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Transient response of viscoplastic rectangular plates

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A LINEARIZED theory of thin rigid-viscoplastic plates of arbitrary shape is postulated. The theory is shown to describe adequately such phenomena as mode transition in the initial phase of the motion, and propagation of rigid zones in the terminal phase of plate motion. In particular, the theory proves useful for the determination of maximum permanent deflection and deflected shape of plates subjected to impulsive or general pulse loading. As an illustration, a clamped rectangular plate with uniform initial velocity is considered. The results obtained are compared with experiments on aluminium and mild steel plates recently reported in the literature.

Zaproponowana jest liniowa teoria sztywno-lepkoplastycznych płyt o dowolnym kształcie. Pokazano, że teoria ta opisuje w zadowalający sposób zmianę postaci pola prędkości w czasie oraz propagację sztywnych stref w końcowym etapie procesu. W szczególności teoria okazuje się przydatna do określenia maksymalnych trwałych ugięć oraz kształtu ugiętych płyt poddanych obciążeniu impulsowemu o nieskończenie krótkim lub skończonym czasie trwania. Jako przykład rozważana jest zamocowana płyta prostokątna z danym rozkładem początkowej prędkości. Otrzymane rezultaty porównane są z ostatnio opublikowanymi w literaturze wynikami doświadczeń dla płyt z aluminium i miękkiej stali.

Предложена линейная теория жестко-вязкопластических пластинок произвольного вида. Показано, что данная теория описывает достаточно хорошо изменения поля скоростей во времени и распространение жестких областей в конечном этапе процесса деформирования пластинки. В частности, данная теория пригодна для определения максимальных остаточных прогибов и профиля прогиба пластинок, подвергнутых импульсным нагрузкам в бесконечно коротком или конечном промежутке времени. В качестве примера рассмотрена закрепленная прямоугольная пластинка с заданным начальным распределением скоростей. Полученные результаты сравниваются с опубликованными в последнее время в литературе результатами опытов, произведенных на пластинках из алюминия и мягкой стали.

1. Introduction

AT PRESENT, many exact and approximate analytical methodes are available to treat dynamic problems of inelastic plates under the condition of circular symmetry. A critical review of existing particular solutions can be found for example in [5]. In extending a general theory describing transient plastic and viscoplastic response of plates to non-symmetric problems, one is faced with several serious difficulties of both a conceptional and a mathematical nature. First, the yield condition, expressed in terms of at least three different stress couples, can no longer be interpreted simply on the plane as in the case of circular plates. Consequently, the flow rule cannot be linearized in a straightforward manner. Next, the governing equations are reduced to the system of partial differential equations in two space dimensions and time, whereas in axially symmetric problems only one space variable appears. Considerations of strain rate effects also add to the complexity of the problem.

It is therefore not surprising that dynamic problems for inelastic plates of arbitrary shape have received little attention in the literature. COX and MORLAND [2] were the first to study dynamic plastic deformation of a square plate. A similar problem for viscoplastic plate was analysed in [10]. However, both these particular solutions could not be easily extended to other plate geometry. JONES *et al.* [3] have recently undertaken an experimental study into the dynamic behaviour of impulsively loaded clamped rectangular plates. They measured the deflected shape of several aluminium and mild steel specimens, and determined the relative importance of geometry changes and strain rate effect in the reduction of permanent vertical deflections. In a subsequent paper, JONES [4] analysed one of these effects and presented an approximate theory for perfectly plastic arbitrarily shaped plates. The approach used in [4] was based on the yield line theory suitably modified to take into account large displacements and transverse inertia terms. Good agreement with experiments on strain rate insensitive plates was obtained. At the same time, the final shape of plates was imposed through the assumption of a velocity pattern stationary in time; hence, the predicted final displacements were far from reality.

In the present paper, an attempt is made to construct a linearized theory of viscoplastic plates and to study the strain rate effect — a second factor which governs the transient response of dynamically loaded structures. It is known from the analysis of beams [6, 9] and circular plates [12] that the dynamic process in viscoplastic structures is characterized by an appreciable mode change of the velocity fields in the initial phase of the motion, and propagation of rigid zones during the terminal phase of the motion. Both these phenomena are icorporated in the present theory. The method presented provides a convenient means for the determination of the permanent shape of the plate and thus, in a sense, complements previous works by JONES [4].

The material behaviour considered here is that of rigid-viscoplastic type. The linearization of constitutive equations is based on similar arguments as in the earlier study by one of the present authors, devoted to axially symmetric shells [13]. All considerations are restricted to infinitesimal strains and small displacements. By way of an example of application, a particular case of fully clamped rectangular plate acted on by a uniformly distributed impulse is studied in detail.

2. Linearization

In the theory of thin plates, the state of stress is essentially plane. Using the rectangular Cartesian coordinate system $x_\alpha, (\alpha, \beta = 1, 2)$, the special case of constitutive equations for rigid viscoplastic material due to PERZYNA takes the form:

$$
\begin{aligned}
&\dot{\varepsilon}_{\alpha\beta} = \gamma\left(\frac{\sqrt{J_2}}{k} - 1\right)\frac{s_{\alpha\beta}}{\sqrt{J_2}} \quad &&\text{for} \quad \sqrt{J_2} > k,\\
&\dot{\varepsilon}_{\alpha\beta} = 0 &&\text{for} \quad \sqrt{J_2} \leqslant k,
\end{aligned}
\tag{2.1}
$$

where $J_2 = \frac{1}{6}[3\sigma_{\alpha\beta}\sigma_{\alpha\beta} - \sigma_{\alpha\alpha}\sigma_{\beta\beta}]$, k is a yield stress in simple shear, γ denotes the viscosity constant, and $s_{\alpha\beta} = \frac{1}{3}[3\sigma_{\alpha\beta} - \sigma_{\gamma\gamma}\delta_{\alpha\beta}]$. An alternative manner of writing Eq. (2.1) is:

$$\text{(2.2)} \qquad \begin{aligned} \dot{\varepsilon}_{\alpha\beta} &= \frac{\gamma}{k}[s_{\alpha\beta} - \overset{\circ}{s}_{\alpha\beta}] \quad &\text{for} \quad \sqrt{J_2} > k, \\ \dot{\varepsilon}_{\alpha\beta} &= 0 \quad &\text{for} \quad \sqrt{J_2} \leqslant k, \end{aligned}$$

where the new stress deviator $\overset{\circ}{s}_{\alpha\beta}$ is defined by

$$\text{(2.3)} \qquad \frac{s_{\alpha\beta}}{\sqrt{J_2}} = \frac{\overset{\circ}{s}_{\alpha\beta}}{\sqrt{\overset{\circ}{J}_2}}, \qquad \overset{\circ}{J}_2 = \frac{1}{2}\overset{\circ}{s}_{\alpha\beta}\overset{\circ}{s}_{\alpha\beta} = k^2.$$

