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Analysis of acceleration waves in material with internal parameters

W. KOSIŃSKI and P. PERZYNA (WARSZAWA)

ACCELERATION waves in a rheological material in the case of one-dimensional theory are investigated. It is assumed that the internal dissipation of a rheological material can be described by n internal scalar parameters. In Sec. 3 the basic theorems for a homothermal acceleration wave are proved. In Sec. 4 a particular case of constitutive equations is introduced. The temperature gradient influences explicitly only the description of a state. The equation for the intrinsic velocity of a general acceleration wave is given. The thermal wave and the homothermal wave for this case of constitutive equations are investigated.

Zbadano fale przyśpieszenia w materiale reologicznym w przypadku jednowymiarowej teorii. Przyjęto, że dysypacja wewnętrzna materiału reologicznego może być opisana przez n skalarnych parametrów wewnętrznych. W p. 3 udowodniono podstawowe twierdzenia dla fali homotermicznej. W p. 4 wprowadzono szczególną postać równań konstytutywnych. Gradient temperatury pozostawiono tylko w opisie stanu. Otrzymano równanie na prędkość ogólnej fali przyśpieszenia. Dla tej postaci równań konstytutywnych zbadano falę termiczną i homotermiczną.

Исследованы волны ускорения в материале с реологическими свойствами. Рассмотрен одномерный случай, кгда диссипация мощности в реологическом материале может быть описана при помощи n скалярных внутренних параметров. В п. 3 доказаны основные теоремы для гомотермической волны. В п. 4 предложен частный вид определяющего уравнения, в котором градиент температуры содержится лишь в описании состояния. Выведено уравнение, описывающее скорость распространения волны ускорения общего вида. Для предложенного частного вида определяющего уравнения исследованы термические и гомотермические волны.

1. Introduction

THE OBJECT of the present paper is an investigation of acceleration waves in a rheological material in the case of one-dimensional theory. It is assumed that the internal dissipation of a rheological material can be described by n internal scalar parameters.

After introducing basic definitions and assumptions, the theorem 3 is proved. This theorem states that in homothermal acceleration waves, the time derivative of internal parameters has no jump discontinuity. The proof of the inverse theorem is also given. Both theorems are of important consequence for subsequent considerations.

In Sec. 3, a homothermal acceleration wave for the general form of constitutive equations is investigated.

In Sec. 4, we assume that all response functions do not explicitly depend on the temperature gradient while we keep the influence of the temperature gradient on the solution of the initial-value problem for the determination of internal parameters. After linearization of this initial-value problem with respect to the temperature gradient the equation for the intrinsic velocity of an acceleration thermo-mechanical wave is obtained. In Sec. 5, the thermal wave, and in Sec. 6 and Sec. 7, the homothermal and mechanical (isothermal) waves are considered.

The basic object of this investigation is to obtain practical information concerning the description of dissipation effects within the framework of internal parameters.

2. Definitions and assumptions

We shall identify a body with an open region $\mathscr{B}$, which is its image in the fixed reference configuration $\varkappa$. A motion of a body is described by a function $x\colon \mathscr{B}\times R \to R$; the value $x(X, t)$ determines the location x at time t of the material point X. By R we denote a real line.

Let us assume that the derivations

$$F(X,t) = \frac{\partial}{\partial X}x(X,t), \quad \dot{x}(X,t) = \frac{\partial}{\partial t}x(X,t), \quad \ddot{x}(X,t) = \frac{\partial^2}{\partial t^2}x(X,t)$$

exist. We call them, respectively, the deformation gradient, the velocity and the acceleration of a particle X at time t.

To describe thermal effects in a body we introduce a function $\theta\colon\mathscr{B}\times R \to R$, the value of which $\theta(X, t)$ is the absolute temperature of a particle X at time t.

Let us introduce a new function of two variables.

$$g\colon\mathscr{B}\times R \to \underbrace{R\times R\times \ldots \times R}_{3+n \text{ times}}. \tag{2.1}$$

We postulate that a thermo-mechanical state of a particle X at time t is described by the value of the function

$$g(X,t) = \{F(X,t), \theta(X,t), \partial_X\theta(X,t), \alpha(X,t)\} \tag{2.2}$$

and by the initial-value problem for differential equation

$$\dot{\alpha}(X,t) = A\big(g(X,t)\big), \quad \alpha(X,t_0) = \alpha_0(X). \tag{2.3}$$

In (2.1)–(2.3), $\alpha(X, t)$ represents n-scalar internal parameters, which are introduced to describe the internal dissipation of a rheological material(1); the function A describes the evolution of α, and $\partial_X\theta(X,t) = \dfrac{\partial}{\partial X}\theta(X,t)$.

We shall assume that the initial-value problem (2.3) has a unique solution. This implies the condition that the function A is the Lipshitz continuous function with respect to α, and the continuous function with respect to its first three arguments.

(1) Cf. [15]. A triple $(F, \theta, \partial_X\theta)$ describes the actual deformation-temperature configuration of a particle X, while $\alpha(X, t)$ together with the evolution equation (2.3) and the initial condition, describe the method of preparation of this actual configuration. The method of preparation supplies the additional information needed to define uniquely for a rheological material a thermomechanical state of particle X at time t. Internal parameters may have different physical interpretation. This depends on the cause of the internal dissipation in a material. For discussion of this problem see [12–15].

We shall introduce a fundamental concept in thermodynamic theory — i.e., the concept of a thermodynamic process.

DEFINITION 1. A local thermodynamic process at a material point X in the interval of time $(t_p, t_k) \subset R$ is a family of functions givien for every $t \in (t_p, t_k)$:

$$\mathscr{P}_X = \{F(X,t), \theta(X,t), \partial_X\theta(X,t), \pi(X,t)\}, \tag{2.4}$$

which satisfies the thermodynamic postulate

$$-\dot{\psi}+T\dot{F}-\eta\dot{\theta}-\frac{1}{\varrho\vartheta}q\cdot\partial_X\theta \geqslant 0, \tag{2.5}$$

where

$$\pi(X,t) = \{\psi(X,t), \eta(X,t), T(X,t), q(X,t)\}$$

represents the specific free energy per unit mass, the specific entropy, the stress and the heat flux in a particle X at time t. We denote by ϱ mass density.

We shall restrict our considerations to homogeneous material.

