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The axisymmetric Boussinesq problem in the micropolar theory of elasticity[1]

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GENERAL solution to the problem of identation of semi-infinite micropolar elastic medium by an axisymmetric rigid punch has been obtained. The Hankel transform is used and the problem is reduced to the solution of the dual integral equations. The discussion of the limit solution for $\alpha \ll 1$ is given.

Uzyskano ogólne rozwiązanie zagadnienia wciskania osiowo-symetrycznego sztywnego narzędzia w półnieskończony mikropolarny ośrodek sprężysty. Zastosowano transformację Hankela i sprowadzono problem do rozwiązania dualnych równań całkowych. Przedyskutowano szczegółowo pewien przypadek rozwiązania granicznego dla $\alpha \ll 1$.

Получено общее решение задачи о вдавливании осесимметричного жесткого инструмента в полубесконечную микропслярную среду. При помощи трансформации Ганкеля задача сведена к решению системы двойственных интегральных уравнений. Подробно обсужден некоторый частный случай предельного решения при $\alpha \ll 1$.

1. Introduction

THE ASYMMETRIC theory of elasticity initiated by VOIGT [1] in 1887 and further developed by E. COSSERAT and F. COSSERAT [2] in 1909, has for several and diverse reasons aroused renewed and growing interest during recent years. The basic equations of the linear micropolar theory of elasticity have been given by KUVCHINSKI and AERO [3], PALMOV [4] and ERINGEN and SUHUBI [5].

The axisymmetric Lamb's problem in a semi-infinite micropolar elastic solid has been solved recently by NOWACKI [6]. PURI [7] and DHALIWAL [8] have obtained solutions respectively for stress concentration and thermoelastic problems for a semi-infinite micropolar elastic solid.

In the present paper, a general solution has been obtained to the problem of indentation of a semi-infinite micropolar elastic medium by an axisymmetric rigid punch.

2. The basic equations

For a homogeneous isotropic centrosymmetric linear-elastic body occupying a region V, we have the following basic equations; the equations of motion:

$$\begin{aligned} \sigma_{ji,j} + \rho X_i &= \rho \ddot{u}_i, \\ \mu_{ji,j} + \varepsilon_{ijk}\sigma_{jk} + JY_i &= J\ddot{w}_i, \end{aligned} \tag{2.1}$$

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the kinematic relations:

$$\beta_{ij} = w_{j,i}, \qquad \gamma_{ij} = u_{j,i} + \varepsilon_{kji} w_k, \tag{2.2}$$

the linear constitutive law:

$$\sigma_{ij} = \lambda \gamma_{kk} \delta_{ij} + 2\mu \gamma_{(ij)} + 2\alpha \gamma_{[ij]}, \qquad \mu_{ij} = \beta \beta_{kk} \delta_{ij} + 2\gamma \beta_{(ij)} + 2\varepsilon \beta_{[ij]}, \tag{2.3}$$

where in these equations, we have used the following notation:

σ_{ij} stress tensor components,
μ_{ij} couple-stress tensor components,
u_i displacement field components,
w_i rotation field components,
X_i body force components,
Y_i body couple components,
γ_{ij} strain tensor components,
β_{ij} curvature twist tensor components,
ε_{ijk} unit antisymmetric tensor,

[ ] and ( ) indicate respectively the skew symmetric and symmetric parts of a tensor, $\lambda, \mu, \alpha, \beta, \gamma, \varepsilon$ are the elastic constants of the micropolar material, ϱ is the density, J is the rotational inertia, and the dots denote the time derivatives.