In the case of plane strain, Eq (2.2) can be inverted and expressed in terms of $\sigma_{\alpha\beta}$ and $\dot{\varepsilon}_{\alpha\beta}$, using the definition introduced for the stress deviations,

$$\text{(2.4)} \qquad \sigma_{\alpha\beta} - \overset{\circ}{\sigma}_{\alpha\beta} = \frac{k}{\gamma}[\dot{\varepsilon}_{\alpha\beta} + \dot{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta}],$$

where $\overset{\circ}{\sigma}_{\alpha\beta}$ is related to $\sigma_{\alpha\beta}$ by

$$\text{(2.5)} \qquad \overset{\circ}{\sigma}_{\alpha\beta} = k\frac{\sigma_{\alpha\beta}}{\sqrt{J_2}}.$$

Integration of Eq. (2.4) across the plate section by means of the Love-Kirchhoff hypothesis leads to the following flow rule:

$$\text{(2.6)} \qquad M_{\alpha\beta} - \overset{\circ}{M}_{\alpha\beta} = \frac{4h^3}{3}\frac{k}{\gamma}(\dot{\varkappa}_{\alpha\beta} + \dot{\varkappa}_{\gamma\gamma}\delta_{\alpha\beta}),$$

where $2h$ is the plate thickness and $\dot{\varkappa}_{\alpha\beta}$ denotes curvature rates corresponding to bending moments $M_{\alpha\beta}$. The flow rule (2.6) is not directly applicable, since $\overset{\circ}{M}_{\alpha\beta}$ is related to $\sigma_{\alpha\beta}$ by means of the nonlinear expression:

$$\text{(2.7)} \qquad \overset{\circ}{M}_{\alpha\beta} = k\int_{-h}^{h}\frac{\sigma_{\alpha\beta}}{\sqrt{J_2}}z\,dz.$$

The above integration constitutes the main difficulty in transforming the flow rule (2.1) to the space of bending moments and curvature rates. For circular plates, where principal directions in all layers coincide, the integration (2.7) can be performed and $\overset{\circ}{M}_{\alpha\beta}$ is expressed in terms of $M_{\alpha\beta}$ as

$$\text{(2.8)} \qquad \overset{\circ}{M}_{\alpha\beta} = M_0\frac{M_{\alpha\beta}}{\sqrt{M_{\alpha\alpha}^2 - M_{\alpha\alpha}M_{\beta\beta} + M_{\beta\beta}^2}},$$

where $M_{\alpha\alpha} = M_r$, $M_{\beta\beta} = M_\theta$ denote respectively radial and circumferential bending moments and $M_{\alpha\beta} = 0$, $\alpha \neq \beta$. For general plates, the relation of the kind (2.8) is no longer true, but following [13] a simple method of linearization can be suggested. To this end an initial-boundary value problem must first be formulated.

Suppose the viscoplastic plate is loaded by a pressure $P(x_\alpha, t)$ variable in space and time. Equations of motion with transverse inertia terms have the form:

$$\text{(2.9)} \qquad T_{\alpha,\alpha} + P = \mu\ddot{w}, \qquad M_{\alpha\beta,\alpha} = T_\beta,$$

where T_α is shearing force, μ is mass density per unit area of the plate middle surface, and w denotes the vertical component of the displacement vector. Differentiation with respect to space and time variables are indicated respectively by a comma and a dot. In the bending theory of plates, the geometric relations are:

$$\dot{\varkappa}_{\alpha\beta} = -\dot{w}_{,\alpha\beta}. \tag{2.10}$$

The solution of the dynamic problem stated above is characterized by time variable fields of moments $M_{\alpha\beta}(t)$, velocities $\dot{w}(t)$ and accelerations $\ddot{w}(t)$. It was found in [11] that while the moment $M_{\alpha\beta}(t)$ changes appreciably in the course of a deformation process following a certain trajectory which lies outside the static yield surface, the moment $\overset{\circ}{M}_{\alpha\beta}$ stays always on the static yield surface and changes very little with time. Consequently, $\overset{\circ}{M}_{\alpha\beta}$ can be taken as constant in a given problem and may be approximated by for example, the moment distribution $M^*_{\alpha\beta}$ of the similar static problem. The validity of this procedure was checked several times on examples of circular plates and rotationally symmetric shells; detailed presentation is given in [13], where the introduced approximation of $\overset{\circ}{M}_{\alpha\beta}$ by $M^*_{\alpha\beta}$ was interpreted on the basis of a non-associated flow rule.

The same approach is extended now to general plate problems in the form of a hypothesis. The present hypothesis not only leads to a full linearization of a flow rule (2.6) but also enables the tensor quantity $M^*_{\alpha\beta}$ to be replaced by a single scalar quantity P^*, which can be identified with a load-carrying capacity of the considered plate made of rigid perfectly plastic material.

Subtracting the equation of static equilibrium

$$M^*_{\alpha\beta,\alpha\beta} + P^* = 0 \tag{2.11}$$

from the equation of motion (2.9), we obtain

$$(M_{\alpha\beta} - M^*_{\alpha\beta})_{,\alpha\beta} + (P - P^*) = \mu\ddot{w}. \tag{2.12}$$

Substituting now the linearized flow rule (2.6), with $\overset{\circ}{M}_{\alpha\beta}$ replaced by $M^*_{\alpha\beta}$, and the geometrical relations (2.10) into (2.12), we obtain the final form of the equation describing motion of a viscoplastic plate

$$\nabla^4\dot{w} + a\ddot{w} - b(P - P^*) = 0, \tag{2.13}$$

where the Laplace operator $\nabla^4\dot{w} = \dot{w}_{,\alpha\alpha\beta\beta}$ and the material constants are $a = 3\gamma\mu/8h^3k$, $b = 3\gamma/8h^3k$.

3. Properties of the governing equation

Before discussing the general properties of Eq. (2.13), it should be noted that this equation applies only in regions of viscoplastic flow. In view of the unloading criterion, the dynamic problem for a rigid-viscoplastic plate includes in general also rigid regions. The location of the boundary between two regions of distinct behaviour is determined by the requirement $\sqrt{J_2} = k$.

