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On fuzzy approach to pricing of example of catastrophe bond

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Abstract

In this paper we discuss the example of catastrophe bond with linear payment function. The approach based on neutral martingale method is used. In order to price the catastrophe bond we use fuzzy parameters and apply Vasicek model under assumption of independence between catastrophe occurrence and behaviour of financial market. Then the Monte Carlo simulations based on the obtained fuzzy pricing formula are carried out. The presented fuzzy sets approach may incorporate expertise knowledge to overcome lack of precise data in the discussed case.

Keywords: catastrophe bond, Vasicek model, neutral martingale method, Monte Carlo simulations, fuzzy sets.

1 Introduction

During last years, the insurance industry face overwhelming risks caused by natural catastrophes. Losses from single catastrophic event could reach even 40 – 60 billion $ (Hurricane Katrina, 2005, see [11]). To cope with dramatic consequences

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of such extreme events special, integrated policy based on applying a whole set of financial and insurance instruments is required.

The classical insurance mechanisms are not prepared to deal with extreme losses caused by natural catastrophes. Even one, single catastrophe could cause problems with reserves for many insurers or even bankruptcy of these enterprises. For example, after Hurricane Andrew more than 60 insurance companies became insolvent (see [11]). The traditional insurance models (see [2]) deal with independent, rather small risks like car accidents. But losses from natural catastrophes have extremely huge values and their sources are strongly dependent in terms of time and localization, e.g. single hurricane could start fire in many houses, leads to robberies, etc. Additionally, classical insurance mechanisms are often criticized because of serious problems with adverse selection and moral hazard. The primary insurers rely on classical reinsurance markets which are affected by pricing cycles connected with occurrence of huge catastrophes or increasing threats of terrorist attacks. Therefore it may be profitable to use new insurance instruments in order to stabilize the portfolio for insurance company or even budget of government.

As it was mentioned, the single catastrophic event could result in damages measured in billions of dollars. Because daily fluctuations on worldwide financial markets reach also tens of billion \$, securitization of losses may be helpful for dealing with results of extreme natural catastrophes (see e.g. [4, 7, 8, 10]). One of possible types of ”packaging” the losses is known as catastrophe bonds (Act-of-God bonds, cat bonds, see e.g. [6, 14, 17]).

In this paper we discuss the example of catastrophe bond with linear payment function. We apply neutral martingale method. In order to price the catastrophe bond we use fuzzy parameters and Vasicek model under assumption of independence between catastrophe occurrence and behaviour of financial market. Then the Monte Carlo simulations based on the obtained fuzzy pricing formula are carried out.

There is a need to take into account possible errors and uncertainties which arise from estimation of rare events with serious, catastrophic consequences like natural catastrophes. Therefore we apply the approach based on fuzzy sets which may also incorporate expertise knowledge of assumptions about future behaviour of catastrophic events or to overcome lack of precise, historical data.

This paper is organized as follows. In Section 2 we present some preliminaries for fuzzy sets. In Section 3 we discuss the general features of catastrophe bonds, present example of catastrophe bond with linear payment function, price this type of catastrophe bond and conduct simulations in order to find appropriate price for fuzzy approach. We conclude the paper in Section 4.

2 Preliminaries

In this section we recall some basic facts about fuzzy sets and numbers.

Let X be a universal set and $\tilde{A}$ be a fuzzy subset of X. We denote by $\mu_{\tilde{A}}$ its membership function $\mu_{\tilde{A}} : X \to [0,1]$, and by $\tilde{A}_\alpha = \{x : \mu_{\tilde{A}} \geq \alpha\}$ the α-level set of $\tilde{A}$, where $\tilde{A}_0$ is the closure of the set $\{x : \mu_{\tilde{A}} \neq 0\}$.

In our paper we assume that $X = \mathbb{R}$.

Let $\tilde{a}$ be a fuzzy number. Then the α-level set $\tilde{a}_\alpha$ is a closed interval, which can be denoted by $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ (see e.g. [20]).

