ON THE USE OF VARIOUS FORMULATIONS OF THE EXTENDED HIGHER-ORDER SHELL THEORY IN DYNAMIC PROBLEMS FOR FUNCTIONALLY GRADED STRUCTURES

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1. Introduction

Improved theories of heterogeneous shells are required as a background for the modeling of guided waves in functionally graded thin-walled structures. The finite element simulation remains the main numerical method [2] while one of the most efficient analytical approaches uses the orthogonal series expansion together with Legendre polynomials [4, 9]. At the same time the generalized shell theory [5] allows the unified variational problem formulation as well for the traditional interpretation of refined 2D shell models as for the finite element discretization across the thickness. The convergence of approximate normal wave frequencies and waveforms based such 2D solutions was analyzed in [3] for homogeneous elastic systems. An improvement of the theory consists in the accounting for the boundary conditions on the faces interpreted as constraints for the 2D variational problem [6, 7]. The dynamic equations accounting for the full boundary reflection can be obtained by the Lagrange multiplier method or as extended Voronets' equations for constrained continuum system. The normal waves in power graded systems are studied here on the basis of various formulations of the shell theory [6, 7].

2. Basics of the extended higher order shell theory

A shell is interpreted as a material surface S furnished by the mechanical properties following from the variational solution of the dimensional reduction problem [5]. The shell model is defined on the 2D manifold within the configuration space $\Omega = \{u_i^{(k)}\}$, $k = 0 \ldots N$, $i = 1 \ldots 3$, where the field variables of the first kind are defined as biorthogonal expansion factors for the displacement vector: $u_i^{(k)} = (u_i, \mathrm{p}^{(k)})_1$, with the base system $\mathrm{p}_{(k)}(\zeta)$ depending on the thickness coordinate $\zeta\in[-1, 1]$ [5] (Legendre polynomials, trigonometric functions, or compact functions resulting in the "finite layer model"), the Lagragian density (1) and the constraints (2) [6, 7]:

$$
\begin{aligned}
\mathcal{L}(u_i^{(k)}, \dot{u}_i^{(k)}, \nabla_\alpha u_i^{(k)}) = \int_S \Big[& \tfrac{1}{2}\rho_{(k)}^{(m)} \dot{u}^i_{(m)} \dot{u}_i^{(k)} - \tfrac{1}{2h}\left(\bar{C}^{i3j\gamma}_{(km)} \nabla_\gamma u_j^{(m)} + \bar{C}^{i3j}_{(km)} u_j^{(m)}\right) D^{(\cdot k)}_{(n \cdot)} u_i^{(n)} \\
& + F^i_{(k)} u_i^{(k)} - \tfrac{1}{2}\left(\bar{\bar{C}}^{\alpha\beta j\gamma}_{(km)} \nabla_\gamma u_j^{(m)} + \bar{C}^{\alpha\beta j}_{(km)} u_j^{(m)}\right)\left(\nabla_\beta u_\alpha^{(k)} + H^{(\cdot k)}_{\beta(n\cdot)} u_\alpha^{(n)} - b_{\alpha\beta} u_3^{(k)}\right) \\
& - \tfrac{1}{2}\left(\bar{\bar{C}}^{3\beta j\gamma}_{(km)} \nabla_\gamma u_j^{(m)} + \bar{C}^{3\beta j}_{(km)} u_j^{(m)}\right)\left(\nabla_\beta u_3^{(k)} + H^{(\cdot k)}_{\beta(n\cdot)} u_3^{(n)} + b^\alpha_\beta u_\alpha^{(k)}\right)\Big] dS + \int_{\partial S} q^i_{B(k)} u_i^{(k)} d\Gamma,
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
& \bar{\bar{C}}^{i\alpha\beta}_{\pm(k)}\left(\nabla_\beta u_\alpha^{(k)} + H^{(\cdot k)}_{\beta(m\cdot)} u_\alpha^{(m)} - b_{\alpha\beta} u_3^{(k)}\right) + \bar{\bar{C}}^{i\beta 3}_{\pm(k)}\left(\nabla_\beta u_3^{(k)} + H^{(\cdot k)}_{\beta(m\cdot)} u_3^{(m)} + b^\alpha_\beta u_\alpha^{(k)}\right) \\
& \quad + \bar{C}^{i3j}_{\pm(k)} h^{-1} D^{(\cdot k)}_{(m\cdot)} u_j^{(m)} - P^i_\pm = 0, \quad i, j = 1, 2, 3;\ \alpha, \beta, \gamma, \mu, \nu = 1, 2;
\end{aligned} \tag{2}
$$

$$
\rho^{(m)}_{(k)} = \left(\rho \mathrm{p}_{(k)}, \mathrm{p}^{(m)}\right)_1, \quad \bar{\bar{C}}^{i\gamma j\beta}_{(km)} = \left(A^{\gamma\cdot}_{\cdot\nu} A^{\beta\cdot}_{\cdot\mu} C^{i\nu j\mu} \mathrm{p}_{(k)}, \mathrm{p}_{(m)}\right)_1, \quad D^{(\cdot k)}_{(n\cdot)} = \left(d\mathrm{p}_{(n)}/d\zeta, \mathrm{p}^{(k)}\right)_1, \ldots
$$

$$
\bar{\bar{C}}^{i\beta j}_{\pm(k)} = \left(\bar{C}^{i3j\beta}|_{\zeta=\pm 1} + \nabla_\gamma h_\pm \bar{\bar{C}}^{i\gamma j\beta}|_{\zeta=\pm 1}\right) \mathrm{p}_{(k)}(\pm 1).
$$