Thus, the unloading condition is still nonlinear, whereas the flow rule itself has been linearized. In fact, PRAGER [8] started the linearization procedure by linearizing the nonlinear loading criterion $\sqrt{J_2} > k$ and replacing it by a set of linear inequalities.

As regards the plate problem, the material becomes rigid if the stress point reaches the static yield surface in the space of moments $M_{\alpha\beta}$. To be consistent with the linearized flow rule, we have to assume that unloading starts when $M_{\alpha\beta} = M^*_{\alpha\beta}$. According to (2.6), this means that $\dot{\varkappa}_{\alpha\beta}+\dot{\varkappa}_{\gamma\gamma}\delta_{\alpha\beta} = 0$, which is equivalent to the requirement that $\dot{\varkappa}_{\alpha\beta} = 0$. In general, we cannot expect that all three components of the curvature rate tensor vanish simultaneously, and hence the unloading criterion can be satisfied only in an approximate manner.

In structural problems for rigid viscoplastic materials solved in velocities, it is advantageous to use a slightly different interpretation of the unloading criteria. Since the model of the material considered is entirely dissipative, the motion of an arbitrary point will cease if a finite value of the energy has been introduced to the body. Usually, we are interested in the moment at which a given point is brought to rest — i.e., $\dot{w} = 0$.

Return now to Eq. (2.13), which is a linear partial differential equation of the parabolic type. Its structure is formally analogous to the equations describing forced vibrations of elastic plates. The difference is in the definition of the stiffness coefficients a and b, and above all in the time derivative of the first term. In the classical elasticity, instead of the velocity $\nabla^4\dot{w}$ we have the displacement ∇w^4. Consequently, the time process is of periodic character and vibrations take place. By contrast, the deformation of a rigid-viscoplastic body is associated with the irreversible dissipation of energy which excludes the occurrence of vibration. This property reflects in Eq. (2.13), where additional time derivative in the first term makes the time process to be exclusively of periodic nature, exp $\left(-\frac{\lambda_{ij}^4}{a}t\right)$. To be more specific, we shall show this by performing a mode analysis of Eq. (2.13). For simplicity, let us assume that the "forcing" term $P(t)$ is held constant. Physically, this corresponds to a rectangular pulse or ideal impulse loading. Under this restriction, the solution to the inhomogeneous Eq. (2.13) can be sought in the form:

$$\dot{w}(x_\alpha, t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij}(t)\psi_{ij}(x_\alpha) - f(x_\alpha), \tag{3.1}$$

where $f(x_\alpha)$ satisfies

$$\nabla^4 f - b(P-P^*) = 0, \tag{3.2}$$

and approximate boundary conditions. The eigenfunctions $\psi_{ij}(x_\alpha, \lambda_{ij})$ form a complete orthogonal system satisfying

$$\nabla^4\psi_{ij} + \lambda_{ij}^4\psi_{ij} = 0, \tag{3.3}$$

and the boundary conditions. The above equation is identical with the case of free vibration of plates; thus, eigenfunctions in the solution of a dynamic problem of a viscoplastic plate are simply natural modes of vibration of elastic plates. Similarly, the eigenvalues λ_{ij} in (3.3) would correspond to those from elastic solution, provided the boundary conditions for velocities and displacements are the same.

The equation for the time function $\lambda_{ij}^4 A + a\dot{A} = 0$ immediately yields the solution:

$$A_{ij}(t) = A_{ij}(0)e^{-\frac{\lambda_{ij}^4}{a}t}, \tag{3.4}$$

which is clearly of the aperiodic type. Coefficients $A_{ij}(0)$ should be so chosen as to satisfy the initial conditions imposed. The decay of the eigenfunctions depends on λ_{ij}, and each is different. Hence, the solution (3.2) describes a mode transition during the plate motion.

The time at which a given point on the plate middlesurface is brought to rest is determined by the condition $\dot{w} = 0$. Since each eigenfunction in the original non-homogeneous Eq. (2.13) vanishes at different time t_{ij}^f, the plate will never stop at all its points simultaneously. On the plane x_α, there exists a moving interface separating regions of viscoplastic and rigid behaviour.

Despite its simplicity, the Eq. (2.13) derived, together with the unloading condition, describe a variety of mechanical phenomena such as plastic and viscous effects, propagation of rigid zones and mode change during the plate motion. A solution to Eq. (3.2) provides a quick method of finding maximum permanent deflection and deflection shape of the structure considered.

4. Impulsive loading of rectangular plate

The second part of the present paper is devoted to a detailed analysis of an exemplary boundary value problem using the method of solution described above.

Consider a fully clamped rectangular plate with dimensions shown in Fig. 1 subjected

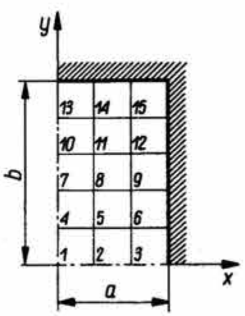

FIG. 1. Dimensions of rectangular plate.

to a uniformly distributed transverse impulse I. The undeformed plate is flat so that the initial conditions are:

$$\dot{w}(x, y, 0) = \frac{I}{\mu}, \quad w(x, y, 0) = 0. \tag{4.1}$$

The boundary conditions are expressed as:

$$\begin{aligned} x &= \pm a, \quad \dot{w} = 0, \quad \frac{\partial \dot{w}}{\partial x} = 0, \\ y &= \pm b, \quad \dot{w} = 0, \quad \frac{\partial \dot{w}}{\partial y} = 0. \end{aligned} \tag{4.2}$$

In the impulsive loading problem, to pressure loading teım is identically equal to zero, $P \equiv 0$, and Eq. (2.13) is reduced to

$$L(\dot{w}) \equiv \frac{2}{3\alpha}\nabla^4\dot{w}+\ddot{w}+\frac{P^*}{\mu} = 0, \tag{4.3}$$

where the Laplace operator in the rectangular Cartesian coordinate system is defined as $\nabla^4\dot{w} = \frac{\partial^4\dot{w}}{\partial x^4}+2\frac{\partial^4\dot{w}}{\partial x^2\partial y^2}+\frac{\partial^4\dot{w}}{\partial y^4}$ and the coefficient α denotes $\alpha = \sqrt{3}\gamma\beta^2\mu a^4/2h^3\sigma_0$. As the static load-carrying capacity of a rigid-perfectly plastic plate we take an approximate expression derived by WOOD [14] in the case of uniformly distributed pressure:

$$P^* = \frac{12M_0\beta^2}{b^2(\sqrt{3+1/\beta^2}-1/\beta^2)^2}, \tag{4.4}$$

where $\beta = b/a$ denotes an aspect ratio.