It is assumed that the density of body force $b(X,t)$ and the heat supply per unit mass and unit time $r(X,t)$ can be uniquely determined by the first Cauchy's law of motion and by the energy balance equation:

$$\varrho\ddot{x}-\partial_X T = \varrho b, \quad \varrho(\dot{\psi}+\eta\dot{\theta}+\dot{\eta}\theta)-T\dot{F}+\partial_X q = \varrho r. \tag{2.6}$$

We postulate that the response of a material — i.e., the thermo-mechanical principle of determinism for a rheological material — is expressed by the constitutive relation(²):

$$\pi(X,t) = \mathscr{R}\big(g(X,t)\big), \tag{2.7}$$

where $\mathscr{R} = \{\Psi, \mathrm{N}, \mathrm{T}, \mathrm{Q}\}$ represents, respectively, the response functions of free energy, entropy, stress and heat flux.

We assume that the function Ψ is of C^3-class and N, T and Q are of C^2-class in their domains.

We shall introduce the following

DEFINITION 2. A local thermodynamic process described by $\mathscr{P}_X$ is said to be admissible in $\mathscr{B}$ if it is compatible with the constitutive assumption (2.7) at each particle X of $\mathscr{B}$.

It is easy to prove(³)

THEOREM 1. *In an admissible local thermodynamic process of a particle X of $\mathscr{B}$ the following relations are satisfied at every time t:*

$$\partial_{\partial_X\theta}\Psi = 0, \quad T(X,t) = \varrho\partial_F\Psi, \quad \eta(X,t) = -\partial_\theta\Psi, \tag{2.8}$$
$$\partial_\alpha\Psi\cdot A+\frac{1}{\varrho\theta}q\cdot\partial_X\theta \leqslant 0.$$

Let us introduce the fundamental definitions concerning the wave(⁴).

(²) In addition to this general constitutive equation, we shall consider simplified equation in which instead of the function $g(,)$ will appear $g^*(,) = \{F(,), \theta(,), \alpha(,)\}$.

(³) Cf. COLEMAN and GURTIN [9] and VALANIS [17].

(⁴) Cf. COLEMAN, GURTIN and HERRERA [4], COLEMAN and GURTIN [5–7, 10] COLEMAN, GREENBERG and GURTIN [8] and TRUESDELL and TOUPIN [16].

Definition 3. The material representation of a wave is a smooth one-parameter family of points $Y_t \in \mathscr{B}$, $t \in (t_p, t_k)$ such that Y_t is a point at which the wave is to be found at time t.

Definition 4. The material trajectory of the wave is the set

$$\Sigma = \{(Y_t, t); t \in (t_p, t_k)\}.$$

We denote by $U = U(t)$ the intrinsic velocity of the wave, which is defined as follows

$$U(t) = \frac{d}{dt} Y_t. \tag{2.9}$$

Let us recall the fundamental Maxwell's theorem.

Theorem 2. *If $f = f(X, t)$ is a continuous function of X and t jointly, and has continuous partial derivatives everywhere except Σ, where these derivatives can have jump discontinuities, then*

$$[\dot{f}] = -U[\partial_X f]. \tag{2.10}$$

We define the jump in $h(X, t)$ across Σ at t by

$$[h] = \lim_{X \to Y_t^-} h(X, t) - \lim_{X \to Y_t^+} h(X, t) \tag{2.11}$$

for a function h: $\mathscr{B} \times R \to R$. By Y_t^-, Y_t^+, we denote respectively the left and right limit.

Let us consider the motion $x(X, t)$ and time dependent fields θ and α on $\mathscr{B} \times R$ in an admissible thermodynamic process for every particle X in $\mathscr{B}$.

Definition 5. It is said that Σ is an acceleration wave if the fields $x(X, t)$, $\theta(X, t)$ and $\alpha(X, t)$ have the following properties:

A1) $x, \dot{x}, F, \theta, \alpha$ are continuous functions of X and t jointly for all X and t;

A2) $\ddot{x}, \dot{F}, \partial_X F, \dot{\theta}_A, \partial_X \theta, \dot{\alpha}, \partial_X \alpha$ have jump discontinuities across Σ but are continuous in X and t jointly everywhere else;

A3) the response functions T, N, Q are C^2-class and Ψ is C^3-class in their domains; the function A is C^1-class in its domain([5]).

We introduce the notations

$$a = [\ddot{x}], \qquad \beta = [\dot{\theta}] \tag{2.12}$$

respectively, for mechanical and thermal amplitudes of the wave. We shall also assume that

$$[\varrho] = [b] = [\dot{b}] = [r] = [\dot{r}] = 0. \tag{2.13}$$

Definition 6. An acceleration wave in which $[\dot{\theta}] = 0$ is called homothermal.

The definition of an acceleration wave implies

$$[\psi] = [T] = [\eta] = 0. \tag{2.14}$$

From the definition of an acceleration wave and from the Maxwell's theorem we obtain

$$a = -U[\dot{F}] = U^2[\partial_X F]; \quad \beta = -U[\partial_X \theta]; \tag{2.15}$$
$$[\dot{\alpha}] = -U[\partial_X \alpha].$$

([5]) This assumption is stronger than that needed for the unique solution of the initial-value problem (2.3).

Let f: $\mathscr{B} \times R \to R$ be a continuous and continuously differentiable function of its variables everywhere except Σ, where it can have jump discontinuity. Then $[f]$ is a function of time t only. Differentiation $[f]$ with respect to time t yields([6])

(2.16) $$\frac{d}{dt}[f] = [\dot{f}] + U[\partial_X f].$$

THEOREM 3. *In a homothermal acceleration wave the time derivative of the internal parameter has no jump discontinuity* — i.e.,

(2.17) $$[\dot{\vartheta}] = 0 \Rightarrow [\dot{\alpha}] = 0.$$

Inversely, if a derivative of the function A with respect to the temperature gradient does not vanish on the wave, and the time derivative of the internal parameter has no jump discontinuity, then the Thomas derivative of the temperature gradient vanishes — i.e.,

(2.18) $$[\dot{\alpha}] = 0 \Rightarrow \frac{d}{dt}[\partial_X \theta] = 0.$$

Proof. The first part of the theorem is a consequence of the definition of an acceleration wave (cf. A3). The proof of the second part of the theorem is as follows. If $[\dot{\alpha}] = 0$, then from (2.10):

(2.19) $$[\ddot{\alpha}] = -U[\partial_X \dot{\alpha}].$$

Using the evolution equation (2.3), we obtain

$$[\ddot{\alpha}] = \partial_F A[\dot{F}] + \partial_\theta A[\dot{\vartheta}] + \partial_{\partial_X \theta} A[\partial_X \dot{\theta}],$$
$$[\partial_X \dot{\alpha}] = \partial_F A[\partial_X F] + \partial_\theta A[\partial_X \theta] + \partial_{\partial_X \theta} A[\partial_X^2 \theta].$$

Substitution of these results into (2.19) gives

$$\partial_{\partial_X \theta} A[\partial_X \dot{\theta}] = -U \partial_{\partial_X \theta} A[\partial_X^2 \theta].$$

Because $\partial_{\partial_X \theta} A \neq 0$, hence $[\partial_X^2 \dot{\vartheta}] = -U[\partial_X^2 \theta]$ or

$$[\partial_X \dot{\theta}] + U[\partial_X^2 \theta] = \frac{d}{dt}[\partial_X \theta] = 0.$$

This completes the proof.