We can now introduce the arithmetic of any two fuzzy numbers. Let $\odot$ be a binary operator $\oplus$, $\ominus$, $\otimes$ or $\oslash$ between fuzzy numbers $\tilde{a}$ and $\tilde{b}$, where the binary operators correspond to $\circ$: $+$, $-$, $\times$ or $/$, according to the "Extension Principle" in [20]. Then the membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\mu_{\tilde{a}\odot\tilde{b}}(z) = \sup_{(x,y):x\circ y=z} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\} \ . \tag{1}$$

Let $\odot_{int}$ be a binary operator $\oplus_{int}$, $\ominus_{int}$, $\otimes_{int}$ or $\oslash_{int}$ between two closed intervals $[a,b]$ and $[c,d]$. Then

$$[a,b] \odot_{int} [c,d] = \{z \in \mathbb{R} : z = x \circ y, \forall x \in [a,b], \forall y \in [c,d]\} \ , \tag{2}$$

where $\circ$ is an usual operation $+, -, \times$ and $/$, if the interval $[c,d]$ does not contain zero in the last case.

Therefore, if $\tilde{a}, \tilde{b}$ are fuzzy numbers, then $\tilde{a} \odot \tilde{b}$ is also the fuzzy number and its α-level set is given by

$$(\tilde{a} \oplus \tilde{b})_\alpha = \tilde{a}_\alpha \oplus_{int} \tilde{b}_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha] \ ,$$

$$(\tilde{a} \ominus \tilde{b})_\alpha = \tilde{a}_\alpha \ominus_{int} \tilde{b}_\alpha = [\tilde{a}^L_\alpha - \tilde{b}^U_\alpha, \tilde{a}^U_\alpha - \tilde{b}^L_\alpha] \ ,$$

$$(\tilde{a} \otimes \tilde{b})_\alpha = \tilde{a}_\alpha \otimes_{int} \tilde{b}_\alpha =$$

$$= [\min\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\}, \max\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\}] \ ,$$

$$(\tilde{a} \oslash \tilde{b})_\alpha = \tilde{a}_\alpha \oslash_{int} \tilde{b}_\alpha =$$

$$= [\min\{\tilde{a}^L_\alpha/\tilde{b}^L_\alpha, \tilde{a}^L_\alpha/\tilde{b}^U_\alpha, \tilde{a}^U_\alpha/\tilde{b}^L_\alpha, \tilde{a}^U_\alpha/\tilde{b}^U_\alpha\}, \max\{\tilde{a}^L_\alpha/\tilde{b}^L_\alpha, \tilde{a}^L_\alpha/\tilde{b}^U_\alpha, \tilde{a}^U_\alpha/\tilde{b}^L_\alpha, \tilde{a}^U_\alpha/\tilde{b}^U_\alpha\}] \ ,$$

if α-level set $\tilde{b}_\alpha$ does not contain zero for all $\alpha \in [0,1]$ in the case of $\oslash$.

Triangular fuzzy number $\tilde{a}$ with membership function $\mu_{\tilde{a}}(x)$ is defined as

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{for} \quad a_1 \leq x \leq a_2 \\ \frac{x-a_3}{a_2-a_3} & \text{for} \quad a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases} \ , \tag{3}$$

where $[a_1, a_3]$ is the supporting interval and the membership function has peak in a_2. Triangular fuzzy number $\tilde{a}$ is denoted as

$$\tilde{a} = (a_1, a_2, a_3) \ .$$

Triangular fuzzy numbers are special case of Left-Right (or L-R) fuzzy numbers (e.g. see [1, 5]), where linear functions used in the definition are replaced by monotonic functions, i.e.

Definition 1. *A fuzzy set $\tilde{A}$ on the set of real numbers is called L-R number if the membership function may be calculated as*

$$\mu_{\tilde{a}}(x) = \begin{cases} L\left(\frac{a_2 - x}{a_2 - a_1}\right) & \mathit{for} \quad a_1 \leq x \leq a_2 \\ R\left(\frac{x - a_2}{a_3 - a_2}\right) & \mathit{for} \quad a_2 \leq x \leq a_3 \\ 0 & \mathit{otherwise} \end{cases} , \tag{4}$$

where L and R are continuous strictly decreasing function defined on $[0, 1]$ with values in $[0, 1]$ satisfying the conditions

$$L(x) = R(x) = 1 \ \ \mathit{if} \ \ x = 0 \ , L(x) = R(x) = 0 \ \ if \ x = 1 \ .$$

The L-R fuzzy number $\tilde{a}$ is denoted as

$$\tilde{a} = (a_1, a_2, a_3)_{LR} \ \ .$$

Next we turn to fuzzy estimation based on statistical approach (see [3]). This approach may be seen as a way to obtain L-R numbers based on statistical data.

