Archives of Mechanics • Archiwum Mechaniki Stosowanej • 24, 4, pp. 539–545, Warszawa 1972

The influence of nonlinear couplings on the behaviour of the solution of the equations of motion of a mechanical system

R. GUTOWSKI (WARSZAWA)

THE SUBJECT matter of this analysis is the influence of nonlinear couplings on the behaviour of the solution of the equations of motion of a mechanical system which may be subdivided into sub-systems non-linearly coupled. The conditions for the motion of the sub-systems to be bounded and capable of being made arbitrarily small are established. Another subject of discussion are bands which can be constructed making use of the solution of a suitable set of linear differential equations, and which contain solution of the subset considered. The width of these bands depends on the estimated values of the nonlinear coupling functions. For synthesis these bands can be made arbitrarily narrow, which means in practice rejection of the nonlinear couplings. The results discussed have been obtained by methods of integral inequalities.

W pracy zbadano wpływ nieliniowych sprzężeń na zachowanie się rozwiązań równań ruchu układu mechanicznego, który można rozdzielić na podukłady sprzężone nieliniowo. Ustalono warunki, przy spełnieniu których ruch podukładów jest ograniczony i może być uczyniony dowolnie małym. Ponadto wskazano pasma, które można zbudować za pomocą rozwiązań odpowiednio dobranego układu równań różniczkowych liniowych, w których przebiegają rozwiązania rozważanych podukładów. Szerokość tych pasm zależna jest od własności nieliniowych funkcji sprzęgających, wyrażonych za pomocą oszacowań tych funkcji. Z punktu widzenia syntezy układu pasma te można uczynić dowolnie wąskimi, co odpowiada możliwości praktycznego pominięcia sprzężeń nieliniowych. Przedstawione rezultaty uzyskano stosując metody nierówności całkowych.

В работе исследовано влияние нелинейных связей на поведение решений уравнений движения механической системы, которую можно разделить на подсистемы с нелинейными взаимными связями. Выведены условия, при выполнении которых движение подсистем ограничено и может быть приведено к произвольно малой величине. Указано, что при помощи решений, соответствующим образом подобранной системы линейных дифференциальных уравнений, можно построить полосы, в которых проходят решения рассматриваемых подсистем. Ширина этих полос зависит от нелинейных свойств сопрягающих функций, выраженных при помощи оценок этих функций. С точки зрения синтеза систем данные полосы можно построить произвольно узкими, что соответствует возможности практического пренебрежения нелинейными связями. Представленные результаты получены по методу интегральных неравенств.

1. The statement of the problem

THE OBJECT of the present paper is to analyse the motion of a complex mechanical system which can be separated into sub-systems with nonlinear couplings, with special reference to the influence of the nonlinear couplings. Sufficient conditions will be established for the solution of the equations of motion to be bounded and capable of being made arbitrarily small. In addition, some linear equations will be discussed, by means of which neighbourhoods can be constructed containing the solutions of the sub-systems with nonlinear couplings under consideration. The extent of these neighbourhoods depends on the properties of the nonlinear coupling functions expressed in terms of appraisals of such functions.

Let us consider a mechanical system with n differential equations. Let us assume in the interests of lucidity that this system can be separated into two sub-systems having l and m $(l+m = n)$ differential equations, respectively, and coupled in a linear manner. The considerations quoted below and concerning this case can be immediately generalized to the case of a system splitting up into an arbitrary number of sub-systems coupled nonlinearly.

Let us consider, therefore a set of differential equations of motion having the form

$$\dot{y} = A(t)y+f(t,x,y)+p(t), \tag{1.1}$$

$$\begin{aligned} \dot{x} &= B(t)x+\varphi(t,x,y)+\pi(t), \\ x(t_0) &= x_0, y(t_0) = y_0, \end{aligned} \tag{1.2}$$

where

$$x = \mathrm{col}[x_1, .., x_l], \quad y = \mathrm{col}[y_1, ..., y_m], \quad l+m = n;$$

$B(t)$ is a square matrix of order l and $A(t)$—a square matrix of order m. These matrices are real and continuous for $t \in [t_0, \infty)$. The matrices $f(t,x,y) = \mathrm{col}\ [f_1, ..., f_m]$, $\varphi(t,x,y) = \mathrm{col}\ [\varphi_1, ..., \varphi_l]$ are real and continuous for $t \in [t_0, \infty)$ and $\|x\|+\|y\| < \infty$ ($\|\cdot\|$—denotes the norm). The matrices $p(t) = \mathrm{col}[p_1, ..., p_m]$, $\pi(t) = \mathrm{col}[\pi_1, ..., \pi_l]$ are also real and continuous for $t \in [t_0, \infty)$.

