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Tests for validation of estimates obtained on the basis of pairwise comparisons with random errors

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Abstract

The paper presents tests for validation of estimates of three relations: preference, equivalence and tolerance obtained on the basis of multiple pairwise comparisons with random errors. The estimates result from appropriate discrete programming tasks, which minimize inconsistencies (differences) between relation form and comparisons. The validation consists of a set of tests for verification assumptions about: distributions of pairwise comparisons, existence of the relation in the set under consideration and the type of relation; some of the tests have been developed by the author. Positive results of validation guarantees high quality of estimates, especially in the case of multiple comparisons of each pair.

Keywords: validation of preference, equivalence and tolerance relation; multiple pairwise comparisons; binary comparisons; multivalent comparisons.

1 Introduction

The estimators of relations: preference, equivalence and tolerance, discussed in Klukowski 1994, 2007, 2010 are based on assumptions about distributions of pairwise comparisons and the crucial assumption – about existence of the relation in a set under consideration. The first group of assumptions have probabilistic nature and, therefore, can be verified with the use of well-known statistical tests. The assumption about existence of a relation is necessary condition of rational estimation and can be also verified with the use of tests – based on estimate errors. Some tests proposed for validation require simulation techniques, because of difficulties with determination of distributions of test statistics. Positive results of verification guarantee reliable estimates, negative – indicate direction of improvement or rejecting it.

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Some results presented in the paper are extensions of earlier investigations of the author, especially test for weak and strict form of the preference relation (see Klukowski 1994, 2006, 2007, 2008b, 2009).

The validation problems are discussed in the literature, e.g. the preference relation - in David (1987), the equivalence and tolerance relation - in Gordon 1999.

The paper consists of four sections. The second section presents formulation of estimation problems for all relation types. Main results, i.e. tests for verification: • assumptions about comparison errors, • existence of the relation, • correct type of the relation, are discussed in the third section. The tests for relation type: equivalence or tolerance and weak or strict form of the preference, have been developed by the author. Further results – based on simulation approach will be presented in forthcoming papers.

2 Formulation of estimation problems

The problem of estimation of the relation form on the basis of pairwise comparisons with random errors can be stated as follows (Klukowski 2010).

Given a finite set of elements $\mathbf{X} = \{x_1, ..., x_m\}$ ($3 \le m < \infty$). There exists in the set $\mathbf{X}$ the equivalence relation $\mathbf{R}^{(e)}$ (reflexive, transitive, symmetric) or the tolerance relation $\mathbf{R}^{(\tau)}$ (reflexive, symmetric) or the preference relation $\mathbf{R}^{(p)}$ (alternative of the equivalence relation and strict preference relation). Each relation generates some family of non-empty subsets $\chi_1^{(\ell)*}, ..., \chi_n^{(\ell)*}$ ($\ell \in \{p, e, \tau\}$; $n \ge 2$); in the case of the preference relation the subsets are ordered.

The equivalence relation generates the family $\chi_1^{(e)*}, ..., \chi_n^{(e)*}$ $(n \ge 2)$ with the following properties:

$$\bigcup_{q=1}^{n} \chi_q^{(e)*} = \mathbf{X}, \tag{1}$$

$$\chi_r^{(e)*} \cap \chi_s^{(e)*} = \{\mathbf{0}\}, \tag{2}$$

where: $\mathbf{0}$ – the empty set,

$$x_i, x_j \in \chi_r^{(e)*} \equiv x_i, x_j - \text{equivalent elements}, \tag{3}$$

$$(x_i \in \chi_r^{(e)*}) \cap (x_j \in \chi_s^{(e)*}) \equiv x_i, x_j - \text{non-equivalent elements for } i \ne j, r \ne s. \tag{4}$$

The tolerance relation generates the family $\chi_1^{(\tau)*}, ..., \chi_n^{(\tau)*}$ with the property (1) and the properties:

$\exists r, s \; (r \ne s)$ such, that $\chi_r^{(\tau)*} \cap \chi_s^{(\tau)*} \ne \{\mathbf{0}\}$,

$x_i, x_j \in \chi_r^{(\tau)*} \equiv x_i, x_j -$ equivalent elements, (5)

$(x_i \in \chi_r^{(\tau)*}) \cap (x_j \in \chi_s^{(\tau)*}) \equiv x_i, x_j -$non-equivalent elements for $i \neq j$ and

$$(x_i, x_j) \notin \chi_r^{(\tau)*} \cap \chi_s^{(\tau)*}, \quad (6)$$

each subset $\chi_r^{(\tau)*}$ $(1 \le r \le n)$ includes an element x_i such that $x_i \notin \chi_s^{(\tau)*}$ $(s \neq r)$. (7)

The preference relation generates the family $\chi_1^{(p)*}, ..., \chi_n^{(p)*}$ with the properties (1), (2) and the property:

$(x_i \in \chi_r^{(p)*}) \cap (x_j \in \chi_s^{(p)*}) \equiv x_i$ is preferred to x_j for $r<s$. (8)

The relations defined by the conditions (1) - (2), (3) - (4), (5) - (8) can be described by some functions $T_\upsilon^{(\ell)}(x_i, x_j)$ $((x_i, x_j) \in \mathbf{X} \times \mathbf{X}; \ell \in \{p, e, \tau\}, \upsilon \in \{b, \mu\})$, where: symbols b, μ denote – respectively: binary and multivalent comparisons, symbols: p, e, τ - relation types. The functions can be defined as follows:

$$T_b^{(e)}(x_i, x_j) = \begin{cases} 0 \; if \;\; exists \; r \; such \, that \, (x_i, x_j) \in \chi_r^{(e)*}, \\ 1 \; otherwise; \end{cases} \quad (9)$$

