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# The properties of a solution of the equations of motion of a mechanical system subject to irregular (singular) perturbations

R. GUTOWSKI (WARSZAWA)

THE SUBJECT matter of the present considerations is the motion of a system described by nonlinear ordinary differential equations of the first order, some of which involve a small parameter with a derivative. If this parameter is zero, the equations take a singular form. Conditions are established sufficient for the solution of the non-degenerate set of equations with a small parameter to tend to the solution of the singular set of equations, if the small parameter tends to zero. The above results have been obtained by means of the method of integral inequalities. They concern cases which differ from those discussed in the existing works on the same subject, above all by the type of the non-linearities admitted.

W pracy rozważono ruch układu opisywany równaniami różniczkowymi zwyczajnymi nieliniowymi pierwszego rzędu, z których część zawiera mały parametr przy pochodnej. Jeśli parametr ten jest równy zeru, wtedy równania przybierają postać osobliwą. W pracy ustalono warunki dostateczne na to, aby rozwiązania układu niezdegenerowanego z małym parametrem zmierzały do rozwiązań układu osobliwego, gdy mały parametr zmierza do zera. Rezultaty otrzymano za pomocą metody nierówności całkowych, przy czym rezultaty te obejmują przypadki odmienne niż w istniejących na ten temat pracach, przede wszystkim ze względu na charakter dopuszczalnych nieliniowości.

В работе рассмотрено движение системы, описываемой нелинейными обыкновенными дифференциальными уравнениями первого порядка, часть которых содержит малый параметр при производной. Если этот параметр равен нулю, то уравнения принимают особую форму. В работе исследованы достаточные условия сходимости решений системы уравнений неособого типа, содержащих малый параметр, к решениям системы особого типа, при стремлении малого параметра к нулю. Полученные результаты выведены при помощи метода интегральных неравенств и охватывают случаи, отличающиеся от ранее рассмотренных в литературе характером допускаемой нелинейности.

## 1. The statement of the problem

LET US consider a mechanical system, the equations of motion of which have the form

$$\mu\dot{z} = A(t)z + F(t, z, \sigma), \tag{1.1}$$

$$\dot{\sigma} = B(t)\sigma + \Phi(t, z, \sigma), \tag{1.2}$$

the initial conditions being $z(t_0) = z_0$, $\sigma(t_0) = \sigma_0$. The following notations have been used

$$z = \mathrm{col}[z_1, \ldots, z_l], \quad \sigma = \mathrm{col}[\sigma_1, \ldots, \sigma_m],$$

where $l+m = n$. The symbols $A(t)$ and $B(t)$ denote square matrices of order $l$ and $m$, respectively, real and continuous for $t \in [t_0, \infty)$. $F = \mathrm{col}[F_1, \ldots, F_l]$, $\Phi = \mathrm{col}[\Phi_1, \ldots, \Phi_m]$ are also matrices real and continuous for $t \in [t_0, \infty)$ and $\|z\| + \|\sigma\| < \infty$ (the symbol $\|\cdot\|$ denoting the norm). The quantity $\mu = \mathrm{const} \geqslant 0$ is a small parameter.

Let us denote by $z(t, \mu)$ and $\sigma(t, \mu)$ the solution of the equations of motion (1.1) and (1.2). Let us consider in addition a system the equations of which have the form

$$A(t)\zeta + F(t, \zeta, \xi) = 0, \quad \dot{\xi} = B(t)\xi + \Phi(t, \zeta, \xi) \tag{1.3}$$

with the initial condition

$$\xi(t_0) = \xi_0 = \sigma_0. \tag{1.4}$$

Such a system is obtained as a result of irregular or, in other words, singular perturbation of the system described by Eqs. (1.1) and (1.2)—that is, if we set $\mu = 0$ in (1.1). Let us denote the solution of the equations of motion (1.3) by

$$\xi = H(t), \quad \zeta = h(t), \quad \|\dot{h}(t)\| \leqslant \delta = \text{const} < \infty. \tag{1.5}$$

This solution will be considered to be known.