Using (2.2) and (2.3) in (2.1) to eliminate σ_{ij}, μ_{ij}, β_{ij} and γ_{ij}, we obtain the system of six differential equations, which we represent in vector form:

$$\begin{aligned} (\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} - (\mu+\alpha)\nabla\times\nabla\times\mathbf{u} + 2\alpha\nabla\times\mathbf{w} + \varrho\mathbf{X} &= \varrho\ddot{\mathbf{u}}, \\ (\beta+2\gamma)\nabla\nabla\cdot\mathbf{w} - (\gamma+\varepsilon)\nabla\times\nabla\times\mathbf{w} + 2\alpha\nabla\times\mathbf{u} - 4\alpha\mathbf{w} + JY &= J\ddot{\mathbf{w}}. \end{aligned} \tag{2.4}$$

These equations are coupled and it is noticed that they become independent when $\alpha = 0$. In this case, Eqs. $(2.4)_1$ reduce to the displacement equations of motion of classical elasticity, while Eqs. $(2.4)_2$ describe a hypothetical elastic body in which only rotations occur. For $\alpha \to \infty$ we get the couple-stress theory conditions $\mathbf{w} = \frac{1}{2}\nabla\times\mathbf{u}$.

Since we are interested in solving a static problem with no body forces, we take $\mathbf{X} = \mathbf{Y} = \ddot{\mathbf{u}} = \ddot{\mathbf{w}} = 0$ and introduce the cylindrical polar coordinates (r, ϕ, z). Equation (2.4) now takes the following form:

$$\begin{aligned}
&(\mu+\alpha)\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi}\right) + (\lambda+\mu-\alpha)\frac{\partial e}{\partial r} + 2\alpha\left[\frac{1}{r}\frac{\partial\omega_z}{\partial\phi} - \frac{\partial\omega_\phi}{\partial z}\right] = 0, \\
&(\mu+\alpha)\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi}\right) + (\lambda+\mu-\alpha)\frac{1}{r}\frac{\partial e}{\partial\phi} + 2\alpha\left(\frac{\partial\omega_r}{\partial z} - \frac{\partial\omega_z}{\partial r}\right) = 0, \\
&(\mu+\alpha)\nabla^2 u_z + (\lambda+\mu-\alpha)\frac{\partial e}{\partial z} + 2\alpha\frac{1}{r}\left[\frac{\partial}{\partial r}(r\omega_\phi) - \frac{\partial\omega_r}{\partial\phi}\right] = 0, \\
&(\gamma+\varepsilon)\left(\nabla^2\omega_r - \frac{\omega_r}{r^2} - \frac{2}{r^2}\frac{\partial\omega_\phi}{\partial\phi}\right) - 4\alpha\omega_r + (\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r} + 2\alpha\left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right) = 0, \\
&(\gamma+\varepsilon)\left(\nabla^2\omega_\phi - \frac{\omega_\phi}{r^2} + \frac{2}{r^2}\frac{\partial\omega_r}{\partial\phi}\right) - 4\alpha\omega_\phi + (\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial\phi} + 2\alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) = 0,
\end{aligned} \tag{2.5}$$

(2.5) [cont.]
$$(\gamma+\varepsilon)\nabla^2\omega_z-4\alpha\omega_z+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z}+2\alpha\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\phi)-\frac{\partial u_r}{\partial\phi}\right]=0,$$

where

$$e=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{r}{1}\frac{\partial u_\phi}{\partial\phi}+\frac{\partial u_z}{\partial z},\qquad \varkappa=\frac{1}{r}\frac{\partial}{\partial r}(r\omega_r)+\frac{1}{r}\frac{\partial\omega_\phi}{\partial\phi}+\frac{\partial\omega_z}{\partial z}.$$

We shall consider a particular case in which the vectors of displacement **u** and of rotation **ω** depend only on the coordinates r, z. In this case, the set of Eqs. (2.5) is decomposed into two mutually independent sets of equations:

$$(\mu+\alpha)\left(\nabla^2u_r-\frac{u_r}{r^2}\right)+(\lambda+\mu-\alpha)\frac{\partial e}{\partial r}-2\alpha\frac{\partial\omega_\phi}{\partial z}=0,$$