• the function $T_b^{(e)}(x_i, x_j)$ expresses the fact if a pair (x_i, x_j) belong to a common subset or not;

$$T_b^{(\tau)}(x_i, x_j) = \begin{cases} 0 \; if \;\; exists \; r, s \; (r = s \; not \, excluded) \; such \, that \, (x_i, x_j) \in \chi_r^{(\tau)*} \cap \chi_s^{(\tau)*}, \\ 1 \; otherwise; \end{cases} \quad (10)$$

• the function $T_b^{(\tau)}(x_i, x_j)$ expresses the fact if a pair (x_i, x_j) belong to any conjunction of subsets (also to the same subset) or not; the condition (7) guarantee uniqueness of the description;

$T_\mu^{(\tau)}(x_i, x_j) = \#(\Omega_i^* \cap \Omega_j^*)$, (11)

where:

Ω_l^* - the set of the form $\Omega_l^* = \{s \mid \; x_l \in \chi_s^{(\tau)*}\}$,

$\#(\Xi)$ - number of elements of the set Ξ;

• the function $T_\mu^{(\tau)}(x_i, x_j)$ expresses the number of subsets of conjunction including both elements; the condition (7) guarantee uniqueness of the description;

$$T_b^{(p)}(x_i, x_j) = \begin{cases} 0 \; if \;\; there \, exists \; r \; such \, that \, (x_i, x_j) \in \chi_r^{(p)*}, \\ \mp 1 \; if \;\; x_i \in \chi_r^{(p)*}, x_j \in \chi_s^{(p)*} \; and \; r < s \; (r > s); \end{cases} \quad (12)$$

(more precisely – the value -1 reflects the case $r<s$, 1 – the case $r>s$);

• the function $T_b^{(p)}(x_i, x_j)$ expresses direction of preference in a pair or equivalency of its elements;

$$T_\mu^{(p)}(x_i, x_j) = d_{ij} \Leftrightarrow x_i \in \chi_r^{(p)*}, x_j \in \chi_s^{(p)*}, d_{ij} = r - s; \tag{13}$$

• the function $T_\mu^{(p)}(x_i, x_j)$ expresses difference of ranks of elements x_i and x_j.

The tolerance relation can be described also (see Klukowski 2007) by the function: $T_\iota^{(\tau)}(x_i, x_j) = n - T_\mu^{(\tau)}(x_i, x_j)$. (14)

General formulation of estimation problem

The relation form has to be estimated on the basis of N ($N \geq 1$) comparisons of each pair $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$, under the following assumptions.

A1. The relation type, i.e.: equivalence or tolerance or preference, is known.

A2. Each pair of elements $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$ is compared N times; any comparison $g_{\upsilon k}^{(\ell)}(x_i, x_j)$ ($\ell \in \{e, \tau, p\}$; $\upsilon \in \{b, \mu, \iota\}$; $k = 1, ..., N$) evaluates the value of the function $T_\upsilon^{(\ell)}(x_i, x_j)$ and can be disturbed by random error. The probabilities of errors $T_\upsilon^{(\ell)}(x_i, x_j) - g_{\upsilon k}^{(\ell)}(x_i, x_j)$ have to satisfy the following conditions:

$$P(T_b^{(\ell)}(x_i, x_j) - g_{bk}^{(\ell)}(x_i, x_j) = 0) \geq 1 - \delta, \quad \delta \in (0, 1/2), \tag{15}$$

$$\sum_{l \leq 0} P(T_\mu^{(\ell)}(x_i, x_j) - g_{\mu k}^{(\ell)}(x_i, x_j) = l) > 1/2, \tag{16}$$

$$\sum_{l \geq 0} P(T_\mu^{(\ell)}(x_i, x_j) - g_{\mu k}^{(\ell)}(x_i, x_j) = l) > 1/2, \tag{17}$$

$$P(T_\mu^{(\ell)}(x_i, x_j) - g_{\mu k}^{(\ell)}(x_i, x_j) = l) \geq P(T_\mu^{(\ell)}(x_i, x_j) - g_{\mu k}^{(\ell)}(x_i, x_j) = l + 1) \quad (l \geq 0), \tag{18}$$

$$P(T_\mu^{(\ell)}(x_i, x_j) - g_{\mu k}^{(\ell)}(x_i, x_j) = l) \geq P(T_\mu^{(\ell)}(x_i, x_j) - g_{\mu k}^{(\ell)}(x_i, x_j) = l - 1) \quad (l \leq 0), \tag{19}$$

$$\begin{aligned} &P((g_{\upsilon k}^{(\ell)}(x_i, x_j) = T_\upsilon^{(\ell)}(x_i, x_j)) \cap (g_{\upsilon l}^{(\ell)}(x_r, x_s) = T_\upsilon^{(\ell)}(x_r, x_s))) = \\ &P(g_{\upsilon k}^{(\ell)}(x_i, x_j) = T_\upsilon^{(\ell)}(x_i, x_j)) P(g_{\upsilon l}^{(\ell)}(x_r, x_s) = T_\upsilon^{(\ell)}(x_r, x_s)) \quad (k \neq l). \end{aligned} \tag{20}$$

The inequalities (15) – (20) reflect the following properties of distributions of comparison errors:

• the probability of correct comparison is greater than incorrect one, in the case of binary comparisons (the inequality (15));
• zero is the median of each distribution of comparison error (the inequalities (15) – (17));
• zero is the mode of each distribution of comparison error (the inequalities (15), (18) – (19));
• *k*-th and *l*-th ($k \neq l$) comparisons $g_{\upsilon k}^{(\ell)}(x_i, x_j)$, $g_{\upsilon l}^{(\ell)}(x_r, x_s)$ are independent (the relationship (20)).