It is known from the theory of vibrations of elastic plates that in the case of the clamped end condition, an effective solution cannot be reached by the eigenvalue method, since it is not possible to express the solution to (3.3) and (4.2) in terms of elementary functions. However, the double series representation of the solution can still be applied by assuming a complete system of functions $\psi_{ij}(x, y)$ satisfying the boundary conditions. The unknown set of time functions $A_{ij}(t)$ will be determined using the familiar Galerkin procedure

$$\int_0^a\int_0^b L(w)\psi_{kl}dy = 0, \quad k, l = 1, 2, \ldots, n. \tag{4.5}$$

We choose now the function $\psi_{ij}(x, y)$ as a product of two functions depending respectively on x and y

$$\psi_{ij}(\xi, \eta) = X_i(\xi)\, Y_j(\eta), \tag{4.6}$$

where $\xi = x/a$ and $\eta = y/b$ denote dimensionless coordinates. In what follows, we shall retain only four terms in the expansion (3.2), namely:

$$\begin{aligned} X_1 &= (\xi^2-1)^2, & Y_1 &= (\eta^2-1)^2, \\ X_2 &= (\xi^2-1)^2\xi^2, & Y_2 &= (\eta^2-1)^2\eta^2. \end{aligned} \tag{4.7}$$

The above functions were used in [1] adequately to predict the static deflection profile of elastic clamped plates.

After substituting (4.6) and (4.7) into (4.5), and time-consuming but straightforward calculation, a system of four linear first order differential equations is obtained for the time variable amplitude $\mathbf{A}(t)$:

$$\mathbf{\Lambda}\dot{\mathbf{A}}+\frac{1}{\alpha}\mathbf{\Gamma}(\beta)\mathbf{A}+\frac{P^*}{\mu}\mathbf{\Phi} = 0, \tag{4.8}$$

where $\mathbf{A}$ is a column vector with the components $[A_{11}, A_{12}, A_{21}, A_{22}]$, $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ denote square matrices of numerical coefficients, and $\mathbf{\Phi}$ is again a column vector. A short derivation of (4.8) and definition of all matrices and vectors appearing in (4.8) and the two following equations are given in the Appendix.

The system of Eqs. (4.8) should be reduced to the normal Cauchy form, convenient in numerical computations:

$$\dot{\mathbf{A}}+\frac{1}{\alpha}\mathbf{\Omega}(\beta)+\frac{P^*}{\mu}\mathbf{\Phi}'=0, \tag{4.9}$$

where $\mathbf{\Omega}(\beta)=\mathbf{\Gamma}(\beta)\mathbf{\Lambda}^{-1}$ and $\mathbf{\Phi}'=\mathbf{\Phi}\mathbf{\Lambda}^{-1}$.

Initial conditions for $\mathbf{A}$ are found by expanding the uniform initial velocity in double series:

$$\dot{w}(\xi,\eta,0)=V_0=\frac{I}{\mu}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}A_{ij}(0)X_iY_j, \tag{4.10}$$

and using the orthogonal properties of functions given by (4.6) and (4.7). It is found that components of $\mathbf{A}(0)$ have the form:

$$\begin{gathered} A_{11}(0)=0.71336I/\mu, \quad A_{22}(0)=31.1418I/\mu,\\ A_{12}(0)=A_{21}(0)=4.1407I/\mu. \end{gathered} \tag{4.11}$$

In order to integrate numerically the system (4.9), all geometrical and mechanical constants appearing in the definitions of α, P^* and μ should be fixed. To enable comparison with existing experimental data, we choose dimensions and parameters of mild steel specimen No. 14 tested by JONES *et al.* [3]

Table 1

σ_0	ϱ	$2h$	V_0	a	b
$\frac{\text{lb}}{\text{in}^2}$	$\frac{\text{lbsec}^2}{\text{in}^4}$	in	$\frac{\text{ft}}{\text{sec}}$	in	in
$3.38\cdot10^2$	$7.68\cdot10^{-4}$	0.098	231.13	1.5	2.53125

The initial value problem (4.9) and (4.11) was solved using the Runge-Kutta-Falson method of integration of the seventh order. All computations were carried out on an ODRA 1204 electronic computer. The main results obtained for three different values of the viscosity constant are presented in Figs. 2–4. It is seen that all amplitudes of the modal function A_{ij} essentially diminish from the same prescribed initial values to zero. We observe a nearly exponential decay for $\gamma=50$ and $\gamma=200$ as predicted by the general eigenvalue analysis (formula 2.4) and almost linear variation for $\gamma=10000$, similarly as in the single mode solution for perfectly plastic material [7]. The plots of the functions A_{11} and A_{12} in Figs. 2–3 exhibit an initial rise for small times, contrary to what might be expected from the general discussion regarding aperiodic motion. This is attributed to the approximate character of the solution, which retains only four terms in the expansion. In fact, the rise observed is not large, but to describe adequately the sudden change in the slope of the A_{12} curve, a numerical procedure of greater accuracy had to be introduced.

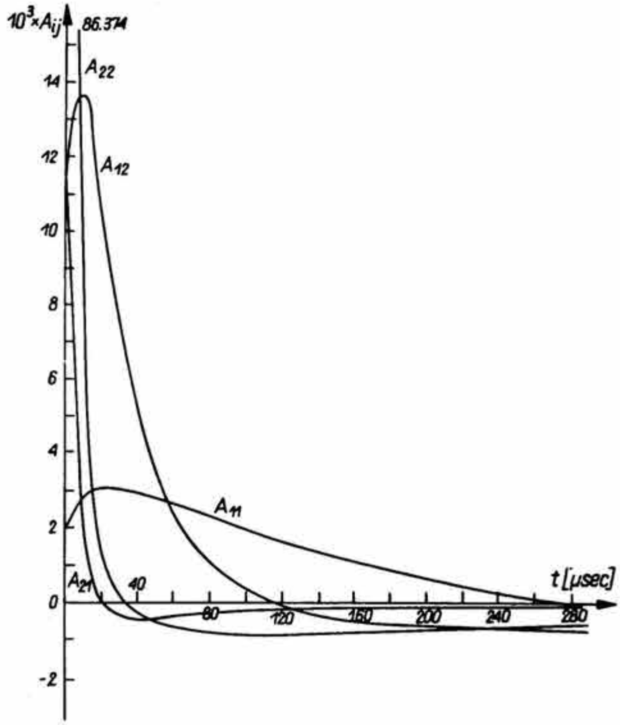

FIG. 2. Time variation of the amplitudes $A_{ij}(t)$ for $\gamma = 50$.