The object of the present paper is to establish conditions sufficient for the relations

$$\|z(t, \mu) - h(t)\| \underset{\mu\to 0}{\to} 0, \quad \|\sigma(t, \mu) - H(t)\| \underset{\mu\to 0}{\to} 0 \tag{1.6}$$

to hold. Let us set in Eqs. (1.1) and (1.2):

$$z(t, \mu) = h(t) + y(t, \mu), \quad \sigma(t, \mu) = H(t) + x(t, \mu). \tag{1.7}$$

Then, we obtain the following differential equations for the functions $y(t, \mu)$ and $x(t, \mu)$:

$$\mu\dot{y} = A(t)y + F[t, h(t)+y, H(t)+x] + A(t)h(t) - \mu\dot{h}(t), \tag{1.8}$$

$$\dot{x} = B(t)x + \Phi[t, h(t)+y, H(t)+x] + B(t)H(t) - \dot{H}(t). \tag{1.9}$$

In agreement with the assumption, the functions $h(t)$ and $H(t)$ satisfy identically the set of Eqs. (1.3)—that is,

$$Ah + F(t, h, H) \equiv 0, \quad \dot{H} \equiv BH + \Phi(t, h, H).$$

Hence,

$$Ah = -F(t, h, H), \quad BH - \dot{H} = -\Phi(t, h, H).$$

It follows that the differential Eqs. (1.8) and (1.9) take the form:

$$\mu\dot{y} = A(t)y + F[t, h(t)+y, H(t)+x] - F[t, h(t), H(t)] - \mu\dot{h}(t), \tag{1.11}$$

$$\dot{x} = B(t)x + \Phi[t, h(t)+y, H(t)+x] - \Phi[t, h(t), H(t)]. \tag{1.12}$$

Let us denote

$$F[t, h(t)+y, H(t)+x] - F[t, h(t), H(t)] = f(t, x, y),$$

$$\Phi[t, h(t)+y, H(t)+x] - \Phi[t, h(t), H(t)] = \varphi(t, x, y),$$

$$-\mu\dot{h}(t) = p(t, \mu).$$

With these notations, the differential Eqs. (1.11) and (1.12) take the form:

$$\mu\dot{y} = A(t)y + f(t, x, y) + p(t, \mu), \tag{1.13}$$

$$\dot{x} = B(t)x + \varphi(t, x, y). \tag{1.14}$$

By virtue of (1.4), (1.5) and (1.7), the initial conditions for the functions $x$ and $y$ are

$$y(t_0) = y_0 = z_0 - h(t_0), \qquad x(t_0) = x_0 = \sigma_0 - H(t_0) = 0.$$

From Eqs. (1.7) it is seen that to show (1.6) it suffices to prove that $||x(t, \mu)|| \to 0$ and $||y(t, \mu)|| \to 0$ if $\mu \to 0$. The problem just stated finds application in the domain of vibrations of nonlinear mechanical systems in which very small masses or very small moments of inertia are involved (hands of instruments for instance), vibrations of nonlinear electric systems with concentrated constants with very small inductions or capacities, and also in systems of automatic control in which some of the time constants are sufficiently small as compared with the remainder.

The problem cited above, and formulated in a similar manner, has already been studied by a number of authors [1–9].

In the present paper will be discussed, by means of the method of integral inequalities [10], these results obtained for cases different from those discussed in the above works.

## 2. Analysis of the properties of the solutions of Eqs. (1.13) and (1.14)

Together with Eqs. (1.13) and (1.14), let us consider the linear differential equations:

(2.1) $$\mu\dot{\eta} = A(t)\eta + p(t, \mu),$$

(2.2) $$\mu\dot{r} = A(t)r,$$

(2.3) $$\dot{q} = B(t)q,$$

where $\eta = \mathrm{col}[\eta_1, \ldots, \eta_l]$, $r = \mathrm{col}[r_1, \ldots, r_l]$, $q = \mathrm{col}[q_1, \ldots, q_m]$.

Let the initial values satisfy the relations:

$$y(t_0) = \eta(t_0) = r(t_0) = y_0, \qquad x(t_0) = q(t_0) = 0.$$

We denote by $R(t)$ and $Q(t)$ the fundamental matrices of solution of Eqs. (2.2) and (2.3), respectively. Let us make the following assumptions:

1. $||R(t)R^{-1}(s)|| \leqslant a e^{-\frac{\gamma}{\mu}(t-s)}$, $||Q(t)Q^{-1}(s)|| \leqslant \alpha e^{-\beta(t-s)}$ for $t_0 \leqslant s \leqslant t$ and $t \in [t_0, \infty)$, where $a, \gamma, \alpha, \beta$ are real positive constants.