(2.6)
$$(\mu+\alpha)\nabla^2u_z+(\lambda+\mu-\alpha)\frac{\partial e}{\partial z}+2\alpha\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\phi)=0,$$

$$(\gamma+\varepsilon)\left(\nabla^2\omega_\phi-\frac{\omega_\phi}{r^2}\right)-4\alpha\omega_\phi+2\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)=0,$$

$$(\mu+\alpha)\left(\nabla^2u_\phi-\frac{u_\phi}{r^2}\right)+2\alpha\left(\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r}\right)=0,$$

(2.7)
$$(\gamma+\varepsilon)\left(\nabla^2\omega_r-\frac{\omega_r}{r^2}\right)-4\alpha\omega_r+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r}-2\alpha\frac{\partial u_\phi}{\partial z}=0,$$

$$(\gamma+\varepsilon)\nabla^2\omega_z-4\alpha\omega_z+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z}+2\alpha\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi)=0,$$

where

$$e=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z},\qquad \varkappa=\frac{1}{r}\frac{\partial}{\partial r}(r\omega_r)+\frac{\partial\omega_z}{\partial z},\qquad \nabla^2=\frac{\partial}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

The sets of Eqs. (2.6) and (2.7) can be considered separately. Here we shall investigate the set of Eqs. (2.6). To the displacement vector $\mathbf{u}=(u_r,0,u_z)$ and to the rotation vector $\boldsymbol{\omega}=(0,\omega_\phi,0)$, is ascribed the following state of force stresses and couple stresses:

(2.8)
$$\begin{bmatrix}\sigma_{rr} & 0 & \sigma_{rz}\\ 0 & \sigma_{\phi\phi} & 0\\ \sigma_{zr} & 0 & \sigma_{zz}\end{bmatrix},\qquad \mu=\begin{bmatrix}0 & \mu_{r\phi} & 0\\ \mu_{\phi r} & 0 & \mu_{\phi z}\\ 0 & \mu_{z\phi} & 0\end{bmatrix},$$

where the particular components of the stress tensors have, according to (2.3), the following form:

$$\sigma_{rr}=2\mu\frac{\partial u_r}{\partial r}+\lambda e,\qquad \sigma_{\phi\phi}=2\mu\frac{u_r}{r}+\lambda e,\qquad \sigma_{zz}=2\mu\frac{\partial u_z}{\partial z}+\lambda e,$$

$$\sigma_{rz}=\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)-\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)+2\alpha\omega_\phi,$$

(2.9)
$$\sigma_{zr}=\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)-2\alpha\omega_\phi,$$

(2.9) [cont.]
$$\mu_{r\phi} = \gamma\left(\frac{\partial\omega_\phi}{\partial r} - \frac{\omega_\phi}{r}\right) + \varepsilon\left(\frac{\partial\omega_\phi}{\partial r} + \frac{\omega_\phi}{r}\right),$$
$$\mu_{\phi r} = \gamma\left(\frac{\partial\omega_\phi}{\partial r} - \frac{\omega_\phi}{r}\right) - \varepsilon\left(\frac{\partial\omega_\phi}{\partial r} + \frac{\omega_\phi}{r}\right),$$
$$\mu_{\phi z} = (\gamma-\varepsilon)\frac{\partial\omega_\phi}{\partial z}, \qquad \mu_{z\phi} = (\gamma+\varepsilon)\frac{\partial\omega_\phi}{\partial z}.$$

3. General solution of Eqs. (2.6)

The three mutually independent functions u_r, u_z and ω_ϕ are involved in the system of Eqs. (2.6). Multiplying (2.6) by $J_0(\xi r)$ and $(2.6)_{1,3}$ by $J_1(\xi r)$, and integrating between the limits 0 to ∞, we find that the system of partial differential equations (2.6) reduces to the following system of ordinary differential equations:

(3.1)
$$[(\mu+\alpha)D^2-(\lambda+2\mu)\xi^2]\tilde{u}_r-(\lambda+\mu-\alpha)\xi D\bar{u}_z-2\alpha D\tilde{\omega}_\phi = 0,$$
$$(\lambda+\mu-\alpha)\xi D\tilde{u}_r+[(\lambda+2\mu)D^2-(\mu+\alpha)\xi^2]\bar{u}_z+2\alpha\xi\tilde{\omega}_\phi = 0,$$
$$2\alpha D\tilde{u}_r+2\alpha\xi\bar{u}_z+[(\gamma+\varepsilon)(D^2-\xi^2)-4\alpha]\tilde{\omega}_\phi = 0,$$

where $\tilde{u}_r$, $\tilde{\omega}_\phi$ and $\bar{u}_z$ denote the Hankel transforms of the functions u_r, ω_ϕ and u_z, respectively,

$$(\tilde{u}_r,\tilde{\omega}_\phi) = \int_0^\infty (u_r,\omega_\phi)\xi J_1(\xi r)dr, \qquad \bar{u}_z = \int_0^\infty u_z\xi J_0(\xi r)dr$$

and $D = d/dz$. The expressions for some of the required transformed components of the stress tensors are obtained as:

(3.2)
$$\bar{\sigma}_{zz} = (\lambda+2\mu)D\bar{u}_z+\lambda\xi\tilde{u}_r,$$
$$\tilde{\sigma}_{zr} = -(\mu-\alpha)\xi\bar{u}_z+(\mu+\alpha)D\tilde{u}_r-2\alpha\tilde{\omega}_\phi,$$
$$\tilde{\mu}_{z\phi} = (\gamma+\varepsilon)D\tilde{\omega}_\phi.$$

The solution of the set of Eqs. (3.1) suitable for the half-space $Z \geqslant 0$ may be written as

(3.3)
$$\tilde{u}_r(\xi,z) = (A_1+A_2z)e^{-\xi z}+A_3e^{-\zeta z},$$
$$\bar{u}_z(\xi,z) = (B_1+B_2z)e^{-\xi z}+B_3e^{-\zeta z},$$
$$\tilde{\omega}_\phi(\xi,z) = (C_1+C_2z)e^{-\xi z}+C_3e^{-\zeta z},$$

where

(3.4)
$$\zeta = \sqrt{\xi^2+m^2}, \qquad m^2 = \frac{4\alpha\mu}{(\mu+\alpha)(\gamma+\varepsilon)},$$

and A_i, B_i, C_i $(i = 1, 2, 3)$ are arbitrary functions of ξ although not all of these are independent. Substituting for $\tilde{u}_r$, $\bar{u}_z$, $\tilde{\omega}_\phi$ from (3.3) into $(3.1)_{1,2}$, and equating the coefficients of $\exp(-\xi z)$, $z\exp(-\xi z)$, $\exp(-\zeta z)$ separately to zero in each of the equations, we obtain a system of six algebraic equations in A_i, B_i, C_i from which we find that:

$$A_1 = A-\frac{\lambda+3\mu}{\lambda+2\mu}\frac{B}{\xi}, \qquad B_1 = A, \quad C_1 = B,$$

$$A_2 = B_2 = \frac{\lambda+\mu}{\lambda+2\mu} B, \qquad C_2 = 0,$$

$$\frac{A_3}{\zeta} = \frac{B_3}{\xi} = -\frac{\gamma+\varepsilon}{2\mu} C, \qquad C_3 = C.$$