It is clear that:
• expected value of any comparison error can differ from zero;
• comparisons $g_{\upsilon k}^{(\ell)}(x_i, x_j)$, $g_{\upsilon k}^{(\ell)}(x_r, x_s)$ (indexes: i, j not the same as: r, s) are not assumed independent;
• number n of subsets of any family $\chi_1^{(\ell)*}, ..., \chi_n^{(\ell)*}$ is not assumed known; therefore it is also estimated parameter;
• the application of estimators proposed below is possible in the case of unknown distributions of comparison errors; it is sufficient that the assumptions about the distributions are satisfied.

The assumptions about comparison errors are weaker than those commonly used in literature. They can be relaxed, because they are sufficient – not necessary.

Two types of estimators have been discussed by the author (see Klukowski 1994, 2007, 2008b). The first one is based on total sum of inconsistencies between relation form (expressed by the function analogous to $T_{\upsilon}^{(\ell)}(x_i, x_j)$) and comparisons $g_{\upsilon k}^{(\ell)}(x_i, x_j)$, the second – based on inconsistencies between relation form and medians from comparisons of each pair. The properties of both estimators are determined under assumption that distributions of comparisons $g_{\upsilon,1}^{(\ell)}(x_i, x_j), ..., g_{\upsilon N}^{(\ell)}(x_i, x_j)$ $((x_i, x_j) \in \mathbf{X} \times \mathbf{X})$ are the same. The assumption simplifies formulas, but can be relaxed.

The first estimator is obtained on the basis of the minimization task:

$$\min_{\chi_1^{(\ell\kappa)}, ..., \chi_{r(\kappa)}^{(\ell\kappa)} \in F_X^{(\ell)}} \{ \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \left| t_{\upsilon}^{(\ell\kappa)}(x_i, x_j) - g_{\upsilon k}^{(\ell)}(x_i, x_j) \right| \}, \quad (21)$$

where:
$F_X^{(\ell)}$ $(\ell \in \{e, \tau, p\})$ - the feasible set (the family of all relations of ℓ–th type in the set **X**),
$\chi_1^{(\ell\kappa)}, ..., \chi_{r^{(\kappa)}}^{(\ell\kappa)}$ - κ-th element of the feasible set.

The second estimator is obtained on the basis of the minimization task (N – uneven):

$$\min_{\chi_1^{(\ell\kappa)},\dots,\chi_{r(\kappa)}^{(\ell\kappa)}\in F_X^{(\ell)}} \{ \sum_{<i,j>\in R_m} \left| t_\upsilon^{(\ell\kappa)}(x_i,x_j) - g_\upsilon^{(\ell,me)}(x_i,x_j) \right| \} \qquad (\ell\in\{e,\tau,p\}), \quad (22)$$

where:

$g_\upsilon^{(\ell,me)}(x_i,x_j)$ - the median from comparisons of each pair, i.e.

$$g_{\upsilon,1}^{(\ell)}(x_i,x_j),\ \dots,\ g_{\upsilon N}^{(\ell)}(x_i,x_j) \quad (<i,j>\in R_m).$$

The optimal solution of each task (21), (22), denoted – respectively - $\hat{\chi}_1^{(\ell)},\dots,\hat{\chi}_{\hat{n}}^{(\ell)}$, $\widehat{\chi}_1^{(\ell)},\dots,\widehat{\chi}_{\hat{n}}^{(\ell)}$ may be not unique, because of its discrete form; the unique solution can be selected in random way or with the use of additional criterion.

The tolerance relation can be also estimated on the basis of compound estimators, i.e. based on different comparisons:

$$\min_{\chi_1^{(\tau\kappa)},\dots,\chi_{r(\kappa)}^{(\tau\kappa)}\in F_X^{(\tau)}} \{ \sum_{<i,j>\in R_m} \sum_{k=1}^{N} \left| t_\mu^{(\tau\kappa)}(x_i,x_j) - g_{\mu k}^{(\tau p)}(x_i,x_j) \right| + \\ + \sum_{<i,j>\in R_m} \sum_{k=1}^{N} \left| t_\iota^{(\tau\kappa)}(x_i,x_j) - g_{\iota k}^{(\tau)}(x_i,x_j) \right| \}, \quad (23)$$

$$\min_{\chi_1^{(\tau\kappa)},\dots,\chi_{r(\kappa)}^{(\tau\kappa)}\in F_X^{(\tau)}} \{ \sum_{<i,j>\in R_m} \left| t_\mu^{(\tau\kappa)}(x_i,x_j) - g_\mu^{(\tau,me)}(x_i,x_j) \right| + \\ + \sum_{<i,j>\in R_m} \left| t_\iota^{(\tau\kappa)}(x_i,x_j) - g_\iota^{(\tau,me)}(x_i,x_j) \right| \}. \quad (24)$$

The compound estimators based simultaneously on binary and multivalent comparisons can be also constructed - in the case of the preference and tolerance relations.

