KIRCHHOFF'S PLATE BENDING ANALYSIS BY EQUILIBRIUM FINITE ELEMENT METHOD

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1. Introduction

The equilibrium approach of the finite element method, despite of some difficulties arising in construction of statically admissible stress field, provides an interesting alternative to the displacement method. This paper presents the application of the equilibrium finite element method to the static problem of Kirchhoff's plate bending. The statically admissible fields of stresses have been constructed with use of the Southwell vector stress function. The possibility of employing this function for seeking the approximate solution for structural analysis of plates was indicated by Zienkiewicz and Fraeijs de Veubeke in [3]. The utilization of this function was proposed by Morley in [4] and Elias in [2].

2. Formulation of the problem

The flat plate bending problem is governed by three types of equations, which will be presented in this paper with use of the indicial notation. Firstly, the differential equilibrium equations

$$M_{\alpha\beta,\beta} - Q_\alpha = 0, \quad Q_{\alpha,\alpha} + q = 0 \tag{1}$$

where $\alpha, \beta = 1, 2$, and $M_{\alpha\beta}$ is the tensor of bending and twisting moments, Q_α the vector of transverse forces and q the transverse distributed load. Next, the geometric relations:

$$\kappa_{\alpha\beta} = -w_{,\alpha\beta} \tag{2}$$

where $\kappa_{\alpha\beta}$ denotes the tensor of curvature and w a function of deflection of plate's middle surface. Finally, the constitutive relation which takes form of the generalized Hooke's law:

$$\kappa_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\, M_{\gamma\delta} \tag{3}$$

where $C_{\alpha\beta\gamma\delta}$ indicates a compliance tensor. Equations (1) – (3) should be supplemented with boundary conditions which may be of kinematic, static or mixed nature. The strong form of the problem presented above may also be set in the variational form as the complementary work principle which can be obtained by multiplication of geometric relation by variation of tensor of bending and twisting moments and integration of it over the plate area. This also corresponds to minimization of the complementary energy functional

$$\mathit{\Pi}_\sigma = \frac{1}{2}\int\limits_{\mathit{\Omega}} M_{\alpha\beta}\, C_{\alpha\beta\gamma\delta}\, M_{\gamma\delta}\,\mathrm{d}x - \int\limits_{\mathit{\Gamma}_\varphi} M_n\,\bar{\varphi}_n\,\mathrm{d}s - \int\limits_{\mathit{\Gamma}_w} (Q_n + M_{ns,s}\,)\,\bar{w}\,\mathrm{d}s \tag{4}$$

on the following set of statically admissible moment fields

$$Y = \left\{ M_{\alpha\beta} \in L^2(\mathit{\Omega}) : M_{\alpha\beta,\alpha\beta} + q = 0 \text{ in } \mathit{\Omega},\;\; M_n = M_0 \;\&\; Q_n + M_{ns,s} = Q_0 \text{ on } \mathit{\Gamma}_\sigma \right\}.$$

where $\bar{\varphi}_n$ and $\bar{w}$ are the angle and deflection functions given on boundaries $\mathit{\Gamma}_\varphi$ and $\mathit{\Gamma}_w$, respectively, and symbol L^2 denotes the space of square integrable functions.

3. Equilibrium approach

The equilibrium equations are satisfied in internal points of plate's area Ω by virtue of Southwell's vector stress function [5], with components U_α. The equilibrium equations will be identically met provided that the bending and twisting moments are determined by Southwell's function's components as follows:

$$M_{\alpha\beta} = \frac{1}{2}\left(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}U_{\gamma,\delta} + \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}U_{\delta,\gamma}\right) - \delta_{\alpha\beta}P_0. \tag{5}$$

where P_0 is the solution of the Poisson equilibrium equation $(1)_2$, which may be either arbitrary presupposed in an analytic form or found numerically. The Southwell stress functions have been approximated with shape functions of class C_0 by use of triangular elements. The solution is found by minimization of the complementary energy and the boundary conditions have been satisfied by use of Lagrange's multiplier method.

4. Numerical example

A square uniformly loaded plate with the bottom edge free and remaining three edges clamped has been considered. The results have been compared with the ones received by means of the Hsieh-Clough-Tocher macro-element, e.g. [1], by the displacement approach. It is shown (see Figure 1) that both methods lead to similar results, but the fact worth highlighting is that the equilibrium approach allows one to find the upper bound of the strain energy when only external forces are prescribed, whereas the displacement method leads to the lower bound. The convergence rate is similar for each of these two elements. With use of both the methods the relative error of approximate solution has been calculated on basis of Synge's hypercycle method. It reaches small values and in the case of the densest mesh barely exceeds 1%.

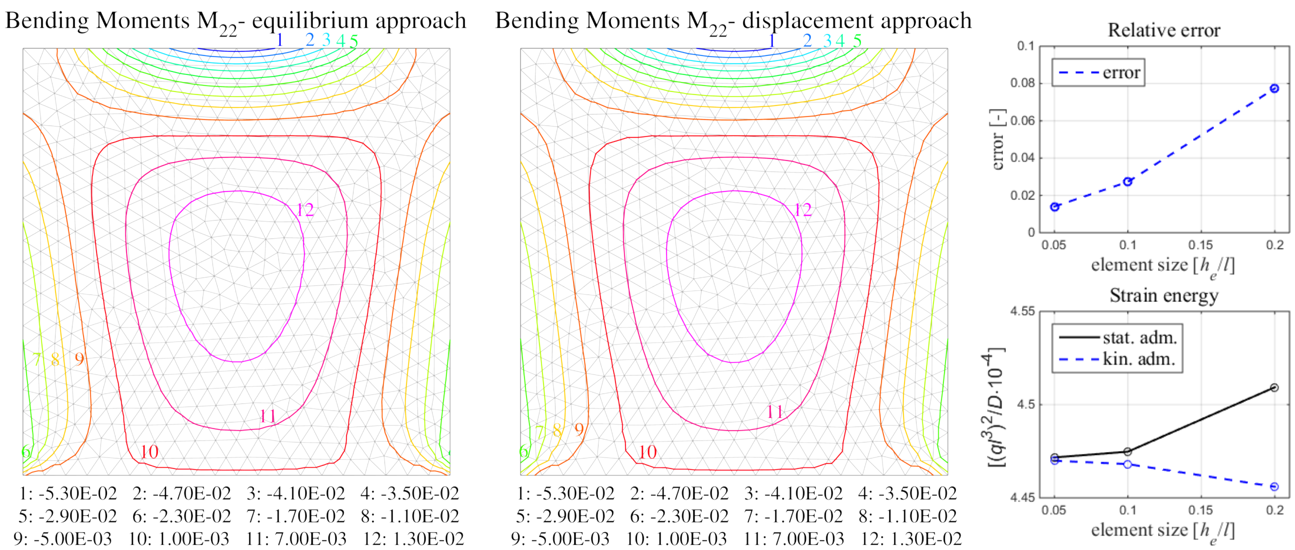

Figure 1: Statically and kinematically admissible solutions (M_{22}), relative error of solution and strain energy

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