Let X be a random variable with probability density function $f_\theta(.)$. Assume that parameter θ is unknown and are to be estimated from a sample $X_1, X_2, \ldots X_n$. Let $\hat{\theta}$ be a statistics based on $X_1, X_2, \ldots X_n$ which is used for such estimation. Then for the given confidence level $0 \leq \beta \leq 1$ we have the $\beta \cdot 100\%$ confidence interval $[\theta_L(\beta), \theta_R(\beta)]$ for θ which is established by the condition

$$\mathrm{P}_{f_\theta}(\theta_L(\beta) \leq \theta \leq \theta_R(\beta)) = \beta \ \ . \tag{5}$$

If we suppose that $[\theta_L(0), \theta_R(0)] = [\hat{\theta}, \hat{\theta}]$ then we could construct fuzzy estimator $\hat{\tilde{\theta}}$ of $\tilde{\theta}$. We place the confidence intervals, one on top of the other, to produce a triangular shaped fuzzy $\tilde{\theta}$ whose α-cuts are the confidence intervals on $\beta = (1 - \alpha)$ confidence levels. Therefore we have

$$\hat{\tilde{\theta}}_\alpha = [\theta_L(1 - \alpha), \theta_R(1 - \alpha)] \tag{6}$$

for e.g. $0.01 \le \alpha \le 1$. In order to "finish" construction of the fuzzy estimator, we suppose that

$$\hat{\tilde{\theta}}_\alpha = [\theta_L(0.99), \theta_R(0.99)] \tag{7}$$

for $0 \le \alpha < 0.01$. It means that we drop the graph of $\hat{\tilde{\theta}}$ straight down to complete its α-cuts (see [3] for additional details).

3 Catastrophe bonds

3.1 Features of catastrophe bonds

Because of problems with coverage of losses by insurance enterprises, dependencies among sources of risks, potentially unlimited losses, problems with adverse selection, moral hazard and reinsurance pricing cycles applying alternative financial or insurance instruments may be profitable. The problem is to "package" natural disasters risk and appropriate losses into classical forms of tradable financial assets, like bonds or options. The most popular catastrophe-linked security is the catastrophe bond, known also as *cat bond* or *Act-of-God* bond (see [6, 9, 14, 17]). Cat bonds were introduced in 1992, and become wider known in April 1997, when USAA, an insurer from Texas, initiated two new classes of cat bonds: A-1 and A-2 (see [12]).

There is one important difference between cat bonds and "classical" bonds – the premiums from cat bond are always connected with additional random variable, i.e. occurrence of some natural catastrophe in specified region and fixed time interval. Such event is called *triggering point* (see [9]). For example, the A-1 USAA bond was connected with hurricane on the east coast of USA between July 15, 1997 and December 31, 1997. If there had been a hurricane in mentioned above region with more than \$1 billion loses against USAA, the coupon of the bond would have been lost. As we can see from this example, the triggering point changes the structure of payments for the cat bond. The other types of cat bonds may be related to various kinds of triggering points — e.g. to magnitude of earthquake, the losses from flood, insurance industry index of losses, etc.

As usually, the structure of payments for cat bonds depends also on some primary underlying asset. In case of A-1 USAA bond, the payment equalled LIBOR plus 282 basis points.

The main aim of cat bonds is to transfer *risk* from insurance markets or governmental budgets to financial markets. Apart from transferring capital, a liquid catastrophe derivatives market allow insurance and reinsurance companies to adjust their exposure to natural catastrophic risk dynamically through hedging with those contracts at lower transaction costs. If the triggering point is connected with industry loss indices or parametric triggers, the moral hazard exposure of bond investors is greatly reduced or eliminated. Cat bonds are often rated by an agency such as Standard & Poor's, Moody's, or Fitch Ratings.

