Archives of Mechanics • Archiwum Mechaniki Stosowanej • 24, 4, pp. 547–554, Warszawa 1972

On sphering thermoelastic annuli

HENRY J. PETROSKI (AUSTIN)

THE DEGREE to which a certain controllable state of elastic heat conductors provides constitutive information is discussed, and it is shown how such information may be employed to solve a problem in nonlinear thermoelasticity. In particular, the tractions, heat fluxes, and other fields necessary to hold an initially flat annulus in the configuration of a heated, pierced spherical cap are exhibited.

Przedyskutowano stopień, do którego pewien kontrolowany stan sprężystych przewodników ciepła zapewnia konstytutywną informację i pokazano, jak ta informacja może być wykorzystana do rozwiązania problemu w nieliniowej termosprężystości. W szczególności wyspecyfikowano siły powierzchniowe, strumienie ciepła i inne pola konieczne do utrzymania wstępnie płaskiego pierścienia w konfiguracji ażurowo sferycznego kubka.

Обсужден вопрос о получении информации о механических свойствах термоупругих материалов, деформирующихся в условиях теплопроводности. Показано, что данную информацию можно использовать для решения задач нелинейной термоупругости. В частности, получены значения поверхностных сил, потоков тепла и других величин, необходимые для поддержания первоначально плоского термоупругого кольца в сферически деформированном состоянии.

Introduction

EXACT solutions to problems in coupled theories of nonlinear thermoelasticity are few not only because the mathematics is complicated but also because the nonlinear behaviour of the material is often unknown. It is usually assumed, however, that any material response functions necessary, to characterize a given material, may be ascertained by some experiment or other, and thus, an analyst proceeds shackled only by his mathematical limitations. If he is asked to produce an experiment sufficient, at least on paper, to characterize the material in question, the analyst is wont to direct the inquirer to the controllable states of the class to which the material belongs. Since a controllable state is possible in all materials of a class, a particular material of the class may be subjected to a controllable state for the purpose of determining its characteristics. E.g., arbitrary homogeneous deformations can be effected in all perfectly elastic materials by the suitable application of surface tractions, and it can easily be seen that they are general enough to characterize completely these materials. It is therefore reasonable to assume that the elastic strain energy function is known when analyzing any noncontrollable state.

This is not the case with thermoelastic materials. It has been shown [1] that there is such a paucity of controllable states of these materials that controllable-state experiments cannot be relied upon to provide all the information that may be necessary for the solution of problems involving seemingly simple, but noncontrollable, states.

Nevertheless, it is possible to exploit the limited information obtained by subjecting an elastic, heat conducting material to the most general thermoelastic controllable state. We illustrate this below by demonstrating expressions which give the surface tractions, heat fluxes, body forces and heat supplies necessary to deform an originally flat circular plate with a hole into a pierced spherical cap. We shall study this deformation coupled with a steady-state heating pattern which includes radial flow and latitudinal flow as special cases. We shall consider such states in homogeneous and isotropic incompressible materials with thermoelastic coupling whose free energy function is assumed to be of a generalized Mooney-Rivlin type and whose heat flux response coefficients are taken to be constants.

The surface and volume distributions we demonstrate are known exactly once the material's response has been determined from the single controllable state experiment. Such possibilities as freeing the spherical surfaces of stress or insulating the edges of the cap may be readily investigated by means of our formulas. The practicability of sphering a particular annulus may be determined from our results.

1. The basic equations

We shall refer the undeformed annulus to a circular cylindrical coordinate system $(X^1, X^2, X^3) = (R, \Phi, Z)$ and the deformed body to a spherical coordinate system $(x^1, x^2, x^3) = (\varrho, \theta, \phi)$. Both these systems are to be tied into the same rectangular Cartesian coordinate system (ξ^1, ξ^2, ξ^3) shown in Fig. 1. We shall use the notation of general tensor analysis. Repeated Roman indices should be summed over 1, 2, 3.