Here h is the shell thickness, $b_{\alpha\beta}$ is the curvature tensor, ∇_α are covariant derivatives on the tangent bundle $T_M S$, $A^{\beta\cdot}_{\cdot\alpha}$ are parallel shifting tensors [5]. The constraints (2) follow from the boundary conditions on the shell faces $S_\pm$ [6, 7], and the initial conditions closing the variational problem statement (1, 2) are written as follows:

$$
u_i^{(k)}(t = t_0) = U_i^{(k)}, \quad \dot{u}_i^{(k)}(t = t_0) = V_i^{(k)}. \tag{3}
$$

3. Spectral problems for graded shells and plates based on the Nth order theory

Let us consider a two-constituent isotropic power graded plate with Young moduli $E_{1,2}$ and mass densities $\rho_{1,2}$:

$$C^{\alpha\beta\gamma\delta}_{(km)} = \left[\lambda_1 a^{\alpha\beta} + \mu_1 \left(a^{\alpha\gamma} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\gamma}\right)\right] \Xi_{(km)};\ C^{3333}_{(km)} = (\lambda_1 + 2\mu_1)\,\Xi_{(km)};\ C^{\alpha\beta 33}_{(km)} = \lambda_1 a^{\alpha\beta} \Xi_{(km)};$$

(4) $$C^{\alpha 3\beta 3}_{(km)} = \mu_1 a^{\alpha\beta} \Xi_{(km)}; \quad \lambda_1 = E_1 \nu \left[(1-2\nu)(1+\nu)\right]^{-1}; \quad 2\mu_1 = E_1 (1+\nu)^{-1}; \quad E_{12} = E_1/E_2;$$

$$\Xi_{(km)} = \left(\varphi(\zeta) \mathrm{p}_{(k)}, \mathrm{p}_{(m)}\right)_1; \quad \varphi(\zeta) = f(\zeta) + (1 - f(\zeta)) E_{12}, \quad f(\zeta) = [(1+\zeta)/2]^p,\ p \in \mathbb{R}_+ \cup 0.$$

Let the waves propagate along the axis Ox_1, and let us introduce the dimensionless variables: $\tau = tc_2/h$, $\xi = x_1/h$, $\tilde{u}_i^{(k)} = u_i^{(k)}/h$. We obtain hence the dynamic equations following from (1) as Lagrange ones [5]:

(5) $$R_{(km)} \partial_\tau^2 \tilde{u}_1^{(m)} = \eta^2 \Xi_{(km)} \partial_\xi^2 \tilde{u}_1^{(m)} - D^{(\cdot n)}_{(k\cdot)} \Xi_{(ns)} \bar{D}^{(\cdot s)}_{(m\cdot)} \tilde{u}_1^{(m)} - \left[D^{(\cdot n)}_{(k\cdot)} \Xi_{(nm)} - \left(\eta^2 - 2\right) \Xi_{(kn)} \bar{D}^{(\cdot n)}_{(m\cdot)}\right] \partial_\xi \tilde{u}_2^{(m)};$$

(6) $$R_{(km)} \partial_\tau^2 \tilde{u}_2^{(m)} = \Xi_{(km)} \partial_\xi^2 \tilde{u}_2^{(m)} + \eta^2 D^{(\cdot n)}_{(k\cdot)} \Xi_{(ns)} \bar{D}^{(\cdot s)}_{(m\cdot)} \tilde{u}_2^{(m)} - \left[\left(\eta^2 - 2\right) D^{(\cdot n)}_{(k\cdot)} \Xi_{(nm)} - \Xi_{(kn)} \bar{D}^{(\cdot n)}_{(m\cdot)}\right] \partial_\xi \tilde{u}_1^{(m)};$$

$$R_{(km)} = \left(\varrho(\zeta) \mathrm{p}_{(k)}, \mathrm{p}_{(m)}\right)_1; \quad \varrho(\zeta) = f(\zeta) - (1 - f(\zeta)) \rho_{12}; \quad \rho_{12} = \rho_1/\rho_2; \quad \eta = c_1/c_2;$$

$c_1^2 = (\lambda_1 + 2\mu_1)\rho_1^{-1}$, $c_2^2 = \mu_1 \rho_1^{-1}$. The constraint equations (2) can be written for a graded plate as follows:

(7) $$\left[\left(\eta^2 - 2\right) \partial_\xi \tilde{u}_1^{(k)} + \eta^2 \bar{D}^{(\cdot k}_{(n\cdot)} \tilde{u}_3^{(n)}\right] \mathrm{p}_{(k)}(\pm 1) = 0; \quad \left(\bar{D}^{(\cdot k}_{(n\cdot)} \tilde{u}_1^{(n)} + \partial_\xi \tilde{u}_3^{(k)}\right) \mathrm{p}_{(k)}(\pm 1) = 0.$$

Substituting $\mathbf{u}^{(k)} = \mathbf{U}^{(k)} \exp[i(\kappa\xi - \omega\tau)]$ into (5, 6), we obtain the spectral problem for the power graded plate. The spectrum can be obtained as a solution of the constrained stationary values problem for two quadratic forms [8] accordingly to [1], by means of the Lagrange multiplier method [7], or transforming the system (5–7) to the generalized equations of Voronets type. The general formulation (5–7) is covariant, therefore it allows one to use any function system being a basis in Hilbert space over $[-1, 1]$ as $\mathrm{p}_{(k)}(\zeta)$. The convergence of phase frequencies and waveforms computed for various wavenumbers and power law indices on the basis of polynomial and compact expansion functions are compared. The analogous formulation for shells, e. g. for cylindrical ones is slightly more complex; such equations are constructing using computer algebra software.

Acknowledgments This investigation is the part of the State Task for Basic Researches GP 14 (0050-2017-0001), the state register number AAAA-A17-117032010136-3, and partially supported by the RFBR under grant No. 16-01-00751-a.

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