This theorem is fundamental for a discussion of acceleration waves in a material with internal state variables. It endows an acceleration wave with an additional property and at the same time solves the question concerning the jump of the time derivatives of the internal parameters.

3. Homothermal acceleration wave

We offer some remarks prior to the discussion of a homothermal acceleration wave in a rheological material.

([6]) A derivative $\frac{d}{dt}[f]$ defined by (2.16) is called the Thomas derivative.

The assumption concerning the differentiability of the free energy function and the results of the theorem 1 yields:

$$\dot{\psi} = \frac{1}{\varrho} T\dot{F} - \eta\dot{\theta} + \partial_\alpha \Psi \cdot \dot{\alpha}. \tag{3.1}$$

R e m a r k. In a local thermodynamic process involving an acceleration wave, the laws of balance of momentum and energy are equivalent to the assertion that for $X \neq Y_t$(⁷)

$$\partial_X T + \varrho b = \varrho \ddot{x}, \tag{3.2}$$

$$\varrho(\dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta}) = T\dot{F} - \partial_X q + \varrho r, \tag{3.3}$$

while for $X = Y_t$

$$[\partial_X T] = \varrho[\ddot{x}], \tag{3.4}$$

$$\varrho[\dot{\psi}] + \varrho\eta[\dot{\theta}] + \varrho\theta[\dot{\eta}] + [\partial_X q] = T[\dot{F}], \tag{3.5}$$

$$[q] = 0. \tag{3.6}$$

These latter results together with (3.1) give

$$\varrho\partial_\alpha \Psi \cdot [\dot{\alpha}] + \varrho\theta[\dot{\eta}] + [\partial_X q] = 0 \tag{3.7}$$

across Σ, and

$$\varrho\partial_\alpha \Psi \cdot \dot{\alpha} + \varrho\theta\dot{\eta} + \partial_X q - \varrho r = 0 \tag{3.8}$$

everywhere else.

Using the definition of an acceleration wave and the smoothness property for the stress function T, we can write

$$[\partial_X T] = \partial_F \mathrm{T}[\partial_X F] + \partial_\theta \mathrm{T}[\partial_X \theta] + \partial_\alpha \mathrm{T} \cdot [\partial_X \alpha]. \tag{3.9}$$

Since on a homothermal acceleration wave $[\theta] = [\partial_X \theta] = 0$, and by theorem 3 $[\dot{\alpha}] = [\partial_X \alpha] = 0$, then instead of (3.9) we have:

$$[\partial_X T] = \partial_F \mathrm{T}[\partial_X F]. \tag{3.10}$$

Substituting (3.10) into (3.4) and using the Maxwell's theorem, we obtain

$$(\partial_F \mathrm{T} - \varrho U^2) a = 0. \tag{3.11}$$

This equation permits to express

Theorem 4. *The intrinsic velocity of a homothermal acceleration wave in a material with internal parameters satisfies*

$$U^2 = \frac{\partial_F \mathrm{T}}{\varrho}, \tag{3.12}$$

where $\partial_F \mathrm{T}$ is taken at the point $\big(F(Y_t, t), \theta(Y_t, t), \alpha(Y_t, t)\big)$.

We shall attempt to find the equation which describes changes of the amplitude $a(t) = [\ddot{x}](t)$. First, we shall show the relation between the amplitudes $a(t)$ and $[\ddot{\theta}](t)$. Since $[\dot{\theta}] = 0$, then by theorem 2 we have

$$[\ddot{\theta}] = -U[\partial_X \dot{\theta}] = U^2[\partial_X^2 \theta]. \tag{3.13}$$

(⁷) Coleman and Gurtin [6].

On a homothermal acceleration wave Σ we have

$$[\partial_X q] = \partial_F \mathrm{Q}[\partial_X F] + \partial_{\partial_X \theta}\mathrm{Q}[\partial_X^2 \theta], \tag{3.14}$$

$$[\dot{\eta}] = \partial_F \mathrm{N}[\dot{F}]. \tag{3.15}$$

Combining the results (3.14) and (3.15) with (2.15) and (3.13) and substituting into the equation

$$\varrho\theta[\dot{\eta}] + [\partial_X q] = 0 \tag{3.16}$$

which describes the energy balance for a homothermal wave, we obtain [8].

THEOREM 5. *In a homothermal acceleration wave with intrinsic velocity U*

$$k_{Y_t}[\ddot{\theta}](t) = (\varphi_{Y_t} + U\zeta_{Y_t}\theta)a(t), \tag{3.17}$$

where

$$k_{Y_t} = -\frac{\partial}{\partial(\partial_X\theta)}\mathrm{Q}\big(F(Y_t, t), \theta(Y_t, t), \partial_X\theta(Y_t, t), \alpha(Y_t, t)\big), \tag{3.18}$$

is the heat conduction modulus on the wave, and

$$\varphi_{Y_t} = \frac{\partial}{\partial F}\mathrm{Q}\big(F(Y_t, t), \theta(Y_t, t), \partial_X\theta(Y_t, t), \alpha(Y_t, t)\big), \tag{3.19}$$

$$\zeta_{Y_t} = -\varrho\frac{\partial}{\partial F}\mathrm{N}\big(F(Y_t, t), \theta(Y_t, t), \alpha(Y_t, t)\big). \tag{3.20}$$

Equations (2.15) and (2.16) yield:

$$2\sqrt{U}\frac{d}{dt}\left(\frac{a}{\sqrt{U}}\right) = [\ddot{x}] - U[\partial_X \dot{F}]. \tag{3.21}$$

Differentation (3.2) with respect to time t gives on the wave

$$[\ddot{x}] = \frac{1}{\varrho}[\partial_X \dot{T}]. \tag{3.22}$$

The jump of $\partial_X \dot{T}$ is given by

$$[\partial_X\dot{T}] = \partial_F \mathrm{T}[\partial_X \dot{F}] + \partial_F^2 \mathrm{T}[\dot{F}\partial_X F] + I_{Y_t}[\partial_X F] + J_{Y_t}[\dot{F}] + \partial_\theta \mathrm{T}[\partial_X \theta] + \partial_\alpha \mathrm{T}[\partial_X \dot{\alpha}], \tag{3.23}$$

where

$$\begin{aligned} I_{Y_t} &= (\partial_\vartheta\partial_F \mathrm{T}\dot{\vartheta})_{Y_t} + (\partial_\alpha\partial_F \mathrm{T}\cdot\dot{\alpha})_{Y_t}, \\ J_{Y_t} &= (\partial_F\partial_\vartheta \mathrm{T}\partial_X\vartheta)_{Y_t} + (\partial_\alpha\partial_F \mathrm{T}\cdot\partial_X\alpha)_{Y_t}. \end{aligned} \tag{3.24}$$