The thermoelastic materials we shall consider belong to the class of elastic heat conductors which are incompressible and unstressed and unheated in their homogeneous and

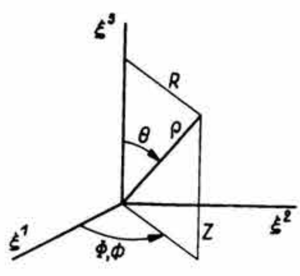

Fig. 1

isotropic reference states and which are characterized by the following constitutive equations for the symmetric stress tensor t^i_j and the heat flux vector q^i:

$$(1.1) \qquad t^i_j = -p\delta^i_j + 2\varrho_0[\bar{c}^{1i}_j \partial_1 \psi - c^i_j \partial_2 \psi]$$

and

$$(1.2) \qquad q^i = [g_{-1}\bar{c}^{1i}_j + g_0\delta^i_j + g_1 c^i_j]\tau^{,j}.$$

In these relations $p = p(x^i)$ is a hydrostatic pressure, δ^i_j is Kronecker's delta, ϱ_0 is the (constant) density of the body in the reference state, and $\tau^{,i}$ is the spatial temperature gradient. The left Cauchy-Green deformation tensor $\bar{c}^{1i}_j$ and its inverse c^i_j are derived from the deformation $x^i = x^i(X^j)$ as follows:

$$\bar{c}^{1ij} = G^{kl}\frac{\partial x^i}{\partial X^k}\frac{\partial x^j}{\partial X^l} \tag{1.3}$$

and

$$c_{ij} = G_{kl}\frac{\partial X^k}{\partial x^i}\frac{\partial X^l}{\partial x^j}, \tag{1.4}$$

where G^{kl} and G_{kl} are the contra- and co-variant components, respectively, of the metric of the (cylindrical) X^j-coordinate system.

The free energy ψ and the heat flux response coefficients g_Γ $(\Gamma = -1, 0, 1)$ are functions of the absolute temperature $\tau \equiv I_0$ and the independent scalar invariants

$$\begin{aligned} &I_1 = \bar{c}^{1i}_i, \quad I_2 = \frac{1}{2}(I_1^2 - \bar{c}^{1i}_j\bar{c}^{1j}_i), \\ &I_3 = \tau^{,i}\tau_{,i}, \quad I_4 = \tau^{,i}\bar{c}^{1j}_i\tau_{,j}, \quad I_5 = \tau^{,i}c^j_i\tau_{,j}, \end{aligned} \tag{1.5}$$

such that

$$\psi = \psi(I_0, I_1, I_2) \tag{1.6}$$

and

$$g_\Gamma = g_\Gamma(I_1, I_2, I_3, I_4, I_5). \tag{1.7}$$

In (1.1) we employ the notation

$$\partial_\alpha\psi \equiv \frac{\partial\psi}{\partial I_\alpha}, \quad \alpha = 1, 2.$$

In discussing the sphering problem in Sec. 4, we shall consider the special case of these materials, where

$$\psi = C_0 I_0 + C_1(I_1 - 3) + C_2(I_2 - 3) \tag{1.6'}$$

with C_0, C_1, and C_2 being constants and where the g_Γ are constants,

$$g_\Gamma = \text{const.} \tag{1.7'}$$

The field equations which all states of these thermoelastic bodies must satisfy, are

$$t^{ij}_{,i} + \varrho_0 b^i = 0 \tag{1.8}$$

and

$$q^i_{,i} - r = 0, \tag{1.9}$$

where $b^i = b^i(x^j)$ is the body force per unit mass and $r = r(x^i)$ is the extrinsic heat supply per unit volume. The reader is referred to [4] for further discussion of these basic equations.

