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Moving load on a solid-solid interface: supersonic regime

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THE PAPER is devoted to the problem of a moving point force on as solid-solid interface. The steady-state solution for supersonic load velocities was obtained through the use of DeHoop modification of Cagniard's technique, and the displacements in the two solids are presented.

Rozpatrzono problem poruszającej się siły punktowej, działającej na powierzchni styku dwóch ciał stałych. Otrzymano stacjonarne rozwiązanie dla naddźwiękowych prędkości ruchu obciążenia. Zastosowano modyfikację DeHoopa metody Cagniarda. Podano przemieszczenia w obu ciałach.

Рассмотрена задача о перемещающемся сосредоточенном усилии, воздействующем на поверхность контакта двух твердых тел. Получено стационарное решение для сверхзвуковой скорости движения нагрузки. При решении применен метод Каньяра в модифицированном виде, предложенном Де Хопом. Даны перемещения в обоих телах.

1. Introduction

IN SEVERAL previous studies [1, 2, 3], the authors have considered the response of a fluid-solid interface to moving point disturbances. The present investigation is devoted to the problem of a moving point force on a solid-solid interface. The steady-state solution for supersonic load velocities was obtained through the use of DeHoop's [4] modification of Cagniard's [5] technique, and the displacements in the two solids are presented.

2. Statement of the problem

Consider a normal point load of magnitude P moving along the plane interface between two different elastic solid half-spaces which have been bonded. The interface lies in the x, y-plane of a rectangular Cartesian coordinate system as shown in Fig. 1. The load P moves with a constant velocity V in the positive x-direction and is considered positive when acting in the positive z-direction. The solid which extends in the positive z-direction will be referred to as solid 1 and the other as solid 2. It is assumed that the solids possess different densities and elastic properties.

After the load has been moving for some time and the transient effects have dissipated, the displacements will appear stationary in a coordinate system moving with the load. Expressions for the displacements for this steady-state problem will be presented for supersonic load velocities.

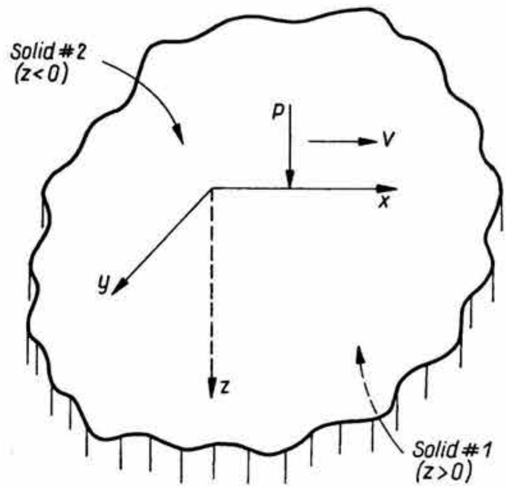

FIG. 1. Moving load on a solid-solid interface.

2.1. Equations of motion

The equations of motion for the two solids are

$$\nabla^2 \varphi_k = \frac{1}{c_{kd}^2} \frac{\partial^2 \varphi_k}{\partial t^2}, \quad \nabla^2 \boldsymbol{\psi}_k = \frac{1}{c_{ke}^2} \frac{\partial^2 \boldsymbol{\psi}_k}{\partial t^2}, \tag{2.1}$$

where

$$c_{kd}^2 = (\lambda_k + 2\mu_k)/\varrho_k, \quad c_{ke}^2 = \mu_k/\varrho_k,$$

$$\boldsymbol{\psi}_k = \psi_{kx}\mathbf{e}_k + \psi_{ky}\mathbf{e}_y + \psi_{kz}\mathbf{e}_z, \quad \nabla \cdot \boldsymbol{\psi}_k = 0, \quad \text{for } k = 1, 2. \tag{2.2}$$

The subscript $k = 1$ refers to solid 1 and $k = 2$ to solid 2. φ_k and ψ_k are the Lamé potentials; c_{kd} and c_{ke} are the dilatational and equivoluminal wave speeds, respectively; λ_k and μ_k are Lamé's constants; ϱ_k is the density of the solid; and $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit vectors in the x, y, and z directions, respectively. The displacements and stresses may be expressed as

$$\begin{aligned} \mathbf{u}_k &= u_{kx}\mathbf{e}_x + u_{ky}\mathbf{e}_y + u_{kz}\mathbf{e}_z = \nabla\varphi_k + \nabla \times \boldsymbol{\psi}_k, \\ \sigma_{kmn} &= \lambda_k \nabla \cdot u_k \delta_{mn} + \mu_k (u_{km,n} + u_{kn,m}), \quad \text{for} \quad m, n = x, y, z \quad \text{and} \quad k = 1, 2, \end{aligned} \tag{2.3}$$

where $\mathbf{u}_k$ is the displacement in solid k, σ_{kmn} is the stress, and δ_{mn} is the Kronecker delta.