The cash flows for catastrophe bond are managed by special tailor-made fund, called a special-purpose vehicle (SPV) (see [18]). The hedger (e.g. insurer) pays an insurance premium in exchange for coverage in case if catastrophic event occurs. The investors purchase an insurance-linked security for cash. The mentioned premium and cash flows are directed to SPV, which issues the catastrophe bonds. Usually, SPV purchases safe securities in order to satisfy future possible demands. Investors hold the issued assets whose coupons and/or principal depend on occurrence of the triggering point. If the pre-specified event occurs during the fixed period, the SPV compensates the insurer and the cash flows for investors are changed, i.e. there is full or partial forgiveness of the repayment of principal and/or interest. If the triggering point does not occur, the investors usually receive the full payment.

3.2 Example of catastrophe bond

We present an example of a catastrophe bond with a piecewise linear stucture of the payoff function. This catastrophe bond was also considered in [16], where it was a part of an insurance company portfolio.

We introduce two finite sequences of constants:

$$0 \leq K_0 = k_0 < K_1 < ... < K_n,$$
$$0 < w_1 < w_2 < ... < w_n, \ \sum_{i=1}^{n} w_i \leq 1, \ n > 1.$$

Definition 2. *We denote by $IB_p(T, Fv)$ catastrophe bond with face value Fv, maturity and payoff date T, and which payoff function has a form*

$$\nu_{IB_p(T,Fv)} = Fv \left(1 - \sum_{j=0}^{n-1} \frac{\tilde{N}_T \wedge K_{j+1} - \tilde{N}_T \wedge K_j}{K_{j+1} - K_j} w_{j+1} \right).$$

The catastrophe bond $IB_p(T, Fv)$ satisfies the following properties.

1. The payoff function is a piecewise linear function of losses $\tilde{N}_T$.

2. If the catastrophe does not occur in the period $[0, T]$, i.e. ($\tilde{N}_T < K_0$), the bondholder receives the payoff equal to the face value Fv.

3. If $\tilde{N}_T \geq K_n$, the bondholder receives $Fv\left(1 - \sum_{i=1}^{n} w_i\right)$.

4. If $K_j \leq \tilde{N}_T \leq K_{j+1}$ dla $j = 0, 1, ..., n$, the payoff is equal to

$$Fv\left(1 - \sum_{0 \leq i < j} w_{i+1} - \frac{\tilde{N}_T \wedge K_{j+1} - \tilde{N}_T \wedge K_j}{K_{j+1} - K_j} w_{j+1}\right).$$

The payoff function of the presented example of a catastrophe bond precisely reflects possible losses caused by a catastrophic event.

3.3 Pricing of catastrophe bond

Let (Ω, F, P) be a probability space.

Let $(W_t)_{t\in[0,T']}$ be a Brownian motion. Let $(U_i)_{i=1}^{\infty}$ be independent, identically distributed random variables with bounded second moment. We assume that each U_i describes the value of losses in case of single catastrophic event.

We define compound Poisson process by formula

$$\tilde{N}_t = \sum_{i=1}^{N_t} U_i \, , t \in \left[0, T'\right],$$

where N_t is Poisson process with intensity κ. Then $\tilde{N}_t$ describes the value of losses caused by catastrophes till time t.

The filtration $(F_t)_{t\in[0,T']}$ is given by formula

$$F_t = \sigma\left(F_t^0 \cup F_t^1\right), F_t^0 = \sigma\left(W_s, s \leq t\right),$$
$$F_t^1 = \sigma\left(\tilde{N}_s, s \leq t\right), t \in \left[0, T'\right].$$

We assume that

$$F_0 = \sigma\left(\{A \in F : P(A) = 0\}\right)$$

and that $(W_t)_{t\in[0,T']}$, $(N_t)_{t\in[0,T']}$ and $(U_i)_{i=1}^{\infty}$ are independent. Then the filtered probability space $\left(\Omega, F, (F_t)_{t\in[0,T']}, P\right)$ satisfies the standard assumptions, i.e. σ-algebra F is P-complete, the filtration $(F_t)_{t\in[0,T']}$ is right continuous, what means that for each $t \in [0, T')$

$$F_{t+} = \bigcap_{s>t} F_s = F_t$$

and F_0 contains all the sets in F of P-probability zero.