2. $||\eta(t)|| \leqslant \mu c$ for $t \in (t_0, \infty)$, where $c$ is a real positive constant.

3. $||f(t, x, y)|| \leqslant k_1 g_1(||x|| + ||y||)$, $||\varphi(t, x, y)|| \leqslant k_2 g_2(||x|| + ||y||)$ for $t \in [t_0, \infty)$ and $||x|| + ||y|| < \infty$; $k_1, k_2$ are non-negative real constants and $g_1(u)$ and $g_2(u)$ are continuous, non-negative, non-decreasing functions for $u \geqslant 0$. We have also $g_1(0) = g_2(0) = 0$.

Let us denote by $g(u)$ a function satisfying the relation:

(2.4) $$g(u) \geqslant \sup_{u \geqslant 0} (g_1, g_2),$$

and let $g(u)$ be a continuous, non-negative, non-decreasing function for $u \geqslant 0$ and $g(0) = 0$. This function is assumed to satisfy the relation:

(2.5) $$\lim_{\vartheta \to 0} \frac{g(\vartheta)}{\vartheta^2} = \nu = \mathrm{const} \geqslant 0,$$

and the constant $\nu = \text{const} < \infty$ is assumed to satisfy the inequality

$$\beta > 4cak_1\nu. \tag{2.6}$$

The conditions for the assumption 1 to be satisfied are mentioned in [3 and 8]. For the assumption 2 it can be easily shown that, if $\|\dot{h}(t)\| \leqslant \delta = \text{const} < \infty$ [cf. (1.5)], then $\|p(t, \mu)\| \leqslant \mu\delta$ and

$$\|\eta\| \leqslant \mu\frac{a\delta}{\gamma} = \mu c < \infty \quad \text{for} \quad t \in (t_0, \infty). \tag{2.7}$$

For $t = t_0$, we have $\|\eta\| \leqslant \|\eta_0\| + \mu c$ and for every $t > t_0$—that is, in the open interval $(t_0, \infty)$—the appraisal (2.7) is valid.

The set of integral equations corresponding to the differential Eqs. (1.13) and (1.14) has the form:

$$y = R(t)R^{-1}(t_0)y_0 + \frac{1}{\mu}\int_{t_0}^{t} R(t)R^{-1}(s)f[s, x(s), y(s)]ds + \frac{1}{\mu}\int_{t_0}^{t} R(t)R^{-1}(s)p(s, \mu)ds.$$

$$x = \int_{t_0}^{t} Q(t)Q^{-1}(s)\varphi[s, x(s), y(s)]ds.$$

The solution of the differential Eq. (2.1) has the form

$$\eta = R(t)R^{-1}(t_0)\eta_0 + \frac{1}{\mu}\int_{t_0}^{t} R(t)R^{-1}(s)p(s, \mu)ds;$$

therefore, the integral equations take the form:

$$y = \eta + \frac{1}{\mu}\int_{t_0}^{t} R(t)R^{-1}(s)f[s, x(s), y(s)]ds, \tag{2.8}$$

$$x = \int_{t_0}^{t} Q(t)Q^{-1}(s)\varphi[s, x(s), y(s)]ds, \tag{2.9}$$

Taking the norms of both members of the inequalities (2.8) and (2.9), we obtain, by virtue of the assumptions 1, 2 and 3, within the interval $(t_0, \infty)$,

$$\|y\| \leqslant \mu c + \frac{1}{\mu}\int_{t_0} ak_1 e^{-\frac{\gamma}{\mu}(t-s)} g_1(\|x\| + \|y\|)ds, \tag{2.10}$$

$$\|x\| \leqslant \int_{t_0}^{t} \alpha k_2 e^{-\beta(t-s)} g_2(\|x\|) + \|y\|)ds. \tag{2.11}$$

Let us denote

$$\|x\| + \|y\| = u.$$

On adding the inequalities (2.10) and (2.11) and taking into consideration the notation (2.4), we obtain:

$$(2.12) \qquad u \leqslant \mu c + \int_{t_0}^{t} \left[ \frac{ak_1}{\mu} e^{-\frac{\gamma}{\mu}(t-s)} + \alpha k_2 e^{-\beta(t-s)} \right] g(u)\, ds.$$