2. The deformation and temperature fields

An undeformed and unheated annular body may be considered bounded by the planes $Z = Z_0$ and $Z = Z_0 + h$ and the cylinders $R = R_1$ and $R = R_2$. We shall assume that such a body is subjected to a thermoelastic state in which planes $Z = \text{const}$ become spherical surface $\varrho = \text{const}$, cylindrical surfaces $R = \text{const}$ become conical surfaces

θ = const, and planes Φ = const remain planes ϕ = const. We can express this class of deformations analytically as

$$\varrho = f(Z), \quad \theta = g(R), \quad \phi = \Phi, \tag{2.1}$$

where f and g are functions which we shall determine presently.

For the deformation (2.1), we calculate

$$\|\bar{c}_j^{1i}\| = \|\bar{c}^{1ik}g_{kj}\| = \operatorname{diag}[f'(Z)^2, f(Z)^2 g'(R)^2, R^{-2}f(Z)^2\sin^2 g(R)], \tag{2.2}$$

where "diag" indicates that the non-zero terms of a diagonal matrix follow, where g_{ij} are components of the metric of the (spherical) x^k-coordinate system, and where a prime denotes differentiation with respect to the primed function's argument.

Following a procedure similar to the one RIVLIN used [3] to study the problem of flexure, we now see what restrictions incompressibility places on the functions f and g. Since the deformation (2.1) is isochoric, if and only if, $\det(\bar{c}_j^{1i}) = 1$, we must have

$$f(Z)^4 f'(Z)^2 R^{-2} g'(R)^2 \sin^2 g(R) = 1$$

for all R and Z within the annulus. Because R and Z are independent variables, this implies

$$f(Z)^4 f'(Z)^2 = a^2, \quad R^{-2}g'(R)^2\sin^2 g(R) = \frac{1}{a^2}, \tag{2.3}$$

where a is a constant.

Equations (2.3) are easily seen to be satisfied by

$$f(Z) = (3aZ)^{1/3}, \quad g(R) = \cos^{-1}\left(1-\frac{R^2}{2a}\right), \tag{2.4}$$

where we have not included constants of integration representing rigid-body motions.

Then, in addition to the deformation

$$\varrho = (3aZ)^{1/3}, \quad \theta = \cos^{-1}\left(1-\frac{R^2}{2a}\right), \quad \phi = \Phi, \tag{2.5}$$

we shall assume that the deformed annulus is heated in such a way that the temperature field has the form

$$\tau = \tau_0 + h\varrho + k\theta, \tag{2.6}$$

where τ_0, h, and k are constants.

We can calculate the components of the deformation tensors (1.3) and (1.4) from the deformation (2.5) and its inverse

$$R = \sqrt{2a(1-\cos\theta)}, \quad \Phi = \phi, \quad Z = \frac{\varrho^3}{3a}. \tag{2.7}$$

We get for the matrices of mixed components

(2.8) and

$$\|\bar{c}_j^{1i}\| = \operatorname{diag}\left[\frac{a^2}{\varrho^4} \quad \frac{2\varrho^2}{a(1+\cos\theta)} \quad \frac{\varrho^2(1+\cos\theta)}{2a}\right]$$

$$\|c_j^i\| = \operatorname{diag}\left[\frac{\varrho^4}{a^2} \quad \frac{a(1+\cos\theta)}{2\varrho^2} \quad \frac{2a}{\varrho^2(1+\cos\theta)}\right]$$

and for the components of the gradient of the temperature field

$$\tau^{,i} = \left(h, \frac{k}{\varrho^2}, 0\right) \text{ and } \tau_{,i} = (h, k, 0). \tag{2.9}$$