On the wave we have the expression

$$[\partial_X\dot{\alpha}] = \partial_F A[\partial_X F] + \partial_{\partial_X\vartheta}A[\partial_X^2\vartheta], \tag{3.25}$$

which after using (3.17) yields

$$[\partial_X\dot{\alpha}] = U^2\left\{\partial_F A + \frac{\partial_{\partial_X\vartheta}A}{\partial_{\partial_X\vartheta}\mathrm{Q}}(U\varrho\vartheta\partial_F\mathrm{N} - \varrho\partial_F\mathrm{Q})\right\}a. \tag{3.26}$$

[8] Cf. COLEMAN and GURTIN [6].

Inserting (3.25) into (3.22), substituting the result into (3.21) and after using (3.17) and (3.26), we have

THEOREM 6. *The amplitude of a homothermal acceleration wave in a material with internal parameters satisfies the equation*

$$2\sqrt{U}\,\frac{d}{dt}\left(\frac{a}{\sqrt{U}}\right) = \frac{\partial_F^2 \mathrm{T}}{\varrho}[\dot{F}\partial_X F] + \frac{1}{\varrho U}\left\{\frac{I_{Y_t}}{U} - J_{Y_t} + \frac{\partial_\alpha \mathrm{T}\cdot\partial_F A}{U} + (\partial_\alpha \mathrm{T}\cdot\partial_{\partial_X\theta}A - \zeta_{Y_t})\left(\frac{\varphi_{Y_t} + U\theta\zeta_{Y_t}}{k_{Y_t}}\right)\right\}a, \tag{3.27}$$

where U satisfies Eq. (3.12), k_{Y_t}, φ_{Y_t}, ζ_{Y_t} *are given by* (3.18)–(3.20) *and* I_{Y_t} *and* J_{Y_t} *by* (3.24).

4. Equation for the velocity of a wave

The object of this Section is to find the equation foı the velocity of an acceleration wave in a rheological material with internal parameters.

We shall solve this problem for a particular case of constitutive equations. We assume the constitutive equation in the form

$$\pi(X, t) = \mathscr{R}(g^*(X, t)), \tag{4.1}$$

where $\pi(X, t)$ is the same as in Sec. 2, and

$$g^*(X, t) = \{F(X, t), \theta(X, t)\alpha(X, t)\}.$$

Let us assume for the response functions $\{\Psi, \mathrm{N}, \mathrm{T}, \mathrm{Q}\}$ the sames moothness properties as in Sec. 2.

We define a local thermodynamic process $(t_p, t_k) \subset R$ as a family of functions given for every time $t \in (t_p, t_k)$:

$$\mathscr{P}_X = \{F(X, t), \theta(X, t), \partial_X\theta(X, t), \pi(X, t)\}. \tag{4.2}$$

Description of a thermo-mechanical state in a particle X at time t is given by the value of the function

$$g(X, t) = \{F(X, t), \theta(X, t), \partial_X\theta(X, t), \alpha(X, t)\}, \tag{4.3}$$

and by the initial-value problem for the differential equation

$$\dot{\alpha}(X, t) = A((g(X, t)), \quad \alpha(X, t_0) = \alpha_0(X). \tag{4.4}$$

Dependence of the description of a thermo-mechanical state on the temperature gradient $\partial_X\theta$ ensures that the Fourier's law of heat conduction can be obtained as a particular case of our constitutive equations.

The assumptions introduced lead to the following relations on an acceleration wave

$$[\partial_X T] = \partial_F \mathrm{T}[\partial_X F] + \partial_\theta \mathrm{T}[\partial_X\theta] + \partial_\alpha \mathrm{T}\cdot[\partial_X\alpha], \tag{4.5}$$

$$[\partial_X q] = \partial_F \mathrm{Q}[\partial_X F] + \partial_\theta \mathrm{Q}[\partial_X\theta] + \partial_\alpha \mathrm{Q}\cdot[\partial_X\alpha], \tag{4.6}$$

$$[\dot{\eta}] = \partial_F \mathrm{N}[\dot{F}] + \partial_\theta \mathrm{N}[\dot{\theta}] + \partial_\alpha \mathrm{N}\cdot[\dot{\alpha}]. \tag{4.7}$$

Additionally, we assume the linearized form of the evolution Eq. (4.4) with respect to the temperature gradient

$$\dot{\alpha}(X,t) = \bar{A}\big(F(X,t), \theta(X,t), \alpha(X,t)\big)\partial_X\theta(X,t) + \bar{B}\big(F(X,t), \theta(X,t), \alpha(X,t)\big). \tag{4.8}$$

This assumption permits to express the jump of $\dot{\alpha}$ by the jump of $\partial_X\theta$:

$$[\dot{\alpha}] = \bar{A}[\partial_X\theta]. \tag{4.9}$$

Substitution of (4.5) into (3.4) and making use of the Maxwell's theorem(9) and (4.9) give

$$(\partial_F \mathrm{T} - \varrho U^2)a + (\partial_\alpha \mathrm{T}\cdot\bar{A} - U\partial_\theta \mathrm{T})\beta = 0. \tag{4.10}$$

In a similar manner, substitution of (4.6) and (4.7) into (3.7) and making use of the Maxwell's theorem and (4.9) yield:

$$(\partial_F \mathrm{Q} - \varrho U\theta\partial_F \mathrm{N})a - (\varrho U\partial_\alpha\Psi\cdot\bar{A} - \varrho U\theta\partial_\alpha \mathrm{N}\cdot\bar{A} - \partial_\alpha \mathrm{Q}\cdot\bar{A} + U\partial_\theta \mathrm{Q} - \varrho U^2\theta\partial_\theta \mathrm{N})\beta = 0. \tag{4.11}$$

Equations (4.10) and (4.11) represent a set of two algebraic equations linear with respect to amplitudes a and β. This set has nontrivial solutions, if and only if, its determinant vanishes (in every particle $X = Y_t$) — i.e.,

$$\begin{aligned}(\partial_F \mathrm{Q} - \varrho U\theta\partial_F \mathrm{N})(\partial_\alpha \mathrm{T}\cdot\bar{A} - U\partial_\theta \mathrm{T})\\ + (\partial_F \mathrm{T} - \varrho U^2)(\varrho U\partial_\alpha\Psi\cdot\bar{A} + \varrho U\theta\partial_\alpha \mathrm{N}\cdot\bar{A} - \partial_\alpha \mathrm{Q}\cdot\bar{A} + U\partial_\theta \mathrm{Q} - \varrho U^2\theta\partial_\theta \mathrm{N}) = 0.\end{aligned} \tag{4.12}$$

THEOREM 7. *The intrinsic velocity U of an acceleration wave obeys* (4.12)(10).