Let us denote the right-hand member by $v(t)$; therefore,

$$(2.13) \qquad v(t) = \mu c + \frac{ak_1}{\mu} e^{-\frac{\gamma}{\mu}t} \int_{t_0}^{t} e^{\frac{\gamma}{\mu}s} g(u)\, ds + \alpha k_2 e^{-\beta t} \int_{t_0}^{t} e^{\beta s} g(u)\, ds \geqslant u.$$

By differentiating this relation with respect to time, we find:

$$(2.14) \qquad \dot{v}(t) = -\left[ \frac{\gamma}{\mu} \left( \frac{ak_1}{\mu} e^{-\frac{\gamma}{\mu}t} \int_{t_0}^{t} e^{\frac{\gamma}{\mu}s} g(u)\, ds \right) + \beta \left( \alpha k_2 e^{-\beta t} \int_{t_0}^{t} e^{\beta s} g(u)\, ds \right) \right] + \left( \frac{ak_1}{\mu} + \alpha k_2 \right) g(u).$$

On denoting

$$\delta_1 = \frac{ak_1}{\mu} \int_{t_0}^{t} e^{-\frac{\gamma^1}{\mu}(t-s)} g(u)\, ds, \qquad \delta_2 = \alpha k_2 \int_{t_0}^{t} e^{-\beta(t-s)} g(u)\, ds,$$

the relations (2.13) and (2.14) can be written

$$(1.15) \qquad v - \mu c = \delta_1 + \delta_2,$$

$$(2.16) \qquad \dot{v} = -\left( \frac{\gamma}{\mu} \delta_1 + \beta \delta_2 \right) + \left( \frac{ak_1}{\mu} + \alpha k_2 \right) g(u) = -\beta \left( \frac{\gamma}{\mu\beta} \delta_1 + \delta_2 \right) + \left( \frac{ak_1}{\mu} + \alpha k_2 \right) g(u).$$

Let $\mu^*$ denote the value for which $\gamma/\mu^*\beta = 1$. For $\mu < \mu^*$, we have:

$$\frac{\gamma}{\mu\beta} \delta_1 + \delta_2 > \delta_1 + \delta_2 = v - \mu c.$$

In addition, by virtue of (2.13) we have $v \geqslant u$. Substituting this in (2.16), we obtain the inequality:

$$(2.17) \qquad \dot{v} \leqslant -\beta(v - \mu c) + \left( \frac{{}^{\prime}ak_1}{\mu} + \alpha k_2 \right) g(v).$$

On changing variables by the relation $\varrho = \frac{1}{\mu c} v - 1$, we find $v = (1+\varrho)\mu c$, and for $t = t_0$, we have $v(t_0) = v_0 = \mu c$, $\varrho_0 = \frac{v_0}{\mu c} - 1 = 0$, $\tau = \beta t$, for $t = t_0$, we have $\tau_0 = \beta t_0$ and $d\tau/dt = \beta$; therefore, the inequality (2.17) takes the form

$$\frac{dv}{dt} = \mu c \frac{d\varrho}{d\tau} \frac{d\tau}{dt} \leqslant -\beta \mu c \varrho + \left( \frac{ak_1}{\mu} + \alpha k_2 \right) g[(1+\varrho)\mu c].$$

Let us denote

$$\varkappa(\mu) = \frac{ak_1 + \mu\alpha k_2}{\beta c}; \quad \text{therefore,} \quad \varkappa(0) = \frac{ak_1}{\beta c}. \tag{2.18}$$

With this notation we have

$$\frac{d\varrho}{d\tau} \leqslant \frac{\varkappa(\mu)}{\mu^2} g[(1+\varrho)\mu c] - \varrho. \tag{2.19}$$

It is assumed that for every $\mu > 0$ there is a $\lambda_0(\mu) < \infty$, which is a continuous function of the parameter $\mu$ such that

$$\frac{\varkappa(\mu)}{\mu^2} g[(1+\lambda_0)\mu c] - \lambda_0 = 0, \tag{2.20}$$

$$\frac{\varkappa(\mu)}{\mu^2} g[(1+\lambda)\mu c] - \lambda > 0 \quad \text{for} \quad \lambda \in [0, \lambda_0). \tag{2,21}$$