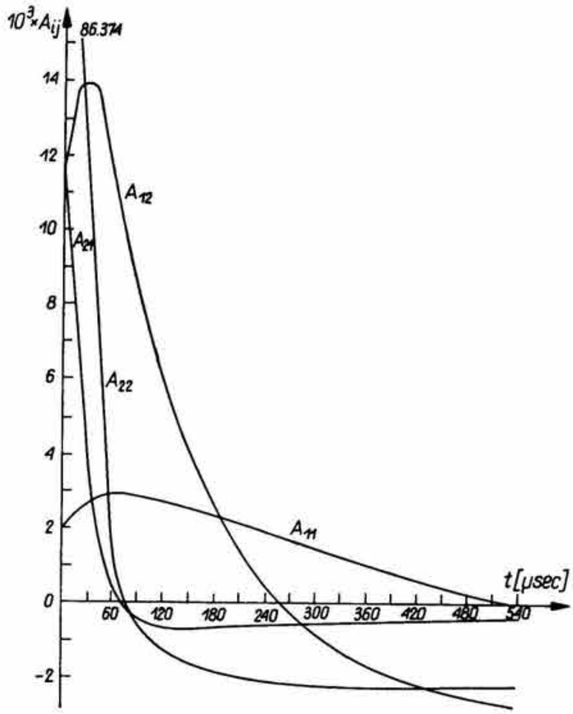

FIG. 3. Time variation of the amplitudes $A_{ij}(t)$ for $\gamma = 200$.

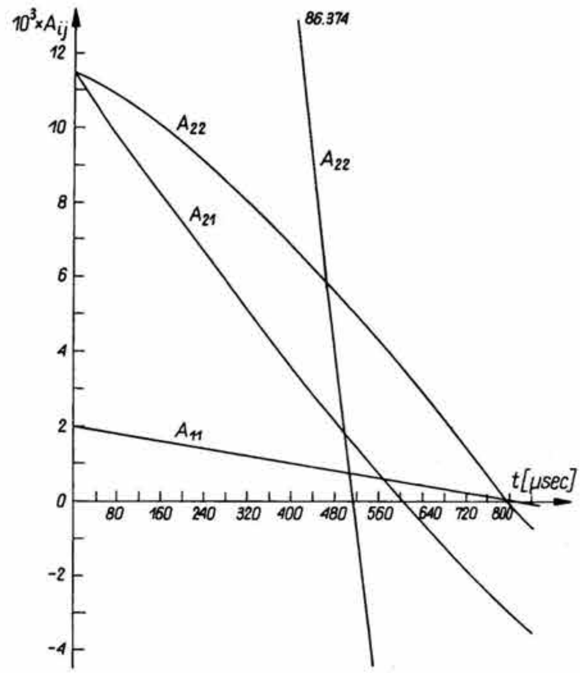

FIG. 4. Time variation of the amplitudes $A_{ij}(t)$ for $\gamma = 10000$.

5. Discussion and comparison with experiments

Having found the time variable amplitude $A_{ij}(t)$ of the subsequent mode functions $\psi_{ij}(x, y)$, the velocity of an arbitrary point of the middle surface of the plate can be obtained by computing $\sum\limits_{i=1}^{2} \sum\limits_{i=1}^{2} A_{ij}\psi_{ij}$, according to (3.2). This has been done numerically at fifteen points of the plate indicated in Fig. 1. The variation of velocities with time is shown in Figs. 5–7. It is now clear that different points in the plate are brought to rest at different times, hence, the intersections of velocity curves with the axis $\dot{w} = 0$ determine the propagation of rigid zones. For example, in the case of $\gamma = 200$ first stop points on the outer edge (curves labeled 15, 14) while the centre of the plate is the last to stop (curve labeled 1). Note that the velocity of the centre of the plate is equal to the plot of the function $A_{11}(t)$.

Values of time to rest of all points considered are gathered in Table 2. For comparison, also given is a response time found from approximate analysis (next paragraph).

Permanent plate deflections are easily found through the integration of velocity diagrams. Numerical values are given in Table 2, together with the approximate solution referred to above. A graphical representation of the deflection profiles at five sections $y =$ = const and three sections $x =$ const is shown in Fig. 8. A full line represents the present solution with four-term approximation, while the broken line denotes a similar solution with only one term retained in the expansion. Experimentally measured deflection profiles are indicated by crosses [3]. Fairly good agreement can be noted over the central

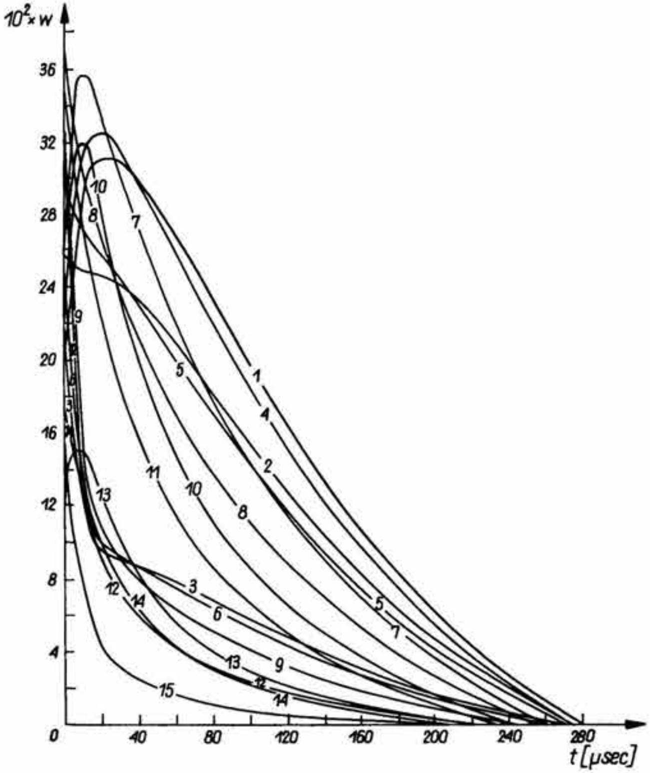

FIG. 5. Velocities and time to rest of different plate points for $\gamma = 50$.