3 Verification of assumptions about: comparisons errors, existence of relation and relation type

The estimators of relations considered are based on assumptions concerning:

- properties of distributions of comparison errors,
- existence of the relation in a set under consideration,
- relation type.

Verification of the assumptions mentioned above is based on appropriate tests. Some tests rest on the comparisons $g_{\upsilon k}^{(\ell)}(x_i,x_j)$ $(k=1,\dots,N;<i,j>\in R_m)$, remaining – on comparisons and the estimates $\hat{t}_{\upsilon}^{(\ell)}(x_i, x_j)$ or $\widehat{t}_{\upsilon}^{(\ell)}(x_i, x_j)$, because the values of the functions $T_{\upsilon}^{(\ell)}(x_i, x_j)$ are unknown.

The results of some tests are true under the assumption about errorless estimate, i.e. $T_{\upsilon}^{(\ell)}(x_i, x_j) = \hat{t}_{\upsilon}^{(\ell)}(x_i, x_j)$ or $T_{\upsilon}^{(\ell)}(x_i, x_j) = \widehat{t}_{\upsilon}^{(\ell)}(x_i, x_j)$ for all pairs $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$. The probability of such event can be determined with the use of simulation approach (see Klukowski – Control and Cybernetics, to appear). The probabilities of errors (the first and second type) in such tests have to be corrected with the value of probabilities $P(\hat{\chi}_1^{(\ell)}, ..., \hat{\chi}_{\hat{n}}^{(\ell)} \equiv \chi_1^{(\ell)*}, ..., \chi_n^{(\ell)*})$ or $P(\widehat{\chi}_1^{(\ell)}, ..., \widehat{\chi}_{\hat{n}}^{(\ell)} \equiv \chi_1^{(\ell)*}, ..., \chi_n^{(\ell)*})$ (see point 3.3 below).

Analysis of the tests results can be treated as data mining process – it confirms examined estimate or allows detection of errors, which can improve the estimate.

The tests necessary in verification process are presented in statistical literature, e.g.: Sheskin (1997), Daniel (1990), Siegel and Castellan (1988), Domański (1990, 1979).

3.1 Verification of assumptions about comparison errors

The assumptions about comparison errors comprise: ∗ independence of comparisons of the same pair, ∗ unimodality of distributions of comparisons errors, ∗ mode and median of the distributions equal zero. Some of tests for these purposes can be used for binary data only, remaining for both types, i.e. binary and multivalent.

Assumption about independence of comparisons $g_{\upsilon,1}^{(\ell)}(x_i, x_j), ..., g_{\upsilon N}^{(\ell)}(x_i, x_j)$ $(< i, j > \in R_m)$ is crucial for properties of estimates. In fact, the hypothesis requires multivariate tests. Construction such tests seem possible for large *N* only. For low or moderate *N* it is feasible to verify independence of comparisons of individual pairs (x_i, x_j); typically individual independence indicates multivariate independence. Some tests for verification of independence require the assumption about identical distribution of comparisons of individual pair. The assumption can be tested with the use of well-known tests for homogeneity of sample. The independence of comparisons of individual pairs can be verified with the use of the tests of randomness (e.g. Sheskin 1997, Test 7). The null hypothesis states randomness, the alternative – non-randomness; the tests do not require the estimates $\hat{t}_{\upsilon}^{(\ell)}(x_i, x_j)$.

The hypotheses about unimodality and zero value of mode and median ((15) – (19)) have to be verified with the use of different tests in the case of binary and multivalent comparisons.
In the case of binary comparisons all the hypotheses are equivalent to the fact that the probability $1-\delta$ of each event $\{T_b^{(\ell)}(x_i,x_j)=g_{bk}^{(\ell)}(x_i,x_j)\}$ $(\ell\in\{e,\tau,p\};\ <i,j>\in R_m;\ k=1,...,N)$ is greater than ½. The basis for the hypotheses are differences $\hat{t}_b^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)$ or $\widehat{t}_b^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)$. The result of such verification are valid in the case $\hat{t}_b^{(\ell)}(x_i,x_j)=T_b^{(\ell)}(x_i,x_j)$ or $\widehat{t}_b^{(\ell)}(x_i,x_j)=T_b^{(\ell)}(x_i,x_j)$ and, therefore, the probabilities of errors in tests have to be corrected by the probability of errorless estimate, i.e. $P(\hat{\chi}_1^{(\ell)},...,\hat{\chi}_{\hat{n}}^{(\ell)}\equiv\chi_1^{*(\ell)},...,\chi_n^{*(\ell)})$ or $P(\widehat{\chi}_1^{(\ell)},...,\widehat{\chi}_{\hat{n}}^{(\ell)}\equiv\chi_1^{*(\ell)},...,\chi_n^{*(\ell)})$. The hypothesis under consideration can be verified on the basis of the binomial distribution. The null and alternative hypotheses assume the form – respectively: H_0: $\delta\geq{}^1{}_2$ and H_1: δ<½. Rejection null hypothesis for all pairs (x_i,x_j) $(<i,j>\in R_m)$ does confirm the assumptions about the probability $1-\delta$. It is obvious that some number of opposite results (errors in the tests) can occur too; the fraction of such results depends on probabilities of errors in tests and the probability of errorless estimate obtaining. The unimodality of the distributions can be also verified on the basis of comparisons $g_{b,1}^{(\ell)}(x_i,x_j),...,g_{bN}^{(\ell)}(x_i,x_j)$ $(<i,j>\in R_m)$, i.e. without estimates $\hat{t}_b^{(\ell)}(\cdot)$ or $\widehat{t}_b^{(\ell)}(\cdot)$.