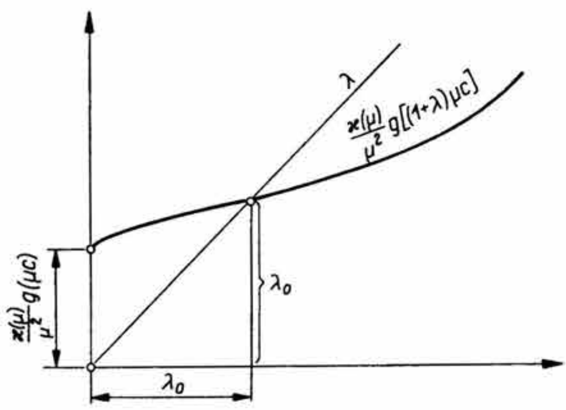


FIG. 1.

Since $\varrho_0 = 0$, therefore, by virtue of (2.21), there exists such a $\tau^* > \tau_0$ that the right-hand side of the inequality (2.19) is positive and we can, within the interval $(\tau_0, \tau^*)$, divide this inequality by its right-hand member. Thus,

$$\int_0^{\varrho(\tau)} \frac{ds}{\frac{\varkappa(\mu)}{\mu^2} g[(1+s)\mu c] - s} \leqslant \tau - \tau_0 \quad \text{for} \quad \tau \in (\tau_0, \tau^*), \mu > 0. \tag{2.22}$$

However (cf. Fig. 1)

$$\text{for } \lambda < \lambda_0 \quad \text{is} \quad \frac{\varkappa(\mu)}{\mu^2} g[(1+\lambda)\mu c) < \lambda_0;$$

therefore,

$$\int_0^{\varrho(\tau)} \frac{ds}{\lambda_0 - s} \leqslant \int_0^{\varrho(\tau)} \frac{ds}{\frac{\varkappa(\mu)}{\mu^2} g[(1+s)\mu c] - s} \leqslant \tau - \tau_0 \tag{2.23}$$

for $\tau \in (\tau_0, \tau^*)$, $\mu > 0$. Let us suppose that $\varrho(\tau^*) = \lambda_0$ for $\tau = \tau^* < \infty$. Then, by virtue of (2.23),

$$\int_0^{\lambda_0} \frac{ds}{\lambda_0 - s} \leqslant \tau^* - \tau_0 \tag{2.24}$$

which is impossible, because the integral on the left-hand side of the inequality (2.24) tends to $+\infty$ and the right-hand side is finite. Thus, the functions $\varrho(\tau)$ satisfying the inequality (2.22) satisfy, for $\tau \in (\tau_0, \infty)$ and $\mu > 0$, the inequality

$$\varrho(\tau) \leqslant \lambda_0(\mu) \tag{2.25}$$

and the equation $\varrho(\tau) = \lambda_0$ is valid for $\tau = \infty$ only (that is $\tau^* = \infty$). As a result we find that

$$\|x\| + \|y\| = u(t) \leqslant v(t) \leqslant [1 + \lambda_0(\mu)]\mu c \tag{2.26}$$

for any sufficiently small $\mu > 0$ and $t \in (t_0, \infty)$. Hence,

$$\|x\| \leqslant [(1 + \lambda_0(\mu)]\mu c, \qquad \|y\| \leqslant [(1 + \lambda_0(\mu)]\mu c \tag{2.27}$$

for any sufficiently small $\mu > 0$ and $t \in (t_0, \infty)$.

To analyse the behaviour of the norms $\|x\|$ and $\|y\|$ for $\mu \to 0$, we must first prove their validity for $\mu = 0$ or, in other words, that the relation (2.25) takes for $\mu \to 0$ the form:

$$\varrho(\tau) \leqslant \lambda_0(0). \tag{2.28}$$

The value $\lambda_0(0)$ can be found from Eq. (2.20), which can be rewritten thus:

$$\varkappa(\mu)[1 + \lambda_0(\mu)]^2 c^2 \frac{g[(1 + \lambda_0(\mu))\mu c]}{(1 + \lambda_0(\mu))^2 \mu^2 c^2} = \lambda_0(\mu). \tag{2.29}$$

On passing to the limit for $\mu \to 0$, we obtain, by virtue of (2.18),

$$\frac{ak_1}{\beta c}[1 + \lambda_0(0)]^2 c^2 \nu = \lambda_0(0). \tag{2.30}$$

If $0 < \nu < \infty$, the relation (2.28) holds, and on transforming (2.30) we have:

$$\lambda_0^2(0) + \left(2 - \frac{\beta}{cak_1\nu}\right)\lambda_0(0) + 1 = 0.$$

By virtue of (2.6), we have $\beta > 4\ cak_1\nu$; therefore, the quadratic equation has two real bounded solutions $\lambda_0(0)$, both of which are positive. We take the smaller one.