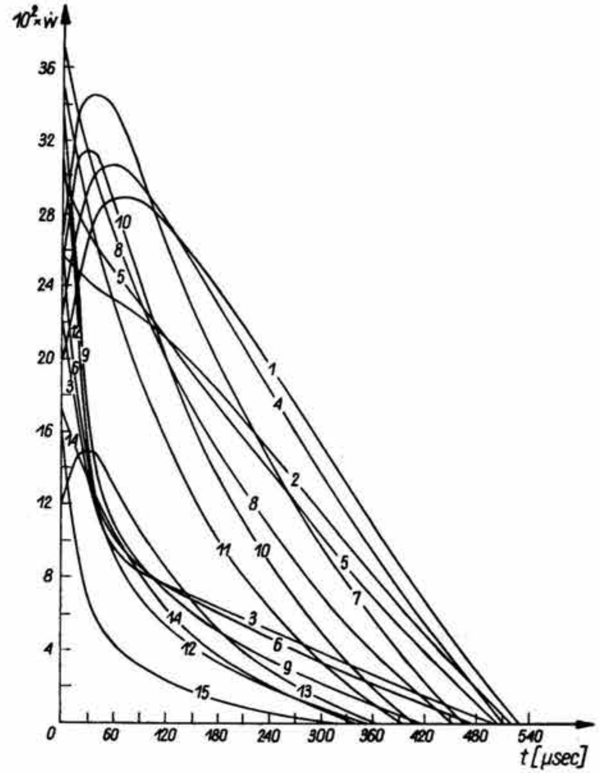

FIG. 6. Velocities and times to rest of different plate points for $\gamma = 200$.

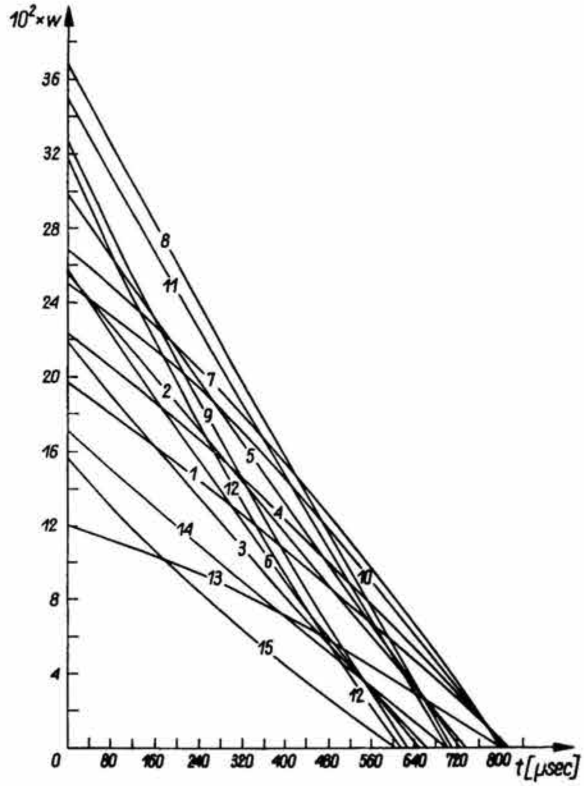

FIG. 7. Velocities and times to rest of different plate points for $\gamma = 10000$.

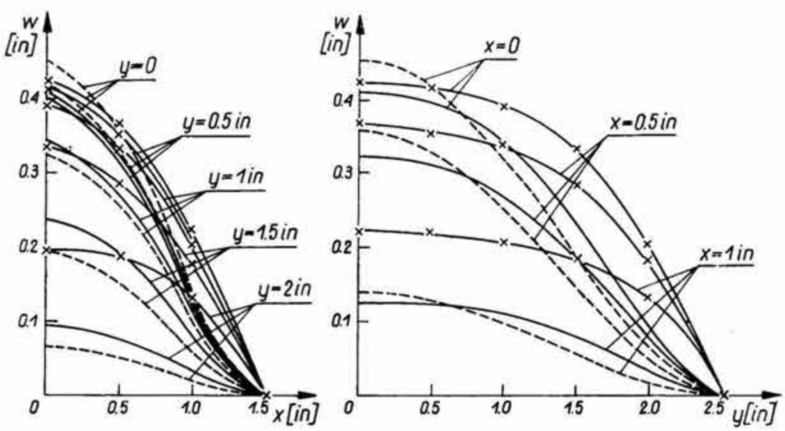

FIG. 8. Theoretical and experimental deflection profiles of a rectangular plate.

Table 2

	No. points of the plate	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\gamma = 50\ \mathrm{sec}^{-1}$	w_f[in] (four terms)	0.4094	0,3245	0.1282	0.3989	0.3136	0.1222	0.3406	0.2722	0.1136	0.2364	0.1920	0.0810	0.0920	0.0764	0.0342
	w_f [in] (one term)	0.4551	0.3596	0.1405	0.4203	0.3321	0.1297	0.3241	0.2561	0.1000	0.1916	0.1514	0.0591	0.0642	0.0508	0.0198
	t_f [μsec] (four terms)	280.02	278.08	272.86	275.27	273.10	266.37	261.46	258.27	248.85	242.39	237.84	224.35	221.19	215.44	197.75
	t_f [μsec] (one term)	265.67														
$\gamma = 200\ \mathrm{sec}^{-1}$	w_f[in] (four terms)	0.8805	0.6982	0.2758	0.8532	0.6782	0.2667	0.5088	0.4058	0.2563	0.5391	0.4507	0.1971	0.2272	0.1876	0.0855
	w_f [in] (one term)	0.9946	0.7858	0.3070	0.9185	0.7257	0.2835	0.7083	0.5597	0.2186	0.4187	0.3308	0.1292	0.1404	0.1109	0.0433
	t_f [μsec] (four terms)	530.65	522.8	499.05	512.10	503.00	475.10	465.00	452.53	416.000	409.05	394.08	352.40	362,46	347.70	406.98
	t_f [μsec] (one term)	480.23														
$\gamma = 10000\ \mathrm{sec}^{-1}$	w_f [in] (four terms)	0.8345	0.9400	0.7085	0.9580	1.0370	0.8093	1.1680	1.3000	0.9680	1.090	1.2120	0.9040	0.5343	0.5885	0.4371
	w_f [in] (one term)	1.7359	1.3715	0.5358	1.6030	1.2666	0.4948	1.2363	0.9768	0.3816	0.7308	0.5774	0.2255	0.2450	0.1936	0.0756
	t_f [μsec] (four terms)	813.5	732.85	663.40	810.25	724.18	648.72	805.68	711.40	627.00	802.86	703.40	613.20	801.34	698.90	604.88
	t_f [μsec] (one term)	729.60														

part of the plate but systematic deviations are observed near the clamped edge. All tests reported in [3] clearly show the occurrence of plastic hinges along the plate circumference. Such discontinuities in the slope of the deflection profile are not admissible in the present theory of viscoplastic plates; hence, the boundary condition (4.2) of the zero slope was assumed as $\dot{w}' = 0$. As regards the deflection profile of circular clamped plates, somewhat similar discrepency between theory and experiment was observed in [12].