2.2. Interface conditions

At the interface ($z = 0$), the normal stress is

$$\sigma_{1zz} = -P\delta(y)\delta(x - Vt) + \sigma_{2zz}, \tag{2.4}$$

where $\delta(\sim)$ is the Dirac delta function. Since the solids are bonded, the shear stresses must be identical

$$\sigma_{1xz} = \sigma_{2xz}, \quad \sigma_{1yz} = \sigma_{2yz} \tag{2.5}$$

and the displacements are also equal,

$$u_{1m} = u_{2m} \quad \text{for} \quad m = x, y, z. \tag{2.6}$$

2.3. Steady-state equations

As was done in earlier studies, e.g. by COLE and HUTH [6], the equations of motion are expressed in terms of the moving coordinate system ($\bar{x} = x - Vt, y, z$) as

$$(2.7)\qquad \begin{aligned} \frac{\partial^2 \varphi_k}{\partial y^2} + \frac{\partial^2 \varphi_k}{\partial z^2} &= (M_{kd}^2 - 1)\frac{\partial^2 \varphi_k}{\partial \bar{x}^2},\\ \frac{\partial^2 \psi_k}{\partial y^2} + \frac{\partial^2 \psi_k}{\partial z^2} &= (M_{ke}^2 - 1)\frac{\partial^2 \psi_k}{\partial \bar{x}^2}, \quad \text{for} \quad k = 1, 2, \end{aligned}$$

where

$$(2.8)\qquad M_{kj} = V/c_{kj} \quad \text{for} \quad j = d, e \quad \text{and} \quad k = 1, 2.$$

Each of the steady-state equations is either elliptic or hyperbolic depending on whether M_{kj} is less than or greater than one. That is, as the load velocity passes from sub-wave speed to super-wave speed, the appropriate equation changes from elliptic to hyperbolic. In this problem the relationship between the wave speeds is chosen such that $M_{2e} > M_{1e} > M_{2d} > M_{1d}$. Also, the Stoneley interface wave is assumed to exist. The conditions under which the Stoneley wave will exist can be found in Ch. 4 of [5].

3. Solution for the supersonic regime

3.1. Method of solution

When the velocity of the load is greater than all the wave speeds, the steady-state equations are all hyperbolic; and it may be assumed that the displacements are all zero ahead of the load (i.e., for $\bar{x} > 0$). It is now convenient to introduce another change of variable

$$(3.1)\qquad x_1 = -\bar{x} = -x + Vt$$

so that the displacements vanish for $x_1 < 0$. This permits a more conventional use of the Laplace transform on x_1 with a Fourier transform on y.

After proceeding in the usual way, the transformed displacements may be obtained but will not be presented here. Their inversion may be accomplished through DeHoop's [4] modification of Cagniard's [5] technique. The essence of Cagniard's method is to deform the path of integration of the Fourier inversion integral in such a manner that the Laplace transform inversion may be performed by inspection. Illustrations of this method may be found in previous studies [1, 3] by the authors.

3.2. Displacements-Solid 1

The displacements in the interior of solid 1 may be expressed as follows:

$$(3.2)\qquad \begin{aligned} u_{1m}(\bar{x}, r, \theta)_{\text{int}} = \frac{P/\mu_1}{4\pi(1-\bar{\mu})}\Bigg\{ &\text{Re}\left[\frac{u_{1md}(L_{1d}(-\bar{x}))}{\Delta_s(L_{1d}(-\bar{x}))}\,\frac{dL_{1d}(-\bar{x})}{d\bar{x}}\right] H(-\bar{x} - B_{1d} r)\\ &+ \text{Re}\left[\frac{u_{1me}(L_{1e}(-\bar{x}))}{\Delta_s(L_{1e}(-\bar{x}))}\,\frac{dL_{1e}(-\bar{x})}{d\bar{x}}\right] H(-\bar{x} - B_{1e} r) \end{aligned}$$

$$+\operatorname{Re}\left[\frac{u_{1me}\big(L_{1d1e}(-\bar{x})\big)}{\Delta_s\big(L_{1d1e}(-\bar{x})\big)}\frac{dL_{1d1e}(-\bar{x})}{d\bar{x}}\right][H(-\bar{x}-\bar{x}_{1d1e})H(\bar{x}+\bar{x}_{2d1e})H(\sin\theta-B_{1d}/B_{1e})+H(-\bar{x}-\bar{x}_{2d1e})H(\bar{x}+B_{1e}r)H(\sin\theta-B_{2d}/B_{1e})]\Bigg\}\quad \text{for}\quad m=x,y,z,$$

where $H(\sim)$ is the Heaviside step function and

$$r = y^2+z^2,\quad \theta = \tan^{-1}(y/z),\quad \bar{\mu} = \mu_2/\mu_1, \tag{3.3}$$