The case of multivalent comparisons is more complex, because the distributions of comparison errors are not determined, in general, by one parameter. The hypotheses about: unimodality and values of mode and median equal zero have to be verified with the use of different tests.

Unimodality of distributions of comparison errors $T_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)$ $(\ell\in\{p,\tau\};\ <i,j>\in R_m;\ k=1,...,N)$ can be verified on the basis of differences $\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)$ or $\widehat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)$ $(k=1,...,N)$. The results have to be corrected by the probability of errorless estimate. The tests for such purpose are based on multinomial distribution, test statistics have asymptotic Gaussian distribution and verify, simultaneously, the value of mode. The hypothesis can be also verified on the basis of comparisons $g_{\mu,1}^{(\ell)}(x_i,x_j)$, …, $g_{\mu N}^{(\ell)}(x_i,x_j)$ only.

The hypothesis that the median of comparison errors is equal zero can be verified similarly way.

3.2 Verification of existence of the relation

Verification of existence of the relation (equivalence or preference or tolerance) in the set **X** has to be done after positive results of tests verifying properties of distributions of comparison errors. The null hypothesis H_0 assumes the form: there exists the relation in estimated form in the set **X**, the alternative H_1 – the relation does not exist in the set **X**. Non-existence of the relation means: randomness or other data structure. The basis for verification are: an estimate (of the relation) and differences between the estimate and comparisons. The simulation approach is useful tool for tests construction.

The case of binary comparisons is simpler than multivalent; the simulation approach is also easier and more efficient in this case.
A basis for "global" verification of null hypothesis is the sum of differences between an estimate and comparisons, i.e.:

$$\frac{1}{N}\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|\hat{t}_b^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)\right| \text{ or } \frac{1}{N}\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|\hat{t}_b^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)\right|. \quad (25)$$

In the case of known δ it is possible to determine the evaluation of expected value and variance of the sums (25) – in similar way, as in Klukowski 2008c, 2010b. The evaluations of expected value and variance of the first sum assume, in the case of errorless estimate and comparisons with values from the set {0, 1}, the form:

$$E\Big(\frac{1}{N}\Big(\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|\hat{t}_b^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)\right|\Big)\Big)\le \frac{1}{2}m(m-1)\delta\,, \quad (26)$$

$$Var\Big(\frac{1}{N}\Big(\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|\hat{t}_b^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)\right|\Big)\Big)\le \frac{1}{N}\Big(\frac{1}{2}m(m-1)+2L_{\mathrm{cov}}\Big)\delta(1-\delta)\,, \quad (27)$$

where:

L_{cov} - the number of positive covariances between comparisons $g_{bk}^{(\ell)}(x_i,x_j)$ and $g_{bk}^{(\ell)}(x_r,x_s)$ $(<i,j>\neq<r,s>)$.

The term $2L_{\mathrm{cov}}$ results from the fact that each covariance between two comparisons is not greater than $\delta(1-\delta)$. The number L_{cov} is lower than $(\frac{1}{2}m(m-1))^2$; in the case of existence of non-correlated comparisons, especially for pairs (x_i,x_j) and (x_r,x_s) $(r\neq i,j;\, s\neq i,j)$ (see Klukowski 1994, Lemma 2) or negatively correlated, it is significantly lower.

The evaluations (26), (27) allow construction of a test based of the Chebyshev inequality:

$$P(\left|X-E(X)\right|\ge\lambda\sigma_x)\le\frac{1}{\lambda^2}\,, \quad (28)$$

where:

X – a random variable,

$E(X), \sigma_x$ - an expected value and standard deviation of a variable X,

k – a coefficient k>0.

The test is constructed in similar way, as in Klukowski (2008c, 2010b); the test statistics assumes the form:

$$\frac{1}{N}\left(\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|\hat{t}_b^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)\right|\right), \tag{29}$$

i.e. expresses differences between an estimate and comparisons. The test rejects the hypothesis about existence of the relation in the case of excessive value of differences (29). The probabilities of errors in the test have to be corrected by the probability of errorless estimate.

The distribution functions of the sums (25) can be also generated with the use of simulation approach. The basis for the approach are: estimates $\hat{t}_b^{(\ell)}(x_i,x_j)$ or $\widehat{t}_b^{(\ell)}(x_i,x_j)$ and the value of δ. The distributions allow construction one-sided critical regions; therefore, tests based on simulation approach should guarantee the power higher, than those based on the inequalities (26) or (27) and the Chebyshev inequality.