We denote by $(B_t)_{t\in[0,T']}$ banking account satisfying the following equation:

$$dB_t = r(t)B_t\,dt,\ B_0 = 1,$$

where r is a risk-free spot interest rate. The solution of the above equation has the form:

$$B_t = \exp\left(\int_0^t r(u)du\right),\ t\in\left[0,T'\right].$$

We denote by $B(t,T)$ the price at the time t zero-coupon bond with maturity date $T \le T'$ and the face value equal to 1.

Definition 3. *$B(t,T)$, $t \le T \le T'$ is called the arbitrage-free family of zero-coupon bond prices with respect to r, if the following conditions are satisfied:*

a) $B(T,T) = 1$ *for each* $T\in[0,T']$.

b) There exists a probability Q,equivalent to P, such that for each $T\in[0,T']$ the process of discounted zero-coupon bond price

$$B(t,T)/B_t,\ t\in[0,T],$$

is a martingale with respect to Q. Then we have the following pricing formula

$$B(t,T) = E^Q\left(e^{-\int_t^T r(u)du}|F_t^Q\right),\ t\in[0,T].$$

Let $\lambda_u = -\lambda$ denote the risk premium for risk-free bonds. The following Radon-Nikodym derivative defines a probability measure Q, equivalent to P:

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda_u dW_u - \frac{1}{2}\int_0^T \lambda_u^2 du\right) P\text{-a.s.},$$

such that $B(t,T)/B_t,\ t\in[0,T]$, is a martingale with respect to Q.

We assume the Vasicek model of the risk-free spot interest rate r. The interest rate satisfies the following equation

$$dr(t) = a(b - r(t))\,dt + \sigma dW_t$$

for positive constants a, b and σ.

We also assume that financial market is independent from the catastrophe risk and investors are neutral toward nature jump risk.

The following theorem and lemma were proved in [16]. In our approach, described in [16], we used methodology from [18].

Theorem 1. *Let $IB\left(0\right)$ be the price of a $IB_p\left(T,Fv\right)$ at time 0. Then*

$$IB\left(0\right)=\mathrm{Fv}\,e^{-TR(T,r(0))}E^{P}\nu_{IB_p(T,Fv)}, \tag{8}$$

where

$$R\left(\theta,r\right)=R_{\infty}-\frac{1}{a\theta}\left\{\left(R_{\infty}-r\right)\left(1-e^{-a\theta}\right)-\frac{\sigma^2}{4a^2}\left(1-e^{-a\theta}\right)^2\right\}$$

and

$$R_{\infty}=b-\frac{\lambda\sigma}{a}-\frac{\sigma^2}{2a^2}.$$

The lemma below is usefull for the numerical computation of the catastrophe bond price.

Lemma 1. *Let*

$$\varphi_m=P\left(\tilde{N}_T\leq K_m\right),m=0,1,...,n \tag{9}$$

and let

$$e_m=E\tilde{N}_T I_{\left\{K_m<\tilde{N}_T\leq K_{m+1}\right\}},m=0,1,...,n-1. \tag{10}$$

The following equality holds

$$\begin{aligned}E^{P}\nu_{IB_p(T,Fv)}=Fv\Big[&1-\left(1-\varphi_n\right)\textstyle\sum_{j=1}^{n}w_j\\&-\textstyle\sum_{m=0}^{n-1}\Big\{\left(\varphi_{m+1}-\varphi_m\right)\sum_{0\leq j<m}w_{j+1}\\&+\frac{e_m-\left(\varphi_{m+1}-\varphi_m\right)K_m}{K_{m+1}-K_m}w_{m+1}\Big\}\Big].\end{aligned}$$

Our aim is to present the catastrophe bond pricing formula in case when the parameters of the spot interest rate are not precisely known. To model this uncertainty we introduce fuzzy numbers $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$ and $\tilde{r}_0$ in place of a, b, σ and $r\left(0\right)$. We also treat the market price of risk as a small fuzzy number. Therefore we replace the parameter λ by its negative fuzzy counterpart $\tilde{\lambda}$.

Let $\mathcal{F}\left(R\right)$ the set of all fuzzy numbers. The proposition below was proved in [19].