The inconsistences indicated above call for a suitable modification of the theory of rigid-viscoplastic materials to allow for zones of localized permanent deformations.

6. A single mode solution

It is possible to obtain a closed form solution to the initial-boundary value problem (4.1)–(4.3) by considering only one term in the expansion (3.2). Such solutions are often of practical value, since they identify certain important parameters and enable discussion of the influence of the viscosity constant on the final central deflection of the plate.

Applying the Galerkin method to the one term approximation, the following ordinary differential equation for the amplitude $A_{11}(t)$ is obtained:

$$f\dot{A}_{11}+cA_{11}+gP^* = 0, \tag{6.1}$$

where $f = 165.119\,\beta^2a^4\mu$, $c = 1307.91\,M_0 2h/\sqrt{3\gamma}$; $g = 284.444\,\beta^2a^4$. Similarly, the initial condition is now given by:

$$A_{11}(0) = 1.723\frac{I}{\mu}. \tag{6.2}$$

Equation (6.1) has a simple analytical solution and by means of the separable form (3.2) the velocity of an arbitrary point of the plate can be determined. However, the values we are mostly interested in are response time t_f and permanent deflections w_f. In the case of a one-degree-of-freedom velocity field, the plate comes to rest at all its point simultaneously and the response time, found from the condition $\dot{w}(0, 0, t_f) = 0$, is equal to:

$$t_f = \frac{A_{11}(0)f}{gP^*}\vartheta\ln\left(1+\frac{1}{\vartheta}\right), \tag{6.3}$$

where $\vartheta = gP^*/A_{11}(0)\cdot c$. The permanent deflections, determined by integrating velocities within the limits $[0, t_f]$, is found to be:

$$w_f(\xi,\eta) = \frac{fA_{11}^2(0)}{2gP^*}\left[2\vartheta-2\vartheta^2\ln\left(1+\frac{1}{\vartheta}\right)\right](1-\xi^2)^2(1-\eta^2)^2. \tag{6.4}$$

Both (6.3) and (6.4) resemble similar expressions obtained in [12] for a clamped circular plate. Terms in brackets involving the parameter ϑ describe the influence of strain rate sensitivity and are responsible for the reduction of the response time and central deflection, Fig. 9. In the limiting case of perfectly plastic material, γ and hence ϑ tend to infinity and the expressions for t_f and w_f reduce to

$$\lim_{\gamma\to\infty} t_f = \frac{fA_{11}(0)}{gP^*} = \frac{I}{P^*} = \frac{V_0a^2\mu}{12M_0}(\sqrt{1/\beta^2+3}-1/\beta)^2. \tag{6.5}$$

$$\lim_{\gamma\to\infty}(w_f/2h) = \frac{\mu b^2 V_0^2}{12 M_0} 0.865(\sqrt{1/\beta^2+3}-1/\beta)^2. \tag{6.8}$$

The approximate solution for t_f (6.5) agrees with the time bound computed by JONES [3] (formula 8), while the maximum central deflection (6.6) differs by a numerical factor 0.865 from Jones's estimate.

For all finite values of γ, the viscosity diminishes the plate deflections relatively to the rigid-perfectly plastic solution. Comparison of present results (full lines) with experiments on mild steel plates and the approximate solution reported in [3] is presented in Fig. 10.

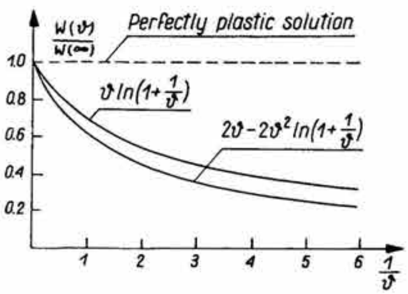

FIG. 9. Reduction of response time and permanent deflections of a rectangular plate due to viscosity of the material.

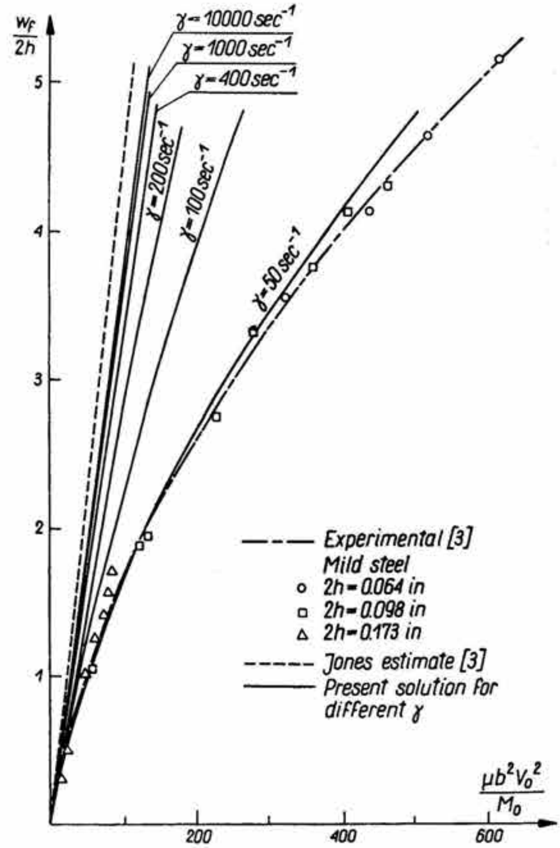

FIG. 10. Central deflection versus applied impulse. Theoretical curves for different γ and experimental points.

Some curves drawn ($\gamma = 50$) closely follow the trend of experimental points over the entire range of the impulse applied. Since the present analysis disregards the important effect of geometry changes and the resulting strengthening of the plate due to membrane action, the correlation cannot be made by considering viscous effects alone. The present theory is valid for small deflection — say of the order of the plate thickness. In this range, a reasonable value of the viscosity constant would be $\gamma = 200$, and this agrees with the value of γ previously found in the case of circular plates [12].