$$u_{1xd}(b) = 2[(1-\bar{\mu})(b^2-1)+M_{1e}](b^2-1-n_{2d}n_{2e})-\bar{\mu}M_{2e}^2(b^2-1+n_{1e}n_{2d}),$$

$$u_{2xd}(b) = [2(1-\bar{\mu})(b^2-1)-\bar{\mu}M_{2e}^2](b^2-1-n_{1d}n_{1e})+M_{1e}^2(b^2-1+n_{1d}n_{2e}),$$

$$u_{1xe}(b) = -n_{1e}[2(1-\bar{\mu})(b^2-1-n_{2d}n_{2e})n_{1d}-\bar{\mu}M_{2e}(n_{1d}+n_{2d})],$$

$$u_{2xe}(b) = -n_{2e}[2(1-\bar{\mu})(b^2-1-n_{1d}n_{1e})n_{2d}+M_{1e}^2(n_{1d}+n_{2d})],$$

$$u_{kyj}(b) = ibu_{kxj}\quad \text{for}\quad j=d,e\quad \text{and}\quad k=1,2,$$

$$u_{kzd}(b) = -(-1)^k n_{kd}u_{kxd}\quad \text{for}\quad k=1,2,$$

$$u_{kze}(b) = -(-1)^k(b^2-1)u_{kxe}/n_{ke}\quad \text{for}\quad k=1,2;$$

$$\Delta_s(b) = (b^2-1)[(b^2-1)+\tfrac{1}{2}(M_{1e}-\bar{\mu}M_{2e})/(1-\bar{\mu})]^2+(b^2-1)n_{1d}n_{2d}n_{1e}n_{2e}-n_{1d}n_{1e}[(b^2-1)-\tfrac{1}{2}\bar{\mu}M_{2e}^2/(1-\bar{\mu})]^2-n_{2d}n_{2e}[(b^2-1)+\tfrac{1}{2}M_{1e}^2/(1-\bar{\mu})]^2-(1/4)\bar{\mu}M_{1e}^2M_{2e}^2(n_{1d}n_{2e}+n_{2d}n_{1e})/(1-\bar{\mu})^2; \tag{3.4}$$

$$n_{kj} = (b+B_{kj}^2)^{1/2}\quad \text{for}\quad j=d,e\quad \text{and}\quad k=1,2,$$

$$B_{kj} = (M_{kj}^2-1)^{1/2}\quad \text{for}\quad j=d,e\quad \text{and}\quad k=1,2;$$

$$L_{kj}(-\bar{x}) = -(-1)^k[(\bar{x}/r)^2-B_{kj}^2]^{1/2}\cos\theta+i(\bar{x}/r)\sin\theta\quad \text{for}\quad j=d,e\quad \text{and}\quad k=1,2,$$

$$L_{1dkj}(-\bar{x}) = i\{[B_{kj}^2-(\bar{x}/r)^2]^{1/2}\cos\theta+(\bar{x}/r)\sin\theta\}\quad \text{for}\quad j=d,e\quad \text{and}\quad k=1,2,$$

$$\bar{x}_{kjk'j'} = r[-(-1)^k(B_{k'j'}-B_{kj})^{1/2}\cos\theta+B_{kj}\sin\theta]\quad \text{for}\quad j,j'=d,e\quad \text{and}\quad k,k'=1,2.$$

The step functions in the displacement expression signify the locations of the wave fronts. $H(-\bar{x}-B_{1d}r)$ corresponds to the dilatational wave, $H(-\bar{x}-B_{1e}r)$ to the equivoluminal wave, $H(-\bar{x}-\bar{x}_{1d1e})H(\bar{x}+\bar{x}_{2d1e})H(\sin\theta-B_{1d}/B_{1e})$ to the head wave generated by the incidence of the solid 1 dilatational wave at the interface, and $H(-\bar{x}-\bar{x}_{2d1e})\times$ $\times H(\bar{x}+B_{1e}r)H(\sin\theta-B_{2d}/B_{1e})$ to the head wave generated by the incidence of the solid 2 dilatational wave at the interface.