We can now calculate the invariants (1.5). They are, for the deformation (2.5) coupled with the temperature field (2.6),

$$\begin{aligned} I_1 &= \frac{a^2}{\varrho^4} + \frac{2\varrho^2}{a(1+\cos\theta)} + \frac{\varrho^2(1+\cos\theta)}{2a}, \\ I_2 &= \frac{\varrho^4}{a^2} + \frac{a(1+\cos\theta)}{2\varrho^2} + \frac{2a}{\varrho^2(1+\cos\theta)}, \\ I_3 &= h^2 + \frac{k^2}{\varrho^2}, \\ I_4 &= \frac{h^2a^2}{\varrho^4} + \frac{2k^2}{a(1+\cos\theta)}, \\ I_5 &= \frac{h^2\varrho^4}{a^2} + \frac{ak^2(1+\cos\theta)}{2\varrho^4}. \end{aligned} \tag{2.10}$$

We shall now interrupt our discussion to observe that if either h or k is zero these invariants are in the same ratios to each other at a given point (ϱ, θ, ϕ) as the invariants of a deformation and temperature field that have been shown [1] to comprise a controllable state. This means that certain information about the free energy and heat flux response functions may be considered knowable through a controllable-state experiment on a particular material to a degree pertinent to the present problem. This is of interest because the controllable states of materials obeying (1.1) and (1.2) are not general enough to characterize completely the functions ψ and g_Γ, i.e., the controllable states do not allow arbitrary independent variation of the arguments I_α of ψ and g_Γ.

3. The pertinent controllable state

The special homogeneous deformation

$$x = (C/\sqrt{A})X - (D/\sqrt{B})Y, \quad y = (D/\sqrt{A})X + (C/\sqrt{B})Y, \quad z = \sqrt{AB}Z, \tag{3.1}$$

where points (X, Y, Z) in the undeformed and (x, y, z) in the deformed body are referred to the same rectangular Cartesian coordinate system and where $A > 0$, $B > 0$, C, and D are constants with $C^2 + D^2 = 1$, coupled with the linear temperature field

$$\tau = \tau_0 + \varkappa z, \tag{3.2}$$

where τ_0 and $\varkappa$ are constants, has been shown [1] to be a controllable state of materials characterized by (1.1) and (1.2). In [2] the constant invariants of this state are exhibited as

$$\begin{aligned} I_1 &= \frac{1}{A} + AB + \frac{1}{B}, \quad I_2 = A + \frac{1}{AB} + B, \\ I_3 &= \varkappa^2, \quad I_4 = AB\varkappa^2, \quad I_5 = \frac{\varkappa^2}{AB}. \end{aligned} \tag{3.3}$$

The stress field corresponding to (3.1) and (3.2) has the components [2]

$$t_x^x = p_0 + \frac{2\varrho_0}{AB}[(BC^2+AD^2-A^2B^2)\partial_1\psi-(A^2BC^2+AB^2D^2-1)\partial_2\psi],$$

$$t_y^y = p_0 + \frac{2\varrho_0}{AB}[(AC^2+BD^2-A^2B^2)\partial_1\psi-(AB^2C^2+A^2BD^2-1)\partial_2\psi], \tag{3.4}$$

$$t_y^x = \frac{CD}{(C^2-D^2)}(t_x^x-t_y^y)$$

and all other $t_j^i = p_0\delta_j^i$, where p_0 is a constant. Measuring any three of t_x^x, t_y^x, t_y^y, t_z^z in the state (3.1)–(3.2) suffices to determine

$$\partial_1\psi = \partial_1\psi(I_0, I_1, I_2), \qquad \partial_2\psi = \partial_2\psi(I_0, I_1, I_2), \tag{3.5}$$

where, however, the arguments are always in the ratios of (3.3).

The only non-zero component of the heat flux field corresponding to (3.1) and (3.2) is

$$\begin{aligned} q_z &= (ABg_{-1}+g_0+g_1/AB)\varkappa \\ &\equiv G\varkappa = \text{const}, \end{aligned} \tag{3.6}$$

where

$$G = G(AB, I_1, I_2, I_3, I_4, I_5), \tag{3.7}$$

since the g_Γ are functions of I_1 through I_5 with the I_α given by (3.3). Measurements of q_z in the controllable state (3.1)–(3.2) therefore give us the values of the function (3.7) but known only for arguments whose ratios are the same as in (3.3).