The case of multivalent comparisons is more complicated, because distributions of comparison errors are usually unknown; a precise estimation of these distributions is often unrealizable. Thus, it is suggested to replace unknown distributions by "quasi-uniform" distributions (see e.g. Klukowski 2008b) defined by the conditions:

- the case $\hat{t}_\mu^{(\ell)}(x_i,x_j)\neq\pm(\hat{n}-1)$

$$\begin{aligned}
&P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)<0)=P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)>0),\\
&P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)=-1)=\ldots=P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-(\hat{n}-1)),\\
&P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)=1)=\ldots=P(\hat{t}_\mu^{(\ell)}(x_i,x_j)+(\hat{n}-1)),\\
&P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)=0)=\\
&=\max\{P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)=1),P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)=-1)\},\\
&\sum_{l=\hat{t}_\mu^{(\ell)}(\cdot)-(\hat{n}-1)}^{\hat{t}_\mu^{(\ell)}(\cdot)+(\hat{n}-1)}P(\hat{t}_\mu^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)=l)=1,
\end{aligned} \tag{30}$$

• the case $\hat{t}_{\mu}^{(\ell)}(x_i, x_j) = \pm(\hat{n}-1)$

$P(\hat{t}_{\mu}^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) = 0) = \frac{1}{2} + \varepsilon \quad (\varepsilon \in (0, \frac{1}{2})),$

$$P(\hat{t}_{\mu}^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) = \pm l) = \frac{1-2\varepsilon}{4(\hat{n}-1)} \quad (l = 1, ..., 2(\hat{n}-1)). \tag{31}$$

Any distribution defined by (30), (31) satisfies the conditions (16) – (19) and has the following features: • symmetry (equal values) of left and right tie, • equality of each probability in the left and right tie. Moreover, such distribution has a variance close to the highest possible one and, therefore, provides the basis for conservative tests. The distributions defined by (30), (31) allow simulations of comparisons and finally obtaining (simulated) distributions of sums:

$$\frac{1}{N} \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \left| \hat{t}_{\mu}^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) \right| \text{ or } \sum_{<i,j> \in R_m} \left| \hat{t}_{\mu}^{(\ell)}(x_i,x_j) - g_{\mu}^{(\ell,me)}(x_i,x_j) \right|. \tag{32}$$

The tests based on distributions of sums (25) or (32) are efficient in the case of good precision of estimation; the values (25), (32) corresponding to such estimates are close to zero. In opposite case they ought to be completed with tests based on another properties of estimates. The examples of such properties are: • positive correlation of comparisons $g_{\upsilon k}^{(\ell)}(x_i, x_r)$ $(x_i \in \hat{\chi}_q; card(\hat{\chi}_q) \geq 2;$ $q = 1, ..., \hat{n};\ 1 \leq k \leq N;\ r = 1, ..., m;)$, i.e. elements belonging to the same subset $\hat{\chi}_q$, • positive correlation of comparisons $g_{\upsilon,1}^{(\ell)}(x_i,x_r), ..., g_{\upsilon N}^{(\ell)}(x_i,x_r)$ $(1 \leq i \leq m;\ r = 1, ..., m)$. The null hypotheses H_0 states lack of correlation (randomness), the alternative H_1 – positive correlation. Rejecting null hypotheses for all correlations validates the estimates of the relations. However, some fraction of non-positive correlations can occur too. It is clear that the approach based on sums (25), (32) verifies global features of relations, while those based on correlations – partial, corresponding to individual subsets of comparisons.

In the case of the preference relation, especially in the case $\hat{n} = m$, the estimates of ranks of individual elements can be applied in verification of relation existence. In particular, ranks of each element, obtained on the basis of individual subsets of comparisons $g_{\upsilon k}^{(p)}(x_i,x_j)$ and $g_{\upsilon l}^{(p)}(x_i,x_j)$ $(k \neq l;\ <i,j> \in R_m)$ have to be positively correlated. Denoting by $\hat{r}_{ik}^{(p)}$ $(i = 1, ..., m; 1 \leq k \leq N)$ ranks of i-th element, obtained on the basis of k-th subset of comparisons $g_{\upsilon k}^{(p)}(x_i,x_j)$ $(1 \leq k \leq N; <i,j> \in R_m)$ one can verify the hypotheses about positive correlation of ranks: $\hat{r}_{1,k}^{(p)}$, …, $\hat{r}_{mk}^{(p)}$ and $\hat{r}_{1,l}^{(p)}$, …, $\hat{r}_{ml}^{(p)}$ $(l \neq k)$. Moreover, it is also possible to verify the hypothesis about positive correlation (concordance) of the matrix of all ranks (see also Raghavachari 2005):

$$\Re' = \begin{bmatrix} \hat{r}_{11}^{(p)} & \cdots & \hat{r}_{1,N}^{(p)} \\ & \cdots & \\ \hat{r}_{m,1}^{(p)} & \cdots & \hat{r}_{mN}^{(p)} . \end{bmatrix}, \tag{33}$$

where: $\mathbf{\Theta}'$ - matrix $\mathbf{\Theta}$ transposed.

The test for concordance hypothesis is based on statistics *W* or *S*, defined as follows:

$$W = \frac{12S}{m^2(N^3 - N)}, \tag{34}$$

$$S = \sum_{k=1}^{N}(R_k - \sum_{k=1}^{N} R_k \Big/ N)^2, \tag{35}$$

$$R_k = \sum_{i=1}^{m} \hat{r}_{ik} .$$

Null hypothesis H_0 states randomness, alternative H_1 positive correlation. Rejecting null hypotheses, especially the case correlation close to one, confirms existence of the relation. The first statistics (34) is appropriate for $m>7$, critical values are determined on the basis of (asymptotic) chi-square distribution. The second one (35) is appropriate for $3 \le m \le 7$ and $3 \le p \le 20$; critical values are presented, e.g. in Siegel (multivariate Kendal's rank test).