We shall study this equation for two particular cases.

Case 1. A material does not conduct heat and the free energy does not depend on internal parameters—i.e., $Q \equiv 0$ and $\partial_\alpha\Psi = 0$. From (4.12), we have:

$$U^2 = \frac{\partial_F \mathrm{T}}{\varrho} - \frac{(\partial_F\partial_\theta\Psi)^2}{\partial_\theta^2\Psi}.$$

Case 2. Let $\partial_\theta \mathrm{T} = 0$ and $\partial_F \mathrm{Q} = 0$. From (4.12) we obtain

$$U_{1,2}^2 = \frac{\partial_F \mathrm{T}}{\varrho},$$

$$U_{3,4}^2\varrho\theta\partial_\theta^2\Psi + U_{3,4}(\partial_\theta \mathrm{Q} + \varrho\partial_\alpha\Psi\cdot\bar{A} - \varrho\theta\partial_\alpha\partial_\theta\Psi\cdot\bar{A}) - \partial_\alpha \mathrm{Q}\cdot\bar{A} = 0.$$

In the case 1, the expression for U shows the influence of thermal effects on the intrinsic velocity of an acceleration wave in an elastic non-conductor. This is an example of a homotropic wave (cf. COLEMAN and GURTIN [6]).

Of interest is the case 2, which exemplifies the situation in which there is no coupling between thermal and mechanical effects and the mechanical and thermal waves propagate separately with finite speeds.

(9) Cf. with the Theorem 2.

(10) A similar result for materials with memory has been obtained by CHEN and GURTIN [3]; cf. also CHEN [1, 2].

5. Thermal wave

We shall consider an acceleration wave without mechanical effects. To this end, let us assume

$$\pi(X, t) = \mathscr{R}\big(g^{\#}(X, t)\big), \tag{5.1}$$

where

$$g^{\#}(X, t) = \{\theta(X, t), \alpha(X, t)\},$$

$$\pi = \{\psi, \eta, q\} \quad \text{and} \quad \mathscr{R} = \{\Psi, \mathrm{N}, \mathrm{Q}\}.$$

A thermal state in a particle X at time t is described by the value of the function

$$g(X, t) = \{\theta(X, t), \partial_X\theta(X, t), \alpha(X, t)\} \tag{5.2}$$

and the following initial-value problem

$$\dot{\alpha}(X, t), = \overline{A}(\theta, \alpha)\partial_X\theta + \overline{B}(\theta, \alpha), \; \alpha(X, t_0) = \alpha_0(X). \tag{5.3}$$

The thermodynamic postulate now yields:

$$\eta(X, t) = -\partial_\theta\Psi(\theta, \alpha), \tag{5.4}$$

$$\partial_\alpha\Psi\cdot(\overline{A}\partial_X\theta + \overline{B}) + \frac{1}{\varrho\theta}\mathrm{Q}\cdot\partial_X\theta \leqslant 0. \tag{5.5}$$

For this case, the equation for the intrinsic velocity (2.12) takes the form([11]):

$$U^2\varrho\partial_\theta^2\Psi + U(\varrho\partial_\alpha\Psi\cdot\overline{A} - \varrho\theta\partial_\alpha\partial_\theta\Psi\cdot\overline{A} + \partial_\theta\mathrm{Q}) - \partial_\alpha\mathrm{Q}\cdot\overline{A} = 0. \tag{5.6}$$

Theorem 8. *The intrinsic velocity of a thermal acceleration wave in a material described by the assumptions* (5.1)–(5.3) *satisfies Eq.* (5.6).

We next intend to obtain the differential equation determining the amplitude of a thermal acceleration wave. To this end, let us differentiate with respect to time t the energy balance equation in the form (3.8). We have

$$\varrho\overline{\partial_\alpha\Psi}^{\,\boldsymbol{\cdot}}\cdot\dot{\alpha} + \varrho\partial_\alpha\Psi\ddot{\alpha} + \varrho\dot{\theta}\dot{\eta} + \varrho\theta\ddot{\eta} + \partial_X\dot{q} - \varrho\dot{r} = 0. \tag{5.7}$$

Using the constitutive equations, we can prove that the following relations are valid in the wave:

$$[\overline{\partial_\alpha\Psi}^{\,\boldsymbol{\cdot}}] = \partial_\theta\partial_\alpha\Psi\cdot[\dot{\theta}] + \partial_\alpha^2\Psi\cdot[\dot{\alpha}], \tag{5.8}$$

$$[\dot{\eta}] = -\partial_\theta^2\Psi[\dot{\theta}] - \partial_\alpha\partial_\theta\Psi\cdot\overline{A}\cdot[\partial_X\theta], \tag{5.9}$$

$$[\ddot{\eta}] = -\partial_\theta^3\Psi[(\dot{\theta})^2] - 2\partial_\alpha\partial_\theta^2\Psi\cdot(\overline{A}[\dot{\theta}\partial_X\theta] + \overline{B}[\dot{\theta}]) - \partial_\theta^2\Psi[\ddot{\theta}] - \partial_\alpha^2\partial_\theta\Psi\cdot[(\dot{\alpha})^2] - \partial_\alpha\partial_\theta\Psi\cdot[\ddot{\alpha}], \tag{5.10}$$

([11]) The analysis of the propagation of a thermal wave in a material with memory has been presented by Gurtin and Pipkin [11] and by Chen [1, 2].