To observe that the sets of invariants (2.10) and (3.3) are in the same ratios when $h = 0$, e.g., just let $A = \varrho^4/a^2$, $B = 2a/(\varrho^2+\varrho^2\cos\theta)$, and $\varkappa = k/\varrho$. A similar comparison shows us that these ratios are maintained also for the case $k = 0$.

We shall henceforth assume that the material response of the annulus is known to the degree delineated in (3.5) and (3.7), and we shall show how this information may be employed in the sphering problem.

4. Stress and heat flux fields for a sphered annulus

The stress field is gotten simply by using (2.8) in (1.1). The only non-zero physical components are

$$t_\varrho^\varrho = -p+2\varrho_0\left[\frac{a^2}{\varrho^4}\partial_1\psi-\frac{\varrho^4}{a^2}\partial_2\psi\right],$$

$$t_\theta^\theta = -p+2\varrho_0\left[\frac{2\varrho^2}{a(1+\cos\theta)}\partial_1\psi-\frac{a(1+\cos\theta)}{2\varrho^2}\partial_2\psi\right], \tag{4.1}$$

$$t_\phi^\phi = -p+2\varrho_0\left[\frac{\varrho^2(1+\cos\theta)}{2a}\partial_1\psi-\frac{2a}{\varrho^2(1+\cos\theta)}\partial_2\psi\right],$$

where the hydrostatic pressure may be determined from the requirement that this stress field satisfy the law of balance of momentum (1.8). We note that it is possible to free the spherical surfaces of tractions by taking

$$p = 2\varrho_0\left[\frac{a^2}{\varrho^4}\partial_1\psi - \frac{\varrho^4}{a^2}\partial_2\psi\right],$$

but then a rather complex body-force field would be necessary to balance momentum. These complications are in contrast to the controllable state (3.1)–(3.2), where, by definition, the stress field (3.4) satisfies the law of balance of momentum regardless of the material function ψ's dependence on the invariants and without requiring that any body force act.

For materials which behave according to the constitutive relation (1.6′), balance of momentum (1.8) requires that

$$\frac{\partial p}{\partial x^i} = \frac{2\varrho_0}{g^{\underline{ii}}}\left\{C_1\bar{c}^{1ij}_{,j} - C_2 c^{ij}_{,j} + \frac{b^i}{2}\right\}, \tag{4.2}$$

where the summation convention is suspended where indices are underlined. Carrying out the covariant differentiations on (2.8), we get the following differential equations for determining $p = p(\varrho, \theta, \phi)$:

$$\frac{\partial p}{\partial \varrho} = -\left(\frac{4a^3+5\varrho^6+(4a^3+2\varrho^6)\cos\theta+\varrho^6\cos^2\theta}{a\varrho^5(1+\cos\theta)}\right)\varrho_0 C_1 + \left(\frac{5a^3-12\varrho^6+(2a^3-12\varrho^6)\cos\theta+a^3\cos^2\theta}{a^2\varrho^3(1+\cos\theta)}\right)\varrho_0 C_2 + \varrho_0 b^\varrho,$$

$$\frac{\partial p}{\partial \theta} = \left(\frac{4+3\cos\theta-3\cos^2\theta-3\cos^3\theta-\cos^4\theta}{a\sin\theta(1+\cos\theta)^2}\right)\varrho_0\varrho^2 C_1 + \left(\frac{1+4\cos\theta-3\cos^2\theta-2\cos^3\theta}{\sin\theta(1+\cos\theta)}\right)\frac{a\varrho_0 C_2}{\varrho^2} + \varrho_0\varrho b^\theta, \tag{4.3}$$