2. Analysis of the properties of a solution of the equations of motion (1.1) and (1.2)

Together with Eqs. (1.1) and (1.2), let us consider the linear differential equations

$$\dot{\eta} = A(t)\eta+p(t), \tag{2.1}$$

$$\dot{r} = A(t)r, \tag{2.2}$$

$$\dot{\xi} = B(t)\xi+\pi(t), \tag{2.3}$$

$$\dot{q} = B(t)q, \tag{2.4}$$

where $\eta = \mathrm{col}[\eta_1, ..., \eta_m]$, $r = \mathrm{col}[r_1, ..., r_m]$, $\xi = \mathrm{col}[\xi_1, ..., \xi_l]$, $q = \mathrm{col}[q_1, ..., q_l]$.

Let the initial values of these functions satisfy the relations

$$y(t_0) = \eta(t_0) = r(t_0) = y_0, \quad x(t_0) = \xi(t_0) = g(t_0) = x_0.$$

Let us denote by $R(t)$ the fundamental solution matrix of Eq. (2.2) and by $Q(t)$ the fundamental solution matrix of Eq. (2.4).

Let us make the following assumptions:

1) $$\|R(t)R^{-1}(s)\| \leqslant \alpha_1 e^{-\beta_1(t-s)}, \quad \|Q(t)Q^{-1}(s)\| \leqslant \alpha_2 e^{-\beta_2(t-s)},$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are real positive constants;

2) $\|\eta(t)\| \leqslant c_1 < \infty, \quad \|\xi(t)\| \leqslant c_2 < \infty \quad$ for $t \in [t_0, \infty)$

where c_1, c_2 are real positive constants;

3) $\|f(t,x,y)\| \leqslant k_1 g_1(\|x\| + \|y\|), \quad \|\varphi(t,x,y)\| \leqslant k_2 g_2(\|x\| + \|y\|),$

for $t \in [t_0, \infty)$ and $\|x\|+\|y\| < \infty$. The symbols k_1, k_2 denote real non-negative constants and $g_1(u)$ and $g_2(u)$—continuous, non-negative, non-decreasing functions for $u \geqslant 0$ and $g_1(0) = g_2(0) = 0$.

The set of integral equations corresponding to the differential Eqs. (1.1) and (1.2) have the form

$$(2.5)\qquad y = R(t)R^{-1}(t_0)y_0 + \int_{t_0}^{t} R(t)R^{-1}(s)f[s, x(s), y(s)]ds + \int_{t_0}^{t} R(t)R^{-1}(s)p(s)\,ds,$$

$$(2.6)\qquad x = Q(t)Q^{-1}(t_0)x_0 + \int_{t_0}^{t} Q(t)Q^{-1}(s)\varphi[s, x(s), y(s)]ds + \int_{t_0}^{t} Q(t)Q^{-1}(s)\pi(s)\,ds.$$

The solutions of the differential Eqs. (2.1) and (2.3) hawe the form

$$(2.7)\qquad \eta = R(t)R^{-1}(t_0)\eta_0 + \int_{t_0}^{t} R(t)R^{-1}(s)p(s)\,ds,$$

$$(2.8)\qquad \xi = Q(t)Q^{-1}(t_0)\xi_0 + \int_{t_0}^{t} Q(t)Q^{-1}(s)\pi(s)\,ds.$$