$$(5.11)\qquad [\partial_X \dot{q}] = \partial_\theta^2 Q[\dot{\theta}\partial_X\theta] + \partial_\alpha\partial_\theta Q\cdot[\dot{\alpha}\partial_X\theta] + \partial_\theta Q[\partial_X\dot{\theta}] + \partial_\theta\partial_\alpha Q\cdot[\dot{\theta}\partial_X\alpha] + \partial_\alpha^2 Q\cdot[\dot{\alpha}\partial_X\alpha] + \partial_\alpha Q\cdot[\partial_X\dot{\alpha}],$$

$$(5.12)\qquad [\dot{\alpha}] = \bar{A}[\partial_X\theta],$$

$$(5.13)\qquad [\ddot{\alpha}] = \partial_\theta\bar{A}[\dot{\theta}\partial_X\theta] + \partial_\alpha\bar{A}\cdot\bar{A}[(\partial_X\theta)^2] + \partial_\alpha\bar{A}\cdot\bar{B}[\partial_X\theta] + \bar{A}[\partial_X\dot{\theta}] + \partial_\theta\bar{B}[\dot{\theta}] + \partial_\alpha\bar{B}\cdot\bar{A}[\partial_X\theta],$$

$$(5.14)\qquad [\partial_X\dot{\alpha}] = \partial_\theta\bar{A}[(\partial_X\theta)^2] - \partial_\alpha\bar{A}\cdot\bar{A}U^{-1}[\partial_X\theta]^2 + \bar{A}[\partial_X^2\theta] + \{\partial_\alpha\bar{A}\cdot\partial_X\alpha^+ - \bar{A}\cdot\partial_\alpha\bar{A}\partial_X\theta^+U^{-1} + \partial_\theta\bar{B} - \bar{A}\cdot\partial_\alpha\bar{B}U^{-1}\}[\partial_X\theta].$$

where the equality

$$(5.15)\qquad [f\cdot h] = [f][h] + f^+[h] + h^+[f]$$

has been used, which is satisfying for two arbitrary functions having jumps across Σ.

Substituting the relations obtained into (5.7), and taking the result on the wave Σ, we have:

$$(5.16)\qquad \mathscr{K}_{Y_t}[(\partial_X\theta)^2] + \mathscr{L}_{Y_t}[\dot{\theta}\partial_X\theta] + \mathscr{M}_{Y_t}[(\dot{\theta})^2] + \mathscr{N}_{Y_t}\beta + S_{Y_t}\beta^2 - \varrho\theta\partial_\theta^2\Psi[\ddot{\theta}] + (\varrho\partial_\alpha\Psi\cdot\bar{A} - \varrho\theta\partial_\alpha\partial_\theta\Psi\cdot\bar{A} + \partial_\theta Q)[\partial_X\dot{\theta}] + \partial_\alpha Q\cdot\bar{A}[\partial_X^2\theta] = 0,$$

where the following notations have been used

$$\mathscr{K}_{Y_t} = \{\varrho\partial_\alpha^2\Psi\cdot(\bar{A})^2 - \varrho\theta\partial_\alpha^2\partial_\theta\Psi\cdot(\bar{A})^2 + \varrho\partial_\alpha\Psi\cdot\partial_\alpha\bar{A}\cdot\bar{A} + \partial_\alpha Q\cdot\partial_\theta\bar{A} - \varrho\theta\partial_\alpha\partial_\theta\Psi\cdot\partial_\alpha\bar{A}\cdot\bar{A} + \partial_\theta\partial_\alpha Q\cdot\bar{A}\}_{Y_t},$$

$$(5.17)\qquad \mathscr{L}_{Y_t} = (\varrho\partial_\theta\bar{A}\cdot\partial_\alpha\Psi - 2\varrho\theta\partial_\alpha\partial_\theta^2\Psi\cdot\bar{A} + \partial_\theta^2 Q - \varrho\theta\partial_\alpha\partial_\theta\Psi\cdot\partial_\theta\bar{A})_{Y_t},$$

$$\mathscr{M}_{Y_t} = -(\varrho\partial_\theta^2\Psi + \varrho\theta\partial_\theta^3\Psi)_{Y_t},$$

$$\mathscr{N}_{Y_t} = \Big\{(\varrho\partial_\alpha\Psi - \varrho\theta\partial_\alpha\partial_\theta\Psi)\cdot(\partial_\theta\bar{B} - U^{-1}\bar{A}\partial_\alpha\bar{B}) + \partial_\alpha Q\cdot(U^{-2}\bar{A}\cdot\partial_\alpha\bar{B} - U^{-1}\partial_\theta\bar{B}) - 2\varrho\theta\partial_\alpha\partial_\theta^2\Psi\cdot B - U^{-1}\{(\varrho\partial_\alpha^2\Psi - \varrho\theta\partial_\alpha^2\partial_\theta\Psi)\cdot 2\bar{A}\bar{B} + \partial_\theta\partial_\alpha Q\cdot\bar{B} + \partial_\alpha^2 Q\cdot(\partial_X\alpha^+ - U^{-1}\dot{\alpha}^+)\bar{A} + (\varrho\partial_\alpha\Psi - \varrho\theta\partial_\alpha\partial_\theta\Psi)\cdot\bar{B}\partial_\alpha\bar{A}\}\Big\}_{Y_t},$$

$$\mathscr{S}_{Y_t} = \{\partial_\alpha\partial_\theta Q\cdot\bar{A}U^{-2} - \partial_\alpha^2 Q\cdot(\bar{A})^2U^{-3} - \partial_\alpha Q\cdot\partial_\alpha\bar{A}\bar{A}U^{-3}\}_{Y_t}.$$

Let us recall the relations

$$(5.18)\qquad \frac{d}{dt}[\dot{\theta}] = [\ddot{\theta}] + U[\partial_X\dot{\theta}], \qquad \frac{d}{dt}[\partial_X\theta] = [\partial_X\dot{\theta}] + U[\partial_X^2\theta].$$

From (5.18) we obtain the equation

$$(5.19)\qquad 2\sqrt{U}\left(\frac{[\dot{\theta}]}{\sqrt{U}}\right)^{\cdot} = [\ddot{\theta}] - U^2[\partial_X^2\theta].$$

If we evaluate $[\partial_X \dot{\theta}]$ from $(5.18)_1$ and $[\partial_X \dot{\theta}]$ from (5 19) and substitute into (5.16), then we have

THEOREM 9. *The amplitude of a thermal acceleration wave in a material described by the assumptions* (5.1)–(5.3) *satisfies the differential equation as follows*:

$$(5.20)\qquad \mathscr{W}_{Y_t}\frac{d\beta}{dt}+(\mathscr{M}_{Y_t}+U^{-2}\mathscr{K}_{Y_t}-U^{-1}\mathscr{L}_{Y_t}+S_{Y_t})\beta^2$$
$$+\left\{2\dot{\theta}\mathscr{M}_{Y_t}-2U^{-1}\partial_X\theta^+ +\mathscr{K}_{Y_t}+\mathscr{N}_{Y_t}+(\partial_X\theta^+ - U^{-1}\dot{\theta}^+)\mathscr{L}_{Y_t}+U^{-3}\frac{dU}{dt}\partial_\alpha \mathrm{Q}\cdot\bar{A}\right\}\beta = 0,$$

where $\mathscr{K}_{Y_t}, \mathscr{L}_{Y_t}, \mathscr{N}_{Y_t}, \mathscr{N}_{Y_t}, S_{Y_t}$ *are given by the expressions* (5.17) *and*

$$\mathscr{W}_{Y_t} = U^{-1}(\varrho\,\partial_\alpha\Psi\cdot\bar{A}+\varrho\,\partial_\alpha\partial_\theta\Psi\cdot\bar{A}+\partial_\theta \mathrm{Q})-2U^{-2}\partial_\alpha \mathrm{Q}\cdot\bar{A}.$$