If $\nu = 0$, it follows from (2.30) that $\lambda_0(0) = 0$ and Eq. (2.25) should be verified. To this end, we use directly Eq. (2.19) which is represented in the form:

$$\frac{d\varrho}{d\tau} \leqslant \varkappa(\mu)(1 + \varrho)^2 c^2 \frac{g[(1 + \varrho)\mu c]}{(1 + \varrho)^2 \mu^2 c^2} - \varrho.$$

By letting $\mu \to 0$, we obtain for $\nu = 0$, by virtue of (2.5),

$$\frac{d\varrho}{d\tau} \geqslant -\varrho, \qquad \text{therefore,} \qquad \varrho(\tau) \leqslant \varrho_0 e^{-(\tau - \tau_0)}.$$

Since now $\varrho_0 = 0$, we have the inequality $\varrho(\tau) \leqslant 0$, which means that Eq. (2.25) remains valid in this case.

Thus, Eq. (2.28) holds for $0 \leqslant \nu < \infty$, and $\lambda_0(0)$ is bounded. Hence, by virtue of (2.27),

$$\lim_{\mu\to 0} \|x(t,\mu)\| = 0, \quad \lim_{\mu\to 0} \|y(t,\mu)\| = 0 \tag{2.31}$$

for $t \in (t_0, \infty)$. As a result, by virtue of (1.7), we find

$$\lim_{\mu\to 0} \|z(t,\mu) - h(t)\| = 0, \quad \lim_{\mu\to 0} \|\sigma(t,\mu) - H(t)\| = 0 \tag{2.32}$$

for $t \in (t_0, \infty)$.

It has been shown above that $\lambda_0(\mu)$ is bounded, therefore, the mutual tendency of the solutions of Eqs. (1.1), (1.2) and (1.3) for $\mu \to 0$ is of order $\mu$ [cf. Eqs. (2.27)]. This is a consequence of the assumption (2.5). If this assumption is weakened by assuming that $\nu = \nu(\mu)$, there are no major difficulties in establishing conditions for $\nu(\mu)$ such that $\lambda_0(\mu)$ may increase indefinitely for $\mu \to 0$ in such a manner that $\mu\lambda_0(\mu) \to 0$ for $\mu \to 0$. These conditions can be obtained directly from Eq. (2.29).

Thus, from the considerations of the present paper it follows that if the assumptions made are satisfied, the system of equations of motion (1.1) and (1.2), which are more complicated, can be replaced by a simple system (1.3), provided that the parameter $\mu$ is sufficiently small.

## References

1. A. N. TIKHONOV, Mat. Sborn., 22 (64), 1968.
2. I. S. GRADSHTEIN, UMN, **6**, 2, 1951.
3. L. FLATTO, N. LEVINSON, J. Rat. Mech. Anal., 4, 1955.
4. D. W. ANOSOV, Mat. Sborn, 50 (92), 1960.
5. W. M. VOŁOSOW, UMN, **17**, 6, 1962.
6. A. B. VASIL'EVA, UMN, **18**, 3, 1963.
7. V. VAZOV, Proceedings of the International Symposium on Nonlinear Vibrations, v. **1**, Izd. Akad. Nauk USSR, Kiev 1963.
8. J. K. HALE, G. SEIFERT, Proceedings of the International Symposium on Nonlinear Vibrations, v. **2**, Izd. Akad. Nauk USSR, Kiev 1963.
9. K. V. ZADIRAKA, *Problems of the theory and history of differential equations*, Naukova Dumka, Kiev 1968.
10. R. GUTOWSKI, B. RADZISZEWSKI, Nonlinear Vibrations Problems, **11**, 1971.

WARSAW TECHNICAL UNIVERSITY
DEPARTMENT OF POWER AND AIRCRAFT ENGINEERING
INSTITUTE OF APPLIED MECHANICS

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