Archives of Mechanics • Archiwum Mechaniki Stosowanej • 24, 4, pp. 655–658, Warszawa 1972

BRIEF NOTES

A new approach to the mode approximation for impulsively loaded rigid-plastic structures

T. WIERZBICKI (WARSZAWA)

THE MODE approximation technique due to MARTIN and SYMONDS is reformulated to satisfy stress boundary conditions and determine uniquelly the mode shape from the properly posed eigenvalue problem. The method is illustrated on the example of impulsively loaded clamped circular plate.

Introduction

IN THE DYNAMIC theory of plasticity valuable information about the actual solution is often obtained by considering a simplifying solution which violates some of the field equations and/or boundary or initial conditions. Among such approximate solution an important role plays so called "mode approximation technique" [2, 3], because of its generality and striking simplicity. Assuming the velocity field $\dot{u}_i\ (x, t)$ in the separable form

$$\dot{u}_i(x, t) = \dot{T}(t)\Phi_i(x), \tag{1}$$

MARTIN and SYMONDS were able to satisfy velocity boundary conditions, yield condition and associated flow rule. The equation of motion was satisfied only in the global sense (integrated over the surface of the structure). The initial condition $\dot{u}_i^0(x)$ and stress boundary conditions were not satisfied but the minimalization procedure was suggested to compute the best value for the initial amplitude $\dot{T}(0)$ of the velocity field, whose shape was imposed by (1)

$$\dot{T}(0) = \frac{\int_s \dot{u}_i^0(x)\Phi_i(x)\,ds}{\int_s \Phi_i(x)\Phi_i(x)\,ds}. \tag{2}$$

The mode shape is not found from the field equation but is chosen more or less arbitrarily. In such formulation this theory bears close resemblance to the kinematical approach in the static limit analysis. While in static problems the freedom in the choice of $\Phi_i(x)$ is advantageous because one of the mode shape gives the exact value of the collapse load, no single mode response exists in general for dynamic problems. The determination of the best function $\Phi_i(x)$ remains the central point in the application of the mode approximation technique and cannot always be easily accomplished.

In the present note an alternative formulation of the Martin and Symonds approach is presented in which the mode shape is computed as a part of the approximate solution. In addition, the present method allows to satisfy all velocity and stress boundary conditions.

Method

CONSIDER a rigid-plastic structure with imposed initial velocity distribution $\dot{u}_i^0(x)$ and assume that during the motion no work is done over the boundary of the structure (mixed boundary conditions). If the generalized notation is used, the equation of motion can be written in the form([1])

$$L_{ij}Q_j = \mu\ddot{u}_i, \tag{3}$$

where the space differential operator L_{ij} is defined for each particular structure. It is further assumed that the yield condition in the Q_j space forms a closed and smooth hypersurface

$$F(Q_j) = 0. \tag{4}$$

In the loading regions the direction of the strain rate vector is uniquelly determined from the associated flow rule

$$\dot{q}_j = \mu\frac{\partial F}{\partial Q_j}. \tag{5}$$

Using (4), the constitutive Eq. (5) can be inverted to give

$$Q_j = \frac{1}{\nu(\dot{q}_i)}\frac{\partial G}{\partial \dot{q}_j}, \tag{6}$$

where $G(\dot{q}_j)$ is a potential function for the stresses, and $\partial G/\partial\dot{q}_j$ is a linear function of $\dot{q}_j$.

As an approximate solution of the field equation (3)–(5), a separable velocity field is assumed

$$\dot{u}_i(x,t) = \dot{T}(t)\Phi_i(x,\lambda), \tag{7}$$

which differs from (1) by the occurrence of the parameter (eigenvalue) λ. The strain rate field, resulting from (7) is also of the separable form

$$\dot{q}_j(x,t) = \dot{T}(t)\Psi_j(x,\lambda), \tag{8}$$

provided the geometrical relations are linear. The functions $\Psi_j(x,\lambda)$ can be computed for each particular structure once the mode shape is known.

As an approximation to (6), the components of the stress tensor are assumed in the form

$$Q_j = \frac{\alpha}{\dot{T}(t)}\frac{\partial G}{\partial \dot{q}_j} = \alpha\frac{\partial G(\Psi_i)}{\partial \Psi_j}, \tag{9}$$

([1]) Repeated indices indicate summation.

where the variable scalar function $\dfrac{1}{\nu(\dot{q}_i)}$ has been replaced by a constant coefficient α. The quantities to be determined are thus $\dot{T}(t)$, $\Phi_i(x, \lambda)$, λ, and α. It is not possible that the approximate solutions (7)–(9) satisfy all governing equations.