Now the expressions for the transformed displacement and rotation components are given by:

$$\tag{3.5} \begin{aligned} \tilde{u}_r &= \left(A - \frac{\lambda+3\mu}{\lambda+2\mu}\frac{B}{\xi} + \frac{\lambda+\mu}{\lambda+2\mu} Bz\right) e^{-\xi z} - \frac{\gamma+\varepsilon}{2\mu} C\zeta e^{-\zeta z}, \\ \overline{u}_z &= \left(A + \frac{\lambda+\mu}{\lambda+2\mu} Bz\right) e^{-\xi z} - \frac{\gamma+\varepsilon}{2\mu} C\xi e^{-\zeta z}, \\ \tilde{\omega}_\phi &= Be^{-\xi z} + Ce^{-\zeta z}. \end{aligned}$$

Substituting from (3.5) into (3.2), we obtain the expressions for the transformed components of stress:

$$\begin{aligned} \overline{\sigma}_{zz}(\xi, z) &= 2\mu\left[\left(-\xi A + \frac{\mu}{\lambda+2\mu} B - \frac{\lambda+\mu}{\lambda+2\mu} B\xi z\right) e^{-\xi z} + \frac{\gamma+\varepsilon}{2\mu} C\zeta\xi e^{-\zeta z}\right], \\ \tilde{\sigma}_{zr}(\xi, z) &= 2\mu\left[\left(-\xi A + B - \frac{\lambda+\mu}{\lambda+2\mu} B\xi z\right) e^{-\xi z} + \frac{\gamma+\varepsilon}{2\mu} C\xi^2 e^{-\zeta z}\right], \\ \tilde{\mu}_{z\phi}(\xi, z) &= -(\gamma+\varepsilon)[B\xi e^{-\xi z} + C\zeta e^{-\zeta z}]. \end{aligned}$$

4. Solution of the Boussinesq problem

We consider a semi-infinite micropolar elastic medium $z \geqslant 0$, indented by an axisymmetric rigid punch on a circular area $r \leqslant 1$ of its free surface $z = 0$.

The boundary conditions for the problem may be written as:

$$\tag{4.1} \sigma_{zr}(r, 0) = \mu_{z\theta}(r, 0) = 0, \qquad r \geqslant 0,$$

$$\tag{4.2} \begin{aligned} \sigma_{zz}(r, 0) &= 0, \qquad r > 1, \\ u_z(r, 0) &= \delta - f(r), \qquad r \leqslant 1. \end{aligned}$$

where $f(r)$ is prescribed by the fact that, referred to the tip as origin, the punch has equation $z = f(r)$ so that $f(0) = 0$; δ is a parameter (as yet unspecified) whose physical significance is that it is the depth to which the tip of the punch penetrates the elastic medium.

The boundary conditions (4.1) will be satisfied if we take

$$\tag{4.3} A = \left(-\frac{\zeta}{\xi^2} + \frac{\gamma+\varepsilon}{2\mu}\xi\right) C, \qquad B = -\frac{\zeta}{\xi} C.$$

Now, if we define the new unknown function $G(\xi)$ by the relation

$$\tag{4.4} G(\xi) = \left[\frac{\lambda+\mu}{\lambda+2\mu} + \frac{\gamma+\varepsilon}{2\mu}\xi(\zeta - \xi)\right] C,$$

we find that:

(4.5) $$\sigma_{zz}(r,0) = 2\mu \int_0^\infty G(\xi)\xi J_0(\xi r)\,d\xi,$$

(4.6) $$u_z(r,0) = \int_0^\infty H(\xi)G(\xi)J_0(\xi r)\,d\xi,$$

where

(4.7) $$H(\xi) = \frac{-2(1-\eta)k^2\zeta}{k^2\zeta+\xi^2(\zeta-\xi)}, \qquad k^2 = \frac{1}{(\eta-1)}\frac{\mu}{\gamma+\varepsilon},$$

and η is Poisson's ratio. We may write $H(\xi)$ in an alternative form:

(4.8) $$H(\xi) = h+H_1(\xi), \qquad h = -4(1-\eta)\frac{k^2}{m^2+2k^2}.$$

From (4.5), (4.6) and (4.8), we find that the boundary conditions (4.2) are satisfied if $G(\xi)$ is the solution of the dual integral equations:

(4.9) $$\begin{aligned} &\int_0^\infty G(\xi)\xi J_0(\xi r)d\xi = 0, \qquad r>1,\\ &\int_0^\infty [h+H_1(\xi)]G(\xi)J_0(\xi r) = \delta-f(r), \qquad r\leqslant 1. \end{aligned}$$

Dual integral equations of the above type have been considered by a number of authors [9, 10]. If we represent $G(\xi)$ by the relation:

(4.10) $$G(\xi) = \frac{2}{\pi}\int_0^1 \psi(t)\cos(\xi t)\,dt,$$

then $(4.9)_1$ is identically satisfied, and $(4.9)_2$ is satisfied if $\psi(t)$ satisfies the Fredholm integral equation:

(4.11) $$h\psi(t)+\frac{1}{\pi}\int_0^\infty K(x,t)\psi(x)\,dx = \left[\int_0^t \frac{\delta-f(x)}{\sqrt{(t^2-x^2)}}\,x\,dx\right], \qquad 0\leqslant t\leqslant 1,$$

where

(4.12) $$K(x,t) = 2\int_0^\infty H_1(\xi)\cos(x\xi)\cos(t\xi)\,d\xi.$$

To evaluate $K(x, t)$, we notice that the denominator of $H_1(\xi)$ has no real zeros and that the zeros are imaginary and unequal or imaginary and equal or complex and equal and opposite in sign according as $k^2 \lesseqgtr 4m^2$. Now, for the case $k^2 > 4m^2$ we find that $H_1(\xi)$ may be written as

(4.13) $$H_1(\xi) = \frac{2(1-\eta)k^2}{m^2+2k^2}\left[\frac{\Gamma_1}{\xi^2+a_1^2}-\frac{\Gamma_2}{\xi^2+a_2^2}-\frac{a_1^2}{a_1^2-a_2^2}\frac{\xi\{(\xi^2+m^2)^{1/2}-\xi\}}{\xi^2+a_1^2}\right.$$
$$\left.+\frac{a_2^2}{a_1^2-a_2^2}\frac{\xi\{(\xi^2+m^2)^{1/2}-\xi\}}{\xi^2+a_2^2}\right],$$

where

$$a_1^2, a_2^2 = \frac{k^2(k^2+2m^2)}{2(2k^2+m^2)} \pm \frac{k^3(k^2-4m^2)^{1/2}}{2(2k^2+m^2)},$$

$$\Gamma_1 = \frac{2a_1^4-(k^2+m^2)a_1^2+k^2m^2}{a_1^2-a_2^2}, \quad \Gamma_2 = \frac{2a_2^4-(k^2+m^2)a_2^2+k^2m^2}{a_1^2-a_2^2}.$$

Substituting for $H_1(\xi)$ from (4.13) into (4.12) and integrating, we find that

$$(4.14) \qquad K(x,t) = \frac{(1-\eta)k^2}{m^2+2k^2}\left[\frac{\pi\Gamma_1}{a_1}\left\{e^{-a_1(x+t)}+e^{-a_1|x-t|}\right\} - \frac{\pi\Gamma_2}{a_2}\left\{e^{-a_2(x+t)}+e^{-a_2|x-t|}\right\}\right.$$
$$\left. - \frac{2}{a_1^2-a_2^2}\{I(a_1,x+t)+I(a_1,|x-t|)-I(a_2,x+t)-I(a_2,|x-t|)\}\right],$$

where

$$(4.15) \qquad I(a,u) = a^2\int_0^\infty \frac{z\{\sqrt{(z^2+m^2)}-z\}}{z^2+a^2}\cos(zu)\,dz$$
$$= \frac{\pi a^2}{2}e^{-au}\{a-\sqrt{(a^2-m^2)}\} - a^2\int_0^m \frac{z(m^2-z^2)^{1/2}}{a^2-z^2}e^{-uz}\,dz.$$

The expressions for $K(x, t)$ for $k^2 \leqslant 4m^2$ may be obtained in a similar way. Numerical values of $K(x, t)$ for $0 \leqslant (x, t) \leqslant 1$ can be obtained, since this involves only the numerical integration of a finite integral (4.15) and hence the integral equation (4.11) may be solved numerically, following the procedure of FOX and GOODWIN [11] as applied by DHALIWAL [12] earlier in a similar problem.