6. The particular case of a homothermal wave

We shall study the case of a homothermal acceleration wave for a material described by Eqs. (4.1)–(4.4) with the additional condition

$$(6.1)\qquad A(F,\theta,\partial_X\theta,\alpha) = \bar{A}(F,\theta,\alpha)\,\partial_X\theta+\bar{B}(F,\theta,\alpha).$$

The set of Eqs. (4.10)–(4.11) for this particular case yields:

$$(6.2)\qquad (\partial_F \mathrm{T}+\varrho U^2)a = 0,$$

$$(6.3)\qquad (\varrho U\theta\,\partial_F\partial_\theta\Psi+\partial_F \mathrm{Q})a = 0.$$

The intrinsic velocity is determined by

$$(6.4)\qquad U^2 = \frac{\partial_F \mathrm{T}}{\varrho}$$

and the additional equation

$$(6.5)\qquad \varrho U\theta\,\partial_F\theta_\theta\Psi = -\partial_F \mathrm{Q}.$$

has to be satisfied.

The condition (6.5) can be treated as an additional restriction for the partial derivatives of the response functions on the wave Σ.

To determine the amplitude of a homothermal acceleration wave for the case considered we use the differential Eq. (3.21) together with (3.22). We evaluate:

$$(6.6)\qquad [\partial_X \dot{T}] = \partial_F \mathrm{T}[\partial_X \dot{F}]+\partial_F^2 \mathrm{T}[\dot{F}\partial_X F]$$
$$+\frac{1}{U}\left\{\frac{I_{Y_t}}{U}-J_{Y_t}+U^{-1}(\partial_F\bar{B}\cdot\partial_\alpha \mathrm{T}+\partial_\alpha \mathrm{T}\,\partial_F\bar{A}\,\partial_X\theta)\right\}a+(\partial_\theta \mathrm{T}-U^{-1}\partial_\alpha \mathrm{T}\cdot\bar{A})[\partial_X\dot{\theta}],$$

where now

$$(6.7)\qquad \begin{aligned} I_{Y_t} &= (\partial_\theta\partial_F \mathrm{T}\,\dot{\theta}+\partial_\alpha\partial_F \mathrm{T}\cdot\dot{\alpha})_{Y_t},\\ J_{Y_t} &= (\partial_F\partial_\theta \mathrm{T}\,\partial_X\theta+\partial_F\partial_\alpha \mathrm{T}\cdot\partial_X\alpha)_{Y_t}.\end{aligned}$$

Substituting (6.6) into (3.22) and combining the result with (3.21), we obtain

$$(6.8)\qquad 2\sqrt{U}\frac{d}{dt}\left(\frac{a}{\sqrt{U}}\right)=\left(\frac{1}{\varrho}\partial_F\mathrm{T}-U^2\right)[\partial_X\dot{F}]+\frac{1}{\varrho}\partial_F^2\mathrm{T}[\dot{F}\partial_X F]$$
$$\frac{1}{\varrho U}\left\{\frac{I_{Y_t}}{U}-J_{Y_t}+U^{-1}(\partial_F\bar{B}+\partial_F\bar{A}\partial_X\theta)\cdot\partial_\alpha\mathrm{T}\right\}a+\frac{1}{\varrho}(\partial_\theta\mathrm{T}-U^{-1}\bar{A}\cdot\partial_\alpha\mathrm{T})\,[\partial_X\dot{\theta}].$$

We should now express $[\partial_X\dot{\theta}]$ by the amplitude a. We shall find this equation by differentiating with respect to time t (3.8) and writing the result on the wave Σ. This gives:

$$(6.9)\qquad (U^{-2}\mathscr{L}_{1Y_t}-U^{-1}\mathscr{L}_{2Y_t})a+\partial_F^2\mathrm{Q}[\dot{F}\partial_X F]-\varrho\theta\partial_F^2\partial_\theta\Psi[(\dot{F})^2]$$
$$+\mathscr{L}_{3Y_t}[\partial_X\dot{\theta}]+\varrho\theta\partial_F\partial_\theta\Psi\frac{d}{dt}\left(\frac{a}{U}\right)=0.$$

Evaluating $[\partial_X\theta]$ from (6.9) and substituting into (6.8), we finally obtain the differential equation for the amplitude of a homothermal acceleration wave in the following form:

$$(6.10)\qquad (2+\varrho\theta\partial_F\partial_\theta\Psi U^{-1}\mathscr{L}_{0Y_t}\mathscr{L}_{3Y_t}^{-1})\frac{da}{dt}-\frac{a}{U}\frac{dU}{dt}(1+\varrho\theta\partial_F\partial_\theta\Psi U^{-1}\mathscr{L}_{0Y_t}\mathscr{L}_{3Y_t}^{-1})$$
$$-\left(\frac{1}{\varrho}\partial_F^2\mathrm{T}-\partial_F\mathrm{Q}\mathscr{L}_{0Y_t}\mathscr{L}_{3Y_t}^{-1}\right)[\dot{F}\partial_X F]-\varrho\theta\partial_F^2\partial_\theta\Psi\mathscr{L}_{0Y_t}\mathscr{L}_{3Y_t}^{-1}[(\dot{F})^2]$$
$$-\frac{1}{\varrho U}\left\{\frac{I_{Y_t}}{U}-J_{Y_t}+U(\partial_F\bar{B}+\partial_F\bar{A}\partial_X\theta)\cdot\partial_\alpha\mathrm{T}-\varrho\mathscr{L}_{0Y_t}\mathscr{L}_{3Y_t}^{-2}U^{-1}\mathscr{L}_{1Y_t}-\mathscr{L}_{2Y_t}\right\}a=0.$$