Proposition 1. *Let $f:R\rightarrow R$, for which the inverse image of any value is compact, induces a fuzzy-valued function $\tilde{f}:\mathcal{F}\left(R\right)\rightarrow\mathcal{F}\left(R\right)$ via the extension principle and the α-level set of $\tilde{f}(\tilde{\Lambda})$ is $\tilde{f}(\tilde{\Lambda})_{\alpha}=\{f(x):x\in\tilde{\Lambda}_{\alpha}\}$.*

Applying Proposition 1, we obtain the following fuzzy version of the pricing formula.

Theorem 2.

$$\tilde{IB}(0) = \mathrm{Fv} \otimes e^{-T\otimes\tilde{R}(T)} \otimes E^P \nu_{IB_p(T,Fv)}, \tag{11}$$

where

$$\tilde{R}(T) = \tilde{R}_\infty \ominus \left\{ \left(\tilde{R}_\infty \ominus \tilde{r}_0\right) \otimes \left(1 \ominus e^{-\tilde{a}\otimes T}\right) \ominus \tilde{\sigma}\otimes\tilde{\sigma} \otimes \left(1 \ominus e^{-\tilde{a}\otimes T}\right) \right. \\ \left. \otimes \left(1 \ominus e^{-\tilde{a}\otimes T}\right) \oslash (4 \otimes \tilde{a}\otimes\tilde{a})\right\} \oslash (\tilde{a} \otimes T) \tag{12}$$

and

$$\tilde{R}_\infty = \tilde{b} \ominus \tilde{\lambda}\otimes\tilde{\sigma} \oslash \tilde{a} \ominus \tilde{\sigma}\otimes\tilde{\sigma} \oslash (2\otimes\tilde{a}\otimes\tilde{a}) \,. \tag{13}$$

To calculate the α-level sets of $\tilde{IB}(0)$ we use formulas similar to (11), (12) and (13), replacing the operators $\oplus$, $\ominus$, $\otimes$, $\oslash$ by $\oplus_{int}$, $\ominus_{int}$, $\otimes_{int}$, $\oslash_{int}$.

3.4 Monte Carlo simulations for fuzzy approach

In order to find the price of the catastrophe bond described in Section 3.2 according to Theorem 2, the appropriate simulations were conducted.

It was necessary to apply two kind of sets in Monte Carlo simulations. First set was used for approximation of parameters (9) and (10) in calculation of value $E^P \nu_{IB_p(T,Fv)}$. In this case we assume that quantity of losses is modelled by Poisson process with expected value κ and the value of each loss is given by random variable from Gamma distribution with scale parameter ζ and shape parameter β. Other types of distributions for modelling the value of losses are also possible, e.g. Weibull distribution.

The second set of simulations was used for generation of trajectories of the risk-free spot interest rate for the Vasicek model with fuzzy parameters. We assume that parameters are described by α-sets which may be derived e.g. from triangular fuzzy numbers or L-R numbers (see Section 2). The approach for applying Monte Carlo simulation in case of intervals based on α-sets was presented in [15].

We assume that for each considered example of catastrophe bond the trading horizon is set on 5 years and we generate $n = 1000000$ (one million) simulations.

In case of *Example I – III* we analyse the estimators of cat bond price for extending limits of α-sets. We assume that $\kappa = 0.05$, $\zeta = 10$, $\beta = 20$, therefore the generated losses have catastrophic nature, i.e. they are rare, but with high value. The catastrophe bond was described by parameters

$$\mathrm{Fv} = 1 \,, \lambda = -0.1 \,, w_1 = 0.3 \,, w_2 = 0.2 \,, \\ K_0 = 5 \,, K_1 = 10 \,, K_2 = 15 \,. \tag{14}$$

Then based on equation (11) we obtained estimators for price of catastrophe bond presented in Table 1. As we could see, the average and median behave rather stably. Minimum, 1% quantile and 5% quantile decrease in constant way. The same applies for first quartile, but in this case the decreasing is not so fast. Maximum, 95% quantile, 99% quantile and third quartile increase in constant way. The same applies for standard deviation which means that the error caused by uncertainty, i.e. widen intervals for α-sets, is also higher. Based on these numerical experiments we may examine the dependency between the level of error for estimators and the width of α-sets connected with uncertainty level α.