It turns out that equations of motion, constitutive equations and geometrical relation can be satisfied exactly, while the yield condition only in the global sense. The Eq. (4) integrated over the volume of the structure will serve to determine the coefficient α.

The remaining unknowns $\dot{T}(t)$, Φ_i, λ are found from the properly posed eigenvalue problem for Eq. 3. Substituting (7) and (9) into (3), we have

$$L_{ij}\left[\alpha \frac{\partial G(\Psi_i)}{\partial \Psi_j}\right] = \mu \ddot{T}(t)\Phi_i(x, \lambda), \tag{10}$$

or

$$\frac{\alpha}{\Phi_i(x, \lambda)} L_{ij}\left[\frac{\partial G(\Psi_i(x, \lambda))}{\partial \Psi_j}\right] = \mu \ddot{T}(t). \tag{11}$$

The left-hand side of Eq. (11) depends solely upon the space coordinate while the right-hand side is a time function; the equality is possible only if these expressions are constant. This leads to the classical eigenvalue problem

$$L_{ij}\left[\frac{\partial G}{\partial \Psi_j}\right] = \lambda \Phi_i, \tag{12}$$

$$\mu \ddot{T}(t) = -\frac{\lambda}{\alpha}. \tag{13}$$

The function $\partial G/\partial \Psi_j$ is a linear function of $\Psi_j(x, \lambda)$ and in view of the geometrical relation depends linearly on $\Phi_i(x, \lambda)$. Equation (12) is a linear ordinary differential equation and the sought mode shape Φ_i is a first eigenfunction. The integration constants and eigenvalue λ are uniquelly determined from the assumed set of boundary conditions.

From Eq. (12) it follows that the amplitude $\dot{T}(t)$ is a linear function of time

$$\dot{T}(t) = \dot{T}(0) - \frac{\lambda}{\alpha \mu} t, \tag{14}$$

but the slope of this line differs in general form that determined in [2, 3].

Comparing the present method with that of Martin and Symonds, it is difficult to say which one leads to better results. In both approaches some equations are satisfied locally and some globally. While the mathematics involved in the original paper [2] is definitely simpler, the present modified method provides means for finding uniquelly the mode shape and satisfying stress boundary conditions.

Example

CONSIDER an impulsively loaded circular plate obeying Huber-Mises yield condition. The mode shape was found to be $\Phi(r, \lambda) = J_0(\lambda r) - \dfrac{J_0(\lambda)}{I_0(\lambda)} I_0(\lambda r)$, where $J_0(\lambda r)$ and $I_0(\lambda r)$ denote Bessel function respectively of real and imaginary arguments. In the case of a simple support $\alpha = 0{,}21$, $\lambda_1 = 2.28$, $\dot{T}(0) = 1.63I/\mu$, where I is the applied impulse. Corresponding values for the clamped plate are $\alpha = 0.495$, $\lambda_1 = 1.96$, $\dot{T}(0) = 1.65\,I/\mu$. The resulting finite central deflection is $\delta = 0{,}93\dfrac{I^2R^2}{8\mu M_0}$ for simply supported plate and $\delta = 0.64\dfrac{I^2R^2}{8\mu M_0}$ for clamped plate which compares favourably with known exact solutions of the same problem [1, 4].

References

1. H. G. HOPKINS, W. PRAGER, *On the dynamics of plastic circular plates*, ZAMP, **4**, 317, 1954.
2. J. B. MARTIN, P. S. SYMONDS, *Mode approximation for impulsively loaded rigid-plastic structures*, J. Eng. Mech. Div., Proc. ASCE, **92**, EM 5, 43, 1966.
3. J. B. MARTIN, *Mode approximations for impulsively loaded structures in the inelastic range*, Proc. Southampton Civil Eng. Conf., 1969, Part 2, Ed. T. Te'eni.
4. A. J. WANG, H. G. HOPKINS, *On the plastic deformation of built-in circular plate under impulsive loading*, J. Mech. Phys. Solids, **3**, 22–37, 1954.

INSTITUTE OF FUNDAMENTAL TECHNICAL RESEARCH
POLISH ACADEMY OF SCIENCES

Received March 28, 1972