In (6.9) and (6.10) we introduced the additional notations

$$\mathscr{L}_{0Y_t}=\frac{1}{\varrho}(\partial_\theta\mathrm{T}-\partial_\alpha\mathrm{T}\cdot\bar{A}U^{-1})_{Y_t},$$
$$\mathscr{L}_{1Y_t}=\{\partial_F\partial_\theta\mathrm{Q}\dot{\theta}+\partial_F\partial_\alpha\mathrm{Q}\cdot\dot{\alpha}+\partial_\alpha\mathrm{Q}\cdot(\partial_F\bar{A}\partial_X\theta+\partial_F\bar{B})\}_{Y_t},$$
$$(6.11)\qquad \mathscr{L}_{2Y_t}=\{\varrho\partial_F\partial_\alpha\Psi\cdot\dot{\alpha}-2\varrho\theta\partial_F\partial_\theta^2\Psi\dot{\theta}-\varrho\partial_F\partial_\theta\Psi\dot{\theta}-2\varrho\theta\partial_F\partial_\theta\partial_\alpha\Psi\cdot\dot{\alpha}$$
$$+\partial_F\partial_\theta\mathrm{Q}\partial_X\theta+\partial_\alpha\partial_F\mathrm{Q}\cdot\partial_X\alpha-(\varrho\theta\partial_\alpha\partial_\theta\Psi-\varrho\partial_\alpha\Psi)\cdot(\partial_F\bar{A}\partial_X\theta+\partial_F\bar{B})\}_{Y_t},$$
$$\mathscr{L}_{3Y_t}=\{\partial_\theta\mathrm{Q}+\varrho\partial_\alpha\Psi\cdot\bar{A}-\varrho\theta\partial_\alpha\partial_\theta\Psi\cdot\bar{A}+\varrho\theta U\partial_\theta^2\Psi-\partial_\alpha\mathrm{Q}\cdot\bar{A}U^{-1}\}_{Y_t}.$$

The results of this section we can gather in the following

THEOREM 10. *A homothermal acceleration wave in a material described by the constitutive assumptions* (4.1)–(4.4), *with the additional condition* (6.1), *propagates with the intrinsic velocity U given by* (6.4). *On the wave, the additional condition* (6.5) *for the response functions has to be satisfied and the amplitude of the wave obeys the differential equation* (6.10).

7. Isothermal wave

We assume that the thermodynamic process considered is isothermal—i.e.:

$$(7.1)\qquad \theta(X,t)=\theta_0=\mathrm{const.}$$

In this case, the function $g(X, t)$ has the form:

$$(7.2)\qquad g(X,t)=\{F(X,t),\ \alpha(X,t)\}.$$

In an isothermal process, a family of functions $\mathscr{P}_X$ is as follows:

$$\mathscr{P}_X = \{F(X,t), \pi(X,t)\}, \tag{7.3}$$

where

$$\pi(X,t) = \{\psi(X,t), \eta(X,t), T(X,t)\}$$

and the thermodynamic postulate has the simple form

$$-\dot{\psi} + \frac{1}{\varrho} T\dot{F} \geqslant 0. \tag{7.4}$$

Equation (2.7) is the constitutive equation for this material. The thermodynamic postulate (7.4) yields

$$T = \varrho\, \partial_F \Psi, \qquad \partial_\alpha \Psi \cdot \dot{\alpha} \leqslant 0. \tag{7.5}$$

The energy balance equation reduces, in this case, to

$$\partial_\alpha \Psi \cdot \dot{\alpha} + \theta_0 \dot{\eta} = 0 \tag{7.6}$$

A chain rule for N implies that, for $X \neq Y_t$, we can write

$$\partial_\alpha \Psi \cdot \dot{\alpha} + \theta_0 \partial_F \mathrm{N} \dot{F} + \theta_0 \partial_\alpha \mathrm{N} \cdot \dot{\alpha} = 0. \tag{7.7}$$

By the Theorem 3, $[\dot{\alpha}] = 0$ and them (7.7) across Σ has the form

$$\theta_0 \partial_F \mathrm{N} [\dot{F}] = 0. \tag{7.8}$$

Because $[\dot{\mathrm{F}}]$ do not vanish on an acceleration wave, we have $\partial_F \mathrm{N} = 0$. Hence, on an acceleration wave the following relation is true

$$-\partial_\alpha \Psi \cdot \dot{\alpha} = \theta_0 \partial_\alpha \mathrm{N} \cdot \dot{\alpha}. \tag{7.9}$$

Under the assumption that $\dot{\alpha} \neq 0$ and $\{\dot{\alpha}^i\}$ are linear independent across Σ, we arrive at:

$$\Psi(F, \alpha) = -\theta_0 \mathrm{N}(\alpha) + \mathrm{C}(F). \tag{7.10}$$

Because in general $\Psi = \varepsilon - \theta \mathrm{N}$, we have for an isothermal acceleration wave the relation

$$\Psi(F, \alpha) = \varepsilon(F) - \theta_0 \mathrm{N}(\alpha), \tag{7.11}$$

where the internal energy E is a function of deformation only and the entropy N depends on the internal parameters α.

Using the smoothness property for the stress function T and the equation of motion (3.4), we obtain:

$$(\partial_F \mathrm{T} - \varrho U^2) a = 0. \tag{7.12}$$

Hence

$$U^2 = \frac{\partial_F \mathrm{T}}{\varrho} \qquad \text{because} \qquad a \neq 0. \tag{7.13}$$

To determine the amplitude of an isothermal acceleration wave, we use the differential equation (3.21) together with (3.22). We evaluate:

$$[\partial_X \dot{\mathrm{T}}] = \partial_F \mathrm{T} [\partial_X \dot{F}] + \partial_F^2 \mathrm{T} [\dot{F} \partial_X F] + \partial_\alpha \mathrm{T} \cdot [\partial_X \dot{\alpha}] + \partial_F \partial_\alpha \mathrm{T} \cdot (\partial_X \alpha [\dot{F}] + \dot{\alpha} [\partial_X F]). \tag{7.14}$$

Substituting (7.14) into (3.22) and combining the result with (3.21), we obtain

$$(7.15)\qquad 2\sqrt{U}\frac{d}{dt}\left(\frac{a}{\sqrt{U}}\right) = \frac{1}{\varrho}\partial_F^2\mathrm{T}[\dot{F}\partial_X F] + \frac{1}{\varrho U}\left(\frac{\partial_F\partial_\alpha\mathrm{T}\cdot\dot{\alpha}}{U} - \partial_F\partial_\alpha\mathrm{T}\cdot\partial_X\alpha + \frac{\partial_F A\cdot\partial_\alpha\mathrm{T}}{U}\right)a.$$

The principal results of this section we summarize in

THEOREM 11. *On an isothermal acceleration wave the free energy is given by the relation* (7.11). *The intrinsic velocity U of an isothermal wave is determined by* (7.13) *and the amplitude of the wave obeys the differential Eq.* (7.15).

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