Table 1: Numerical estimators for price of catastrophe bond in *Example I, II, III*

	Example I	*Example II*	*Example III*
a_α	[0.02,0.03]	[0.015,0.035]	[0.01,0.04]
b_α	[0.05,0.06]	[0.045,0.065]	[0.04,0.07]
σ_α	[0.01,0.02]	[0.005,0.025]	[0.005,0.03]
$r_\alpha(0)$	[0.05,0.07]	[0.045,0.075]	[0.04,0.08]
Average	0.6511	0.652089	0.651655
First quartile	0.635761	0.628966	0.620808
Median	0.650879	0.651559	0.650661
Third quartile	0.666293	0.674975	0.682049
Standard deviation	0.0177032	0.0266988	0.0355123
Minimum	0.618231	0.601944	0.586333
1% quantile	0.621041	0.606673	0.591928
5% quantile	0.62389	0.611391	0.597849
95% quantile	0.678962	0.69423	0.70817
99% quantile	0.682045	0.699594	0.715074
Maximum	0.685385	0.706107	0.723928

In case of *Example I* and *Example IV – V* we analyse the estimators of cat bond price for increasing values of ζ and β, i.e. higher expected value and variance of single catastrophic event. We assume that α-set were described by intervals

$$a_\alpha = [0.02, 0.03]\ , b_\alpha = [0.05, 0.06]\ ,$$
$$\sigma_\alpha = [0.01, 0.02]\ , r_\alpha(0) = [0.05, 0.07] \quad (15)$$

and the catastrophe bond was the same as in previous scenarios, i.e. parameters were given by (14). The obtained estimators may be found in Table 2. As we could see, all of the estimators, including average, are lower for higher values of scale and shape parameters. However, the differences are not so significant.

Table 2: Numerical estimators for price of catastrophe bond in *Example I, IV, V*

	Example I	*Example IV*	*Example V*
ζ	10	15	20
β	20	30	40
Average	0.6511	0.651101	0.651012
First quartile	0.635761	0.635729	0.635649
Median	0.650879	0.650856	0.65074
Third quartile	0.666293	0.666311	0.666261
Standard deviation	0.0177032	0.0177192	0.0177149
Minimum	0.618231	0.618144	0.618102
1% quantile	0.621041	0.621078	0.620989
5% quantile	0.62389	0.623936	0.623834
95% quantile	0.678962	0.679018	0.678894
99% quantile	0.682045	0.682084	0.681977
Maximum	0.685385	0.685491	0.685286

Table 3: Numerical estimators for price of catastrophe bond in *Example I, VI, VII*

	Example I	*Example VI*	*Example VII*
K_0	5	5	5
K_1	10	15	20
K_2	15	30	40
Average	0.6511	0.651484	0.651512
First quartile	0.635761	0.636083	0.635955
Median	0.650879	0.651268	0.651253
Third quartile	0.666293	0.666758	0.666555
Standard deviation	0.0177032	0.0177386	0.017723
Minimum	0.618231	0.618443	0.618275
1% quantile	0.621041	0.621393	0.621271
5% quantile	0.62389	0.624239	0.624124
95% quantile	0.678962	0.679363	0.679218
99% quantile	0.682045	0.682444	0.68227
Maximum	0.685385	0.685815	0.6858

In case of *Example I* and *Example VI – VII* we analyse the estimators of cat bond price for increasing values of triggering points K_1 and K_2. We assume that α-set were described by intervals (15) and the parameters of catastrophe bond were given by (14) apart from values of triggering points. The obtained estimators may be found in Table 3. As we could see, all of the estimators, including average, are higher for higher values of triggering points K_1 and K_2. However, the differences are not so significant. It may be explained by the fact, that first triggering point K_0 – the most often realized event in case of catastrophe – was the same for all scenarios.

4 Conclusions

In this paper we discuss the example of catastrophe bond with linear payment function. The approach based on neutral martingale method is used. We use fuzzy parameters and apply Vasicek model under assumption of independence between catastrophe occurrence and behaviour of financial market. Because of possible errors and uncertainties which arise from estimation of rare events with serious, catastrophic consequences and lack of precise, historical data, the fuzzy set approach is applied. Then appropriate simulations for the obtained fuzzy pricing formula are generated. We analyse output from some numerical experiments for various sets of parameters.

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The papers presented in this Volume 2 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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