Substituting for $G(\xi)$ from (4.10) into (4.5), and interchanging the order of integration, we obtain:

$$(4.16) \qquad \sigma_{zz}(r,0) = \begin{cases} -\dfrac{4\mu}{\pi}\dfrac{1}{r}\dfrac{d}{dr}\displaystyle\int_r^1 \frac{t\psi(t)\,dt}{\sqrt{(t^2-r^2)}}, & 0 \leqslant r \leqslant 1, \\ 0, r > 1. \end{cases}$$

The total load P which must be applied to the punch to maintain the prescribed displacement below it is given by:

$$(4.17) \qquad P = -2\pi\int_0^1 \sigma_{zz}(r,0)\,r\,dr = 8\mu\int_0^1 dr\frac{d}{dr}\int_r^1 \frac{t\psi(t)\,dt}{\sqrt{(t^2-r^2)}}$$
$$= 8\mu\left[\int_r^1 \frac{t\psi(t)\,dt}{\sqrt{(t^2-r^2)}}\right]_{r=0}^{r=1} = -8\mu\int_0^1 \psi(t)\,dt.$$

5. Solution for $\alpha \ll 1$

For small values of α, m^2 will be small and hence, disregarding m^4 and higher powers of m, we find that

$$(5.1) \qquad \zeta = (\xi^2+m^2)^{1/2} = \xi\left(1+\frac{m^2}{2\xi^2}\right)+0(m^4),$$

and hence

$$H(\xi) = h + 0(m^4). \tag{5.2}$$

Now using (5.2) and the representation (4.10), we find that the system of dual integral Eqs. (4.9) reduces to the Abel integral equation

$$\int_0^r \frac{\psi(t)\,dt}{\sqrt{(r^2-t^2)}} = \frac{\pi}{2h}[\delta - f(r)], \quad r \leqslant 1, \tag{5.3}$$

for the determination of the function $\psi(t)$. The solution of this integral equation is known to be

$$\psi(t) = \frac{1}{h}\left[\delta - \frac{d}{dt}\int_0^t \frac{rf(r)\,dr}{\sqrt{(t^2-r^2)}}\right], \quad t \leqslant 1, \tag{5.4}$$

which may be written as:

$$\psi(t) = \frac{1}{h}\left[\delta - t\int_0^t \frac{f'(r)}{\sqrt{(t^2-r^2)}}\,dr\right]. \tag{5.5}$$

In the case of a punch with a smooth profile, it has been shown by SNEDDON [13] that for $\sigma_{zz}(r, 0)$ to remain finite as $r \to 1-$, we must have

$$\psi(1) = 0. \tag{5.6}$$

Using (5.6) and (5.5), we obtain the total depth of penetration of the tip of the punch:

$$\delta = \int_0^1 \frac{f'(r)\,dr}{\sqrt{(t^2-r^2)}}, \tag{5.7}$$

which is the same as in the corresponding classical problem [13].

Substituting for $\psi(t)$ from (5.5) in (4.17) and making use of (5.7) to eliminate δ, we obtain the formula for the total load:

$$P = \frac{-8\mu}{h}\int_0^1 \frac{r^2 f'(r)\,dr}{\sqrt{(1-r^2)}}, \tag{5.8}$$

which reduces to the corresponding classical expression [13], since $h \to -2(1-\eta)$ as $m \to 0$ (i.e. $\alpha \to 0$).

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