Since it has been assumed that $\eta(t_0) = \eta_0 = y_0$ and $\xi(t_0) = \xi_0 = x_0$, therefore, the set of integral Eqs. (2.5) and (2.6) takes the form

$$(2.9)\qquad y = \eta + \int_{t_0}^{t} R(t)R^{-1}(s)f[s, x(s), y(s)]ds,$$

$$(2.10)\qquad x = \xi + \int_{t_0}^{t} Q(t)Q^{-1}(s)\varphi[s, x(s), y(s)]ds.$$

Taking the norm of both members of these equations, we obtain, by virtue of the assumptions 1), 2), 3),

$$(2.11)\qquad \|y\| \leqslant c_1 + \int_{t_0}^{t} k_1\alpha_1 e^{-\beta_1(t-s)} g_1(\|x\| + \|y\|)\,ds,$$

$$(2.12)\qquad \|x\| \leqslant c_2 + \int_{t_0}^{t} k_2\alpha_2 e^{-\beta_2(t-s)} g_2(\|x\| + \|y\|)\,ds.$$

Let us denote

$$\|x\| + \|y\| = \zeta,\ k\alpha = \max(k_1\alpha_1, k_2\alpha_2),\ \beta = \min(\beta_1, \beta_2),\ c = c_1 + c_2,$$

$g(\zeta) \geqslant \max_{\zeta \geqslant 0}(g_1, g_2)$, $g(0) = 0$ and g is a continuous, non-negative, non-decreasing function for $\zeta \geqslant 0$.

By adding the inequalities (2.11) and (2.12), we obtain

$$(2.13)\qquad \zeta \leqslant c + \int_{t_0}^{t} k\alpha e^{-\beta(t-s)} g\big(\zeta(s)\big)\,ds.$$

Let us denote the right-hand member of this inequality by $u(t)$—that is,

$$u = c+\alpha k e^{-\beta t}\int_{t_0}^{t} e^{\beta s} g(\zeta)\,ds. \tag{2.14}$$

On differentiating with respect to time, we obtain

$$\dot{u} = -\beta(u-c)+\alpha k g(\zeta(t)).$$

Hence, by virtue of (2.13) and (2.14) and the properties of the function g, we obtain the inequality

$$\dot{u} \leqslant \beta c-\beta u+\alpha k g(u). \tag{2.15}$$

Let us change variables by setting $\varrho = \frac{1}{c}u-1$—that is, $u = (1+\varrho)c$. For $t = t_0$ we have $u = u_0 = c$, therefore $\varrho_0 = \frac{u_0}{c}-1 = 0$, $\tau = \beta t$. For $t = t_0$ we have $\tau_0 = \beta t_0$ and $\frac{d\tau}{dt} = \beta$. Thus, the inequality (2.15) takes the form

$$\frac{du}{dt} = c\frac{d\varrho}{d\tau}\frac{d\tau}{dt} \leqslant -\beta c\varrho+\alpha k g[(1+\varrho)c],$$

$$\frac{d\varrho}{d\tau} \leqslant \frac{\alpha k}{\beta c} g[(1+\varrho)c]-\varrho. \tag{2.16}$$

We assume that there exists a λ_0 such that

$$\frac{\alpha k}{\beta c} g[(1+\lambda_0)c]-\lambda_0 = 0, \tag{2.17}$$

$$\frac{\alpha k}{\beta c} g[(1+\lambda)c]-\lambda > 0 \quad \text{for} \quad \lambda\in[0, \lambda_0). \tag{2.18}$$

Since $\varrho_0 = 0$, therefore, by virtue of (2.18), there exists a $\tau^* > \tau_0$ such that the right-hand member of the inequality (2.16) is positive for $\tau \in[\tau_0, \tau^*)$ and we can divide in this in-

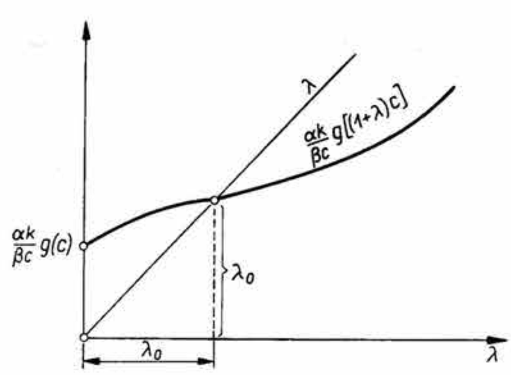

FIG. 1

terval both members of the inequality (2.16) by its right-hand member. Thus, we obtain

$$G(\varrho) = \int_0^{\varrho(\tau)} \frac{ds}{\frac{\alpha k}{\beta c} g[(1+s)c]-s} \leqslant \tau-\tau_0 \quad \text{for} \quad \tau \in [\tau_0, \tau^*). \tag{2.19}$$