In the case under consideration, i.e. the preference relation with $\hat{n} = m$, many other features of estimate of the relation can be applied in validation process, e.g.: the mode and median of comparisons $g_{\mu,1}^{(p)}(x_i, x_r), \ldots, g_{\upsilon N}^{(p)}(x_i, x_r)$ $(x_i \in \hat{\chi}_q^{(p)}; 1 \le q \le \hat{n}; r > i)$ are lower than comparisons $g_{\mu,1}^{(p)}(x_j, x_r), \ldots, g_{\mu N}^{(p)}(x_j, x_r)$ $(x_j \in \hat{\chi}_s^{(p)})$ for $(s > q)$.

The tests described above can be also applied in the case of multiple solutions of the tasks (21), (22); they allow rejecting invalid estimates. Positive results of all tests, confirm existence of the relation, negative - the opposite conclusion.

3.3 Testing relation type - weak or strict preference relation

The estimators of the relations discussed are based on the assumption that the type of the relation is known. Sometimes, the knowledge about relation type may be incomplete; typical problem of that type is distinguishing between: the equivalence relation or tolerance relation (non-overlapping or overlapping classification - see Gordon 1999, Chapt. 5; Klukowski 2006). A similar problem is distinguishing of weak and strict form of the preference relation; an example of the problem is a set of κ realizations of ν random variables with different expected values. If $\kappa = \nu$ then strict form of the relation exist, in a case $\kappa > \nu$ - weak form.

The idea of a test for distinguishing strict and weak form of the preference relation, proposed in this point, is close to those applied to equivalence and tolerance relations in the case of binary comparisons.

The test statistics is a function of inconsistencies (differences) between comparisons $g_{bk}^{(p)}(x_i,x_j)$ and estimates $\hat{t}_b^{(pw)}(x_i,x_j)$, $\hat{t}_b^{(ps)}(x_i,x_j)$ or $\widehat{t}_b^{(pw)}(x_i,x_j)$, $\widehat{t}_b^{(ps)}(x_i,x_j)$ expressing - respectively - weak and strict form of the relation, resulting from one of the estimators defined. Strict form is obtained on the basis of the task (21) or (22), with the feasible set satisfying the condition $n = m$. Weak form is obtained on the basis of the task with the feasible set satisfying the condition $n \le m-1$. The differences are determined for the set $\hat{I}_p = \{<i,j> \mid \hat{t}_b^{(pw)}(x_i,x_j) \ne \hat{t}_b^{(ps)}(x_i,x_j)\}$ or $\widehat{I}_p = \{<i,j> \mid \widehat{t}_b^{(pw)}(x_i,x_j) \ne \widehat{t}_b^{(ps)}(x_i,x_j)\}$. The test statistics $\hat{S}_N^{(\gamma)}$, corresponding to the estimates $\hat{t}_b^{(pw)}(x_i,x_j)$, assumes the form:

$$\hat{S}_N^{(\gamma)} = \frac{1}{N}\frac{1}{\#(\hat{I}_p)} \sum_{<i,j>\in \hat{I}_p} \sum_{k=1}^{N} \eta(\hat{S}_{ijk}^{(\gamma)}), \tag{36}$$

where:

$$\hat{S}_{ijk}^{(\gamma)} = \left|\hat{t}_b^{(pw)}(x_i,x_j) - g_{bk}^{(p)}(x_i,x_j)\right| - \left|\hat{t}_b^{(ps)}(x_i,x_j) - g_{bk}^{(p)}(x_i,x_j)\right|$$
$$(1 \le k \le N;\ <i,j>\in \hat{I}_p), \tag{37}$$

and:

γ - index indicating actual relation type – weak *w* or strict *s*,

$\#(\hat{I}_p)$ – number of elements of the set $\hat{I}_p$,

$\eta(\hat{S}_{ijk}^{(\gamma)})$ - the function defined as follows:

$$\eta(\hat{S}_{ijk}^{(\gamma)}) = \begin{cases} -1 \ \ if\ \hat{S}_{ijk}^{(\gamma)} \le 0; \\ \ \ 1 \ \ if\ \hat{S}_{ijk}^{(\gamma)} > 0. \end{cases}$$

The properties of the statistics $\hat{S}_N^{(\gamma)}$ depend on actual relation type in the set **X**. In the case of weak form of the relation $\chi_1^{*(pw)},...,\chi_{n_w}^{*(pw)}$ and errorless result of estimation, i.e. $\hat{\chi}_1^{(pw)},...,\hat{\chi}_{\hat{n}_w}^{(pw)} \equiv \chi_1^{*(pw)},...,\chi_{n_w}^{*(pw)}$, the equalities $\hat{t}_b^{(pw)}(x_i,x_j) = T_b^{(pw)}(x_i,x_j)$ $(<i,j>\in I_w)$ hold; the probability of such event is denoted $P(\hat{\chi}_1^{(pw)},...,\hat{\chi}_{\hat{n}_w}^{(pw)} \equiv \chi_1^{*(pw)},...,\chi_{n_w}^{*(pw)} \mid \mathbf{R}^{(pw)})$. In the case of strict form of relation $\chi_1^{*(ps)},...,\chi_{n_s}^{*(ps)}$ and errorless result of estimation, i.e. $\hat{\chi}_1^{(ps)},...,\hat{\chi}_{\hat{n}_s}^{(ps)} \equiv \chi_1^{*(ps)},...,\chi_{n_s}^{*(ps)}$, the equalities $\hat{t}_b^{(ps)}(x_i,x_j) = T_b^{(ps)}(x_i,x_j)$ $(<i,j>\in \hat{I}_p)$ hold; the probability of such event is denoted $P(\hat{\chi}_1^{(ps)},...,\hat{\chi}_{\hat{n}_s}^{(ps)} \equiv \chi_1^{*(ps)},...,\chi_{n_s}^{*(ps)} \mid \mathbf{R}^{(ps)})$. The variables $\eta(\hat{S}_{ijk}^{(\gamma)})$ $(<i,j>\in \hat{I}_p)$ assume values from the set $\{-1,1\}$. It can be shown, in the same way as in Klukowski 2006, that in the case of weak form of relation, the expected value and variance of the variables $\hat{S}_{ijk}^{(\gamma)}$ satisfy the inequalities:

$$E(\hat{S}_{ijk}^{(w)}) \le -1+2\delta, \tag{38}$$

$$Var(\hat{S}_{ijk}^{(w)}) \le 4\delta(1-\delta). \tag{39}$$

The expected value and the variance of the variable $\hat{S}_N^{(w)}$ fulfill the relationships:

$$E(\hat{S}_N^{(w)}) \le -1+2\delta, \tag{40}$$

$$Var(\hat{S}_N^{(w)}) \le \tfrac{4}{N}(1-2L(\hat{I}_p)/(\#(\hat{I}_p))^2)\delta(1-\delta), \tag{41}$$

where:

$L(\hat{I}_p)$ - number of non-correlated comparisons $g_{bk}^{(p)}(x_i,x_j)$ and $g_{bk}^{(p)}(x_r,x_s)$ $(r\ne i,j;\ s\ne i,j;\ i,j,r,s\in I_p)$.

In the case of strict relation and errorless estimation result, the variance of the variable satisfy the inequality (41), while expected value $E(\hat{S}_N^{(s)})$ satisfies the condition:

$$E(\hat{S}_N^{(s)}) \ge 1-2\delta. \tag{42}$$

The inequalities (38) – (42) and the Chebyshev's inequality (28) indicate the form of tests for both forms of the relation. The null and alternative hypotheses for weak preference relation, versus strict relation, assumes the form:

$$H_{w,0}: E(\hat{S}_N^{(w)}) \le -1+2\delta, \tag{43}$$

$$H_{w,1}: E(\hat{S}_N^{(w)}) > 1-2\delta, \tag{44}$$

with a critical region:

$$\Lambda_N^{(w)} = \{\hat{S}_N^{(w)} \mid \hat{S}_N^{(w)} > 2\delta-1+\lambda\sigma_S\}, \tag{45}$$

where:

λ - the parameter obtained on the basis of Chebyshev inequality (28) (see Klukowski 2006),

σ_S - square root of the expression $\frac{4}{N}(1-2L(I_w)/(\#(I_w))^2)\delta(1-\delta)$.

The hypotheses verifying strict relation, versus weak one, assumes the form:

$$H_{s,0}: E(\hat{S}_N^{(s)}) \geq 2\delta - 1, \quad (46)$$

$$H_{s,1}: E(\hat{S}_N^{(s)}) < -1 + 2\delta, \quad (47)$$

with a critical region:

$$\Lambda_N^{(s)} = \{\hat{S}_N^{(s)} \mid \hat{S}_N^{(s)} < 1 - 2\delta - \lambda\sigma_S\}. \quad (48)$$

The evaluation of probability of the first type error of the test for weak preference relation assume the form (see Klukowski 2006):

$$1 - (1 - \alpha_w) P(\hat{\chi}_1^{(pw)}, \ldots, \hat{\chi}_{\hat{n}_w}^{(pw)} \equiv \chi_1^{*(pw)}, \ldots, \chi_{n_w}^{*(pw)} \mid \mathbf{R}^{(pw)}),$$

where: α_w the probability resulting from the evaluation (28).

The evaluation of probability of the second type error of the test for weak preference relation assume the form (see Klukowski 2006):

$$1 - (1 - \beta_w) P(\hat{\chi}_1^{(pw)}, \ldots, \hat{\chi}_{\hat{n}_w}^{(pw)} \equiv \chi_1^{*(pw)}, \ldots, \chi_{n_w}^{*(pw)} \mid \mathbf{R}^{(pw)}), \quad (49)$$

where: β_w the probability resulting from the evaluation (28).

The evaluations of probabilities of both errors require the probabilities of errorless estimation results, because the relationships (38) – (42) are valid in the case of errorless estimation result. The probabilities can be determined with the use of simulation approach (see Klukowski – to appear). It is rational to verify null hypothesis about relation form, which provides lower value of criterion function.

The test based on medians instead of individual comparisons is similar to the case N=1, with appropriate probabilities of comparison errors (see Klukowski 2006).

4 Conclusions

The tests mentioned in the paper allow versatile verification of the assumptions about: • comparison errors, • existence of a relation in a set of elements, • relation type. In the case of negative result of verification it is possible to detect errors or to reject an incorrect estimate. Especially, it possible to reject incorrect estimates in the case of multiple solutions of the optimization tasks. Statistical tools – tests and estimators – allow computerization of inference.

Some tests proposed for validation of an estimate are based on comparison errors - resulting from estimates. Therefore, distributions of test statistics or their parameters have to be corrected by probability of errorless estimate. The probabilities can be determined with some precision only - on the basis of simulation approach. However, such tools provide progress in comparison with heuristic approaches.

It should be emphasized that the hypotheses considered in the paper can be verified with the use of many different tests - based on parametric or non-parametric statistics. Therefore, estimates which satisfy all tests used have to be considered as trustworthy and.

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The papers presented in this Volume 2 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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