 \text{ if } g(x_i,x_j)=0 \text{ for } \langle i,j\rangle\varepsilon\, \mathfrak{K}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)\\ 2 \text{ if } g(x_i,x_j)=-1 \text{ for } \langle i,j\rangle\varepsilon\, \mathfrak{K}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)\ .\end{cases} \tag{54}$$

However, in this case the corresponding mathematical programming problem cannot be written in the form given above.

It should be emphasized that the same reasoning can be used in the case of N experts. If this is the case one has to define the variable W in the form

$$W=\sum_{k=1}^{N}\sum_{i=1}^{3} W_i^{(k)}(\mathfrak{X}_1,\dots,\mathfrak{X}_r)\ . \tag{55}$$

It seems possible to obtain analogous results for the case when comparisons $g(x_i,x_j)$; x_i, $x_j\varepsilon\mathfrak{X}$ assume the same values as the function T.

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