However, (cf. Fig. 1), we have

$$\frac{\alpha k}{\beta c} g[(1+\lambda)c] < \lambda_0$$

for $\lambda < \lambda_0$, therefore

$$\int_0^{\varrho(\tau)} \frac{ds}{\lambda_0 - s} \leqslant \int_0^{\varrho(\tau)} \frac{ds}{\frac{\alpha k}{\beta c} g[(1+s)c]-s} \leqslant \tau-\tau_0. \tag{2.20}$$

Let us assume that for $\tau = \tau^* < \infty$ we have $\varrho(\tau^*) = \lambda_0$. Then, by virtue of (2.20), we have

$$\int_0^{\lambda_0} \frac{ds}{\lambda_0 - s} \leqslant \tau^*-\tau_0, \tag{2.21}$$

which is impossible, because the integral on the left-hand side of the inequality (2.21) is divergent and the right-hand side is finite. Therefore the functions $\varrho(\tau)$ satisfying the inequalities (2.19) satisfy for $\tau \in [\tau_0, \infty)$ the inequality

$$\varrho(\tau) \leqslant \lambda_0 \tag{2.22}$$

and the equation $\varrho(\tau) = \lambda_0$ holds for $\tau = \infty$ only (that is $\tau^* = \infty$). Thus,

$$\|x\|+\|y\| = \zeta(t) \leqslant u(t) \leqslant (1+\lambda_0)c. \tag{2.23}$$

Hence, the following appraisals:

$$\|x\| \leqslant (1+\lambda_0)c, \qquad \|y\| \leqslant (1+\lambda_0)c. \tag{2.24}$$

It will be shown that $\|x\|$ and $\|y\|$ can be made arbitrarily small if $\alpha k/\beta$ and c are sufficiently small.

Indeed, by virtue of (2.17), we have:

$$\lambda_0 = \frac{\alpha k}{\beta} \frac{1}{c} g[(1+\lambda_0)c]. \tag{2.25}$$

If $c = \text{const} > 0$, then, for $\alpha k/\beta \to 0$, we have $\lambda_0 \to 0$. In other words, if $\alpha k/\beta$ is sufficiently small, which depends on either the matrices $A(t)$ and $B(t)$ (the ratio α/β), or the functions f, and φ being small (constant k), then, λ_0 can be made arbitrarily small for $c = \text{const} > 0$.

Let us assume, in addition, that the function g satisfies the condition:

$$\lim_{\vartheta \to 0} \frac{g(\vartheta)}{\vartheta} = \Omega. \tag{2.26}$$

By virtue of (2.25), we have:

$$\frac{\lambda_0}{1+\lambda_0} = \frac{\alpha k}{\beta}\frac{g[(1+\lambda_0)c]}{(1+\lambda_0)c}.$$

If $c \to 0$, then, by virtue of (2.26), we obtain:

$$\lambda_0 = \frac{\dfrac{\alpha k}{\beta}}{1-\dfrac{\alpha k}{\beta}\Omega} \quad \text{for} \quad \frac{\alpha k}{\beta}\Omega < 1 \quad \text{and} \quad c \to 0.$$

If, therefore, $\alpha k/\beta \to 0$, we have also $\lambda_0 \to 0$ for $c \to 0$ and $(\alpha k/\beta)\Omega < 1$. It results, by virtue of (2.24), that $||x||$ and $||y||$ can be made arbitrarily small if $\alpha k/\beta$ and c are sufficiently small and $(\alpha k/\beta)\Omega < 1$.

We shall now determine the neighbourhoods in which the solutions for the sub-systems considered are contained, coupled in a nonlinear manner by the functions f and φ.