3.3. Displacements — Solid 2

The displacements in the interior of solid 2 may be expressed as

$$(3.5)\quad u_{2m}(x, r, \theta)_{\rm int} = \frac{-P/\mu_1}{4\pi(1-\bar{\mu})}\Bigg\{\mathrm{Re}\left[\frac{u_{2md}(L_{2d}(-\bar{x}))}{\Delta_s(L_{2d}(-\bar{x}))}\frac{dL_{2d}(-\bar{x})}{d\bar{x}}\right]H(-\bar{x}-B_{2d}r)$$
$$+\mathrm{Re}\left[\frac{u_{2me}(L_{2e}(-\bar{x}))}{\Delta_s(L_{2e}(-\bar{x}))}\frac{dL_{2e}(-\bar{x})}{d\bar{x}}\right]H(-\bar{x}-B_{2e}r)$$
$$+\mathrm{Re}\left[\frac{u_{2md}(L_{1d\,2d}(-\bar{x}))}{\Delta_s(L_{1d\,2d}(-\bar{x}))}\frac{dL_{1d\,2d}(-\bar{x})}{d\bar{x}}\right]H(-\bar{x}$$
$$-\bar{x}_{1d\,2d})H(\bar{x}+B_{2d}r)H(\sin\theta-B_{1d}/B_{2d})+\mathrm{Re}\left[\frac{u_{2me}(L_{1d\,2e}(-\bar{x}))}{\Delta_s(L_{1d\,2e}(-\bar{x}))}\frac{dL_{1d\,2e}(-\bar{x})}{d\bar{x}}\right]\big[H(-\bar{x}$$
$$-\bar{x}_{1d\,2e})H(\bar{x}+\bar{x}_{2d\,2e})H(\sin\theta-B_{1d}/B_{2e})$$
$$+H(-\bar{x}-\bar{x}_{2d\,2e})H(\bar{x}+\bar{x}_{1e\,2e})H(\sin\theta-B_{2d}/B_{2e})$$
$$+H(-\bar{x}-\bar{x}_{1e\,2e})H(\bar{x}+B_{2e}r)H(\sin\theta-B_{1e}/B_{2e})\big]\Bigg\}\quad \text{for}\quad m = x, y, z.$$

Again, the wave fronts are signified by the step functions. $H(\bar{x}-B_{2d}r)$ corresponds to the solid 2 dilatational wave, $H(\bar{x}-B_{2e}r)$ to the equivoluminal wave, $H(-\bar{x}-\bar{x}_{1d\,2d})\times \times H(\bar{x}+B_{2d}r)\,H(\sin\theta-B_{1d}/B_{2d})$ and $H(-\bar{x}-x_{1d\,2e})\,H(\bar{x}+\bar{x}_{2d\,2e})\,H(\sin\theta-B_{1d}/B_{2e})$ to the head waves generated by the incidence of the solid 1 dilatational wave at the interface, $H(-\bar{x}-\bar{x}_{2d\,2e})\;H(\bar{x}+\bar{x}_{1e\,2e})\;H(\sin\theta-B_{2d}/B_{2e})$ to the head wave generated by the incidence of the solid 2 dilatational wave at the interface, and $H(-\bar{x}-\bar{x}_{1e2e})\;H(\bar{x}+B_{2e})\;H(\sin\theta- -B_{1e}/B_{2e})$ to the head wave generated by the incidence of the solid 1 equivoluminal wave at the interface. The whole system of wavefronts for both solids is shown in Fig. 2.

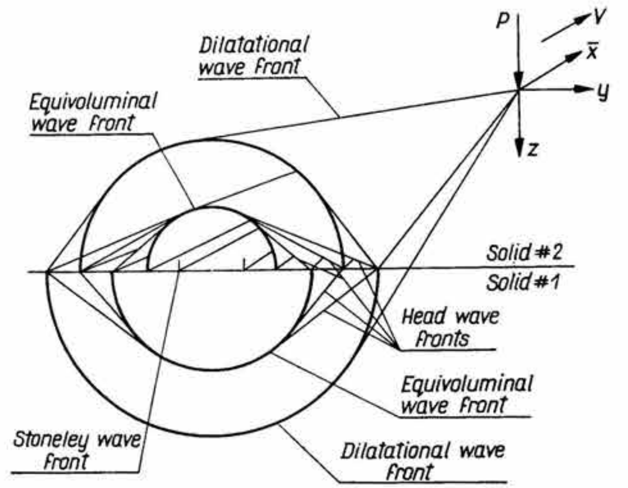

FIG. 2. Wave front system. Supersonic regime.

The displacements at the interface may be expressed as

$$(3.6)\quad u_m(\bar{x}, y, 0) = u_{1m}(\bar{x}, r, \theta)_{\text{int}}\big|_{\theta=\pi/2} - \frac{P/\mu_1}{4(1-\bar{\mu})}\,\text{Im}\left[\frac{u_{1md}(-iB_s)+u_{1me}(-iB_s)}{\left.\dfrac{d\Delta_s(b)}{db}\right|_{b=-iB_s}}\right]\delta(-\bar{x}-B_s y), \quad \text{for} \quad m = x, y, z,$$

where

$$(3.7)\qquad B_s = (M_s^2-1)^{1/2}, \quad M_s = V/c_s;$$

c_s is the velocity of the Stoneley interface wave.

A more detailed treatment of this and related problems can be found in a forthcoming Ph. D. dissertation [7].

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