7. Conclusions

It is shown in this paper that an extremely simple linearly-viscoplastic model of material behaviour proves satisfactory for the description of certain important features of transient response of non-circular plates subject to high intensity loading. It was assumed that permanent deflections and response time can be predicted by considering the medium of entirely dissipative character. Such a behaviour was ensured by imposing unique relations between stresses and strain rates and introducing a suitably formulated unloading condition. In the case of an impulsively loaded rectangular plate, the linearized governing equation was shown to yield satisfactory qualitative and quantitative results. However, the occurrence of plastic hinges in experiments on strain rate sensitive plates indicates that the existing theory of viscoplastic behaviour should be revised to allow for such discontinuities.

Appendix

Substituting (4.6) and (4.7) into (4.5), we obtain a system of four equations involving 68 coefficients. These coefficients can be expressed in terms of 11 constants:

$$\text{(A.1)}\qquad \begin{aligned} a_{ij} &= \int_0^1 X_i X_j d\xi = \int_0^1 Y_i Y_j d\eta, & b_{ij} &= \int_0^1 X_i X_j'' d\xi = \int_0^1 Y_i X_j'' d\eta,\\ c_{ij} &= \int_0^1 X_i X_j^{\mathrm{IV}} d\xi = \int_0^1 Y_i Y_j^{\mathrm{IV}} d\eta, & d_i &= \int_0^1 X_i d\xi = \int_0^1 Y_i d\eta. \end{aligned}$$

Numerical values of a_{ij}, b_{ij}, c_{ij} and d_i are:

$$\text{(A.2)}\qquad \begin{aligned} &a_{11} = \frac{128}{3\cdot 105}, \quad a_{12} = a_{21} = \frac{128}{33\cdot 105}, \quad a_{22} = \frac{128}{143\cdot 105},\\ &b_{11} = \frac{-128}{105}, \quad b_{12} = b_{21} = 0, \quad b_{22} = \frac{-128}{11\cdot 105},\\ &c_{11} = \frac{64}{5}, \quad c_{12} = c_{21} = \frac{64}{35}, \quad c_{22} = \frac{3\cdot 64}{35}, \quad d_1 = \frac{8}{15}, \quad d_2 = \frac{8}{105}. \end{aligned}$$

The components of a symmetric square matrix $\mathbf{\Lambda}$ and the column vector $\mathbf{\Phi}$ are:

$$\text{(A.3)}\qquad \mathbf{\Lambda} = \begin{bmatrix} 165.119 & 15.011 & 15.011 & 1.363\\ 15.011 & 5.464 & 1.365 & 0.315\\ 15.011 & 1.365 & 3.464 & 0.315\\ 1.355 & 0.315 & 0.315 & .073 \end{bmatrix}, \qquad \mathbf{\Phi} = \begin{bmatrix} 284{,}444\\ 40{,}635\\ 40{,}635\\ 5{,}805 \end{bmatrix},$$

The square matrix $\mathbf{\Gamma}$ is also symmetric, but its components depend on the dimension ratio β:

(A.4)
$$\begin{aligned}
\Gamma_{11} &= 3467.126\beta^2+1981{,}235+3467.126\beta^{-2},\\
\Gamma_{22} &= 72.737\beta^2+180.112+1485.924\beta^{-2},\\
\Gamma_{33} &= 1485.924\beta^2+180.112+72.737\beta^{-2},\\
\Gamma_{44} &= 31.173\beta^2+16.374+31.173\beta^{-2},\\
\Gamma_{12} &= \Gamma_{21} = 315.195\beta^2+495.303\beta^{-2},\\
\Gamma_{13} &= \Gamma_{31} = 495.303\beta^2+315.195\beta^{-2},\\
\Gamma_{14} &= \Gamma_{41} = 45.027\beta^2+45.027\,1/\beta^2 = \Gamma_{23} = \Gamma_{32},\\
\Gamma_{24} &= \Gamma_{42} = 10.391\beta^2+45.027\beta^{-2},\\
\Gamma_{34} &= \Gamma_{43} = 135.083\beta^2+10.391\beta^{-2}.
\end{aligned}$$

The matrix $\mathbf{\Omega}$ is given by:

(A.5)
$$\begin{aligned}
\Omega_{11} &= 12.98762\beta^2+34.71693+12.98762\beta^{-2},\\
\Omega_{12} &= 0.13412\beta^2-14.6497-61.37868\beta^{-2},\\
\Omega_{13} &= -61.37868\beta^2-14.64970+9.13412\beta^{-2},\\
\Omega_{14} &= 1.52501\beta^2+6.60058+8.84945\beta^{-2},\\
\Omega_{21} &= 2{,}09154\beta^2-161.14832+87.85821\beta^{-2},\\
\Omega_{22} &= 11.98175\beta^2+158.42436+706.08737\beta^{-2},\\
\Omega_{23} &= 19.94927\beta^2+72.65769-1.20872\beta^{-2},\\
\Omega_{24} &= -73.59151\beta^2-70.14372-93.4231\beta^{-2},\\
\Omega_{31} &= 87.85821\beta^2-161.14832+2.09154\beta^{-2},\\
\Omega_{32} &= -1.20872\beta^2+72.65760+19.94927\beta^{-2},\\
\Omega_{33} &= 706.08736\beta^2+158.42436+11.98175\beta^{-2},\\
\Omega_{34} &= -14.21532\beta^2-70.14372-109.91834\beta^{-2},\\
\Omega_{41} &= -20.24409\beta^2+798.85582-20.24409\beta^{-2},\\
\Omega_{42} &= 96.57476\beta^2-771.75236-196.81023\beta^{-2},\\
\Omega_{43} &= -196.81023\beta^2-771.75236+96.57426\beta^{-2},\\
\Omega_{44} &= 817.05963\beta^2+746.93133+1202.91438\beta^{-2},
\end{aligned}$$

whereas the components of the column vector $\mathbf{\Phi}'$ are

(A.6) $$\Phi_1' = 0.71336; \quad \Phi_2' = \Phi_3' = 4.14047; \quad \Phi_4' = 31.14185.$$

For fixed value of the dimension ratio $\beta = 1.6875$, the components of the matrix $\mathbf{\Omega}$ become:

(A.7)
$$\mathbf{\Omega} = \begin{bmatrix} 76.26204 & -35.82150 & -189.3876 & 14.05084 \\ -124.34007 & 440.49383 & 129.04177 & -312.51303 \\ 89.77557 & 76.22106 & 217.32154 & -149.22289 \\ 734.09883 & -565.85313 & -1298.28582 & 3496.04656 \end{bmatrix}.$$

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