By virtue of (2.9) and (2.10), and the assumptions 1) and 3), we have

$$||y-\eta|| \leqslant k_1\alpha_1 \int_{t_0}^{t} e^{-\beta_1(t-s)} g_1(||x||+||y||)\,ds,$$

$$||x-\xi|| \leqslant k_2\alpha_2 \int_{t_0}^{t} e^{-\beta_2(t-s)} g_2(||x||+||y||)\,ds.$$

By virtue of (2.23), these inequalities take the form:

$$||y-\eta|| \leqslant k_1\alpha_1 g_1[(1+\lambda_0)c] \int_{t_0}^{t} e^{-\beta_1(t-s)}ds = \frac{k_1\alpha_1}{\beta_1} g_1[(1+\lambda_0)c]\,(1-e^{-\beta_1(t-t_0)}),$$

$$||x-\xi|| \leqslant k_2\alpha_2 g_2[(1+\lambda_0)c] \int_{t_0}^{t} e^{-\beta_2(t-s)}ds = \frac{k_2\alpha_2}{\beta_2} g_2[(1+\lambda_0)c]\,(1-e^{-\beta_2(t-t_0)}).$$

Hence,

$$||y-\eta|| \leqslant \frac{k_1\alpha_1}{\beta_1} g_1[(1+\lambda_0)c] = \delta_1, \qquad ||x-\xi|| \leqslant \frac{k_2\alpha_2}{\beta_2} g_2[(1+\lambda_0)c] = \delta_2$$

— that is, if for instance the norm is assumed to be a sum of absolute values of the elements of the matrices,

$$\eta_i(t)-\delta_1 \leqslant y_i(t) \leqslant \eta_i(t)+\delta_1 \quad i = 1, \ldots, m, \tag{2.27}$$

$$\xi_j(t)-\delta_2 \leqslant x_j(t) \leqslant \xi_j(t)+\delta_2, \quad j = 1, \ldots, l, \tag{2.28}$$

where x_j, ξ_j, y_i, η_i ($j = 1, \ldots, l$, $i = 1, \ldots, m$) are elements of the column matrices x, ξ, y, η.

From the above considerations it is seen that we can determine the neighbourhoods (2.27) and (2.28) in which the solution $x(t)$, $y(t)$ is contained if we have information contained in the assumptions 3) on the coupling functions f and φ. From the point of view of synthesis of a system, the results of the considerations above show how the matrices A and B, the coupling functions f and φ and the functions p and π should be selected in order that the neighbourhoods (2.27) and (2.28) may be arbitrarily narrow. In other words,

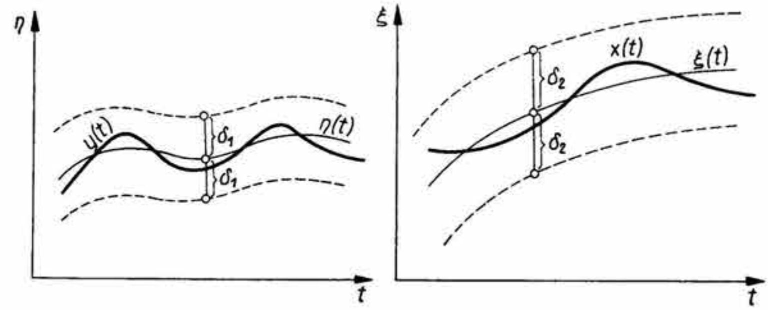

FIG. 2

the above results provide conditions sufficient for determining the manner in which the synthesis is to be performed for a system — so that nonlinear couplings may in practice be rejected.

References

1. R. GUTOWSKI, B. RADZISZEWSKI, *Appraisal of solutions of the equations of motion of a non-autonomic system*, Nonlinear Vibrations Problems, **11**, 1971.
2. E. A. BARBASHIN, *Lyapunov functions*, Nauka, Moscow 1970.

WARSAW TECHNICAL UNIVERSITY
DEPARTMENT OF POWER AND AIRCRAFT ENGINEERING
INSTITUTE OF APPLIED MECHANICS

Received August 2, 1971