$$\frac{\partial p}{\partial \phi} = \varrho_0\varrho\sin\theta b^\phi,$$

where $(b^\varrho, b^\theta, b^\phi)$ are the physical components of the body force field. If $b^\phi = 0$, $(4.3)_3$ tells us that $p = p(\varrho, \theta)$ only, and the stress field is axisymmetric. Since the material constants C_1 and C_2 may be considered known, as discussed in Sec. 3, the Eqs. (4.3) are the defining differential equations for the hydrostatic pressure p which must be present for a given body force field $(b^\varrho, b^\theta, b^\phi)$. For the general case of independent constants C_1 and C_2 and a constant g gravity-field body force $(b^\varrho, b^\theta, b^\phi) = (-g\cos\theta, g\sin\theta, 0)$, there exists no smooth solution to (4.3). This is merely an expression of the fact that momentum cannot be balanced in a given body for arbitrarily chosen deformation (i.e., stress) fields and body force fields. Nevertheless, given specific values of C_1, and C_2, it may be possible to integrate (4.3) for a realistic body force field.

We shall now consider the heat flux field necessary to sphere the annulus. The components of q^i are gotten by using (2.8)–(2.9) in (1.2), and therefore

$$
\begin{aligned}
q^{\varrho} &= \left[\frac{a^2}{\varrho^4} g_{-1} + g_0 + \frac{\varrho^4}{a^2} g_1\right] h,\\
q^{\theta} &= \left[\frac{2\varrho^2}{a(1+\cos\theta)} g_{-1} + g_0 + \frac{a(1+\cos\theta)}{2\varrho^2}\right]\frac{k}{\varrho},\\
q^{\phi} &= 0
\end{aligned}
\tag{4.4}
$$

are the physical components. We observe that at any point (ϱ, θ, ϕ) the material constants g_Γ occur in exactly the combination of (3.6).

For the case of constant g_Γ which we are considering, an experiment based on the controllable state discussed in Sec. 3 will yield the values of the individual g_Γ. In fact, we need only record the heat flux q_z corresponding to three different pairs of the independently variable temperature gradient $\varkappa$ and deformation measure AB in (3.6). Thus, we may consider the g_Γ known from here on.

Now energy is not automatically balanced with the heat flow (4.4), and an extrinsic heat supply r may have to be provided. We may check this point by noting that the balance of energy expression (1.9) requires a heat supply field r whenever q^i is not solenoidal. Thus, we have, using (4.4) in (1.9):

$$
r = \left\{-\frac{2a^2h}{\varrho^5} + \frac{2k}{a\sin\theta(1+\cos\theta)}\right\} g_{-1} + \left\{\frac{2h}{\varrho} + \frac{k\cos\theta}{\varrho^2\sin\theta}\right\} g_0 + \left\{\frac{6h\varrho^3}{a^2} + \frac{ak(2\cos^2\theta + \cos\theta - 1)}{2\varrho^4\sin\theta}\right\} g_1 .
\tag{4.5}
$$

In general, this does not vanish identically. Thus, considering the material constants g_Γ determined from the controllable state of Sec. 3, (4.5) is the heat supply necessary to effect the deformation (2.5) and the temperature field (2.6) in that material.

Global heat fluxes, forces, and moments necessary to sphere a particular annulus may, of course, be computed from the local fields (4.1) and (4.4).

References

1. H. J. PETROSKI and D. E. CARLSON, Arch. Ration. Mech. Analysis, **31**, 127, 1968.
2. H. J. PETROSKI and D. E. CARLSON, J. Appl. Mech., **37**, 1151, 1970.
3. R. S. RIVLIN, Proc. Royal Soc., London, A **195**, 463, 1949.
4. C. TRUESDELL and W. NOLL, The non-linear field theories of mechanics, Handbuch der Physik, III/3. Springer, Berlin 1965.

DEPARTMENT OF AEROSPACE ENGINEERING
AND ENGINEERING MECHANICS
THE UNIVERSITY OF TEXAS AT AUSTIN

Received September 6, 1971