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Limit state of an elliptical plate

D. NIEPOSTYN (WARSZAWA)

THE PAPER presents a complete solution for an elliptical plate simply supported at the boundary and uniformly loaded, made of an isotropic material yielding according to the Johansen condition. The equation of equilibrium of the plate is written in curvilinear coordinates of the principal moment trajectories. After integration a solution is obtained which determines the limit moment value and the initially unknown set of trajectories. The static solution leads to the same result as the kinematic solution. Basic results are given following from the approximate solution of the Abel differential equation with a parameter; plates with various axial ratios of the ellipse $\beta = b/a$ are considered.

W pracy podano rozwiązanie zupełne dla płyty eliptycznej, przegubowo opartej na obwodzie, obciążonej równomiernie, wykonanej z materiału izotropowego, uplastyczniającego się zgodnie z warunkiem Johansena. Równania równowagi płyty zapisano we współrzędnych krzywoliniowych, za jakie przyjęto trajektorie momentów głównych. Po scałkowaniu równań otrzymano równanie określające wartość momentu granicznego oraz początkowo nieznany układ trajektorii. Rozwiązanie statyczne prowadzi do identycznego rozwiązania jak i rozwiązanie kinematyczne. Podano podstawowe wyniki uzyskane na podstawie numerycznego rozwiązania równania różniczkowego Abela z parametrem dla płyty eliptycznej o różnym stosunku długości pół osi elipsy $\beta = b/a$.

В работе дано полное решение для равномерно нагруженной, шарнирно опёртой по контуру, эллиптической пластинки, выполненной из изотропного материала, подчиняющегося условию текучести Иогансена. Уравнения равновесия пластинки записаны в криволинейных координатах, какими являются траектории главных моментов. В результате интегрирования уравнений получено решение, определяющее величину предельного момента, а также первоначально неизвестную систему траекторий. Статическое решение приводит к такому же результату, как и кинематическое решение. Приведены основные результаты, полученные при численном решении дифференциального уравнения Абеля с параметром. Вычисления относятся к эллиптическим пластинкам с различными отношениями длин полуосей эллипса $\beta = b/a$.

1. Introduction

1.1. Limit state of a structure

STRUCTURES made of perfectly elastic-plastic materials and loaded by external forces can work in both the elastic and elastic-plastic states. One of the characteristic features of the structure must be its geometric invariability. Under a corresponding increase of the load intensity the yield zones will propagate, and once a certain limit of the load intensity is reached, the yield zones will make the system geometrically variable. The least value of the load which transforms the structure into a mechanism is called the limit load. The state which is attained by the structure immediately after the application of the limit load is called the limit state.

The limit load intensity depends on the form and dimensions of the structure, the load distribution, the yield limit and the yield condition. The plates which will be dealt with in the present paper obey the K. W. Johansen yield condition for isotropic materials

$$\max(|M_1|, |M_2|) = M_0, \tag{1.1}$$

M_1, M_2 denoting the principal moments, and M_0 the moment leading to total yielding of a unit cross-section of the plate.

Our considerations will be based on the technical theory of thin plates and small deflections; all assumptions of that theory will be valid throughout the paper.

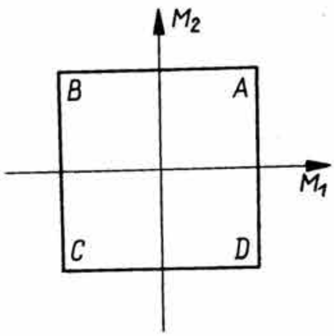

FIG. 1.

The solutions will be based on the theory of small elastic-plastic deformations and on the perfectly rigid-plastic model of the body; it will enable us to deal with the concept of plate deflection instead of the deflection increment.

The physical relations are in agreement with the associated flow rule and the existence of a plastic potential.

1.2. Approximate solutions

A complete solution of the problem of load carrying capacity of a structure consisting in the simultaneous determination of

(a) the limit load intensity,

(b) distribution of the generalized internal forces,

(c) mechanism of collapse of the structure

is known for certain particular types of structures only. It is not easy to find the complete — i.e., the correct (within the framework of the assumptions introduced) — solution. This difficulty necessitates seeking approximate solutions, including those determined in a static or kinematic way.

The static solution in the case of plates consists, in principle, in assumption of a certain distribution of moments compatible with the boundary conditions, satisfying the equations of equilibrium and the yield condition in certain regions. Such a solution allows for the determination of the limit load P_{0s}, certainly approximate.

The kinematic solution consists in assumption of a certain mechanism of collapse of the plate and in determination of the limit load P_{0k}, which is also approximate. It is known that the actual limit load P_0 is contained in the interval:

$$P_{0s} \leqslant P_0 \leqslant P_{0k}. \tag{1.2}$$

The bounds of the interval make it simultaneously possible to decide which of two different solutions, corresponding to two different statically admissible moment distributions, or two different collapse mechanisms, lies closer to the exact solution. These are obviously the solutions which contract the interval.

The majority of papers dealing with the load carrying capacity of plates are limited to the kinematic solution. Such collapse mechanisms are sought for which correspond to the least intensity of the limit load.

In the case of a complete solution, such collapse mechanism and such moment distributions are determined which correspond to equal values of the collapse load. The left and right-hand bounds of the interval (1.2) coincide,

$$P_{0k} = P_{0s} = P_0, \tag{1.3}$$

and hence such solution is exact within the framework of the assumptions made.

In the present paper is given the solution for a plate simply supported along the entire boundary and with a uniform load, having at least one axis of symmetry. For simplicity, it is assumed that it is located on the horizontal axis.

2. Complete solution

2.1. Equation of equilibrium of the plate in curvilinear coordinates

It is convenient to write the equation of equilibrium of the plate in curvilinear coordinates.

The curves

$$x = x(\alpha, \beta), \quad y = y(\alpha, \beta) \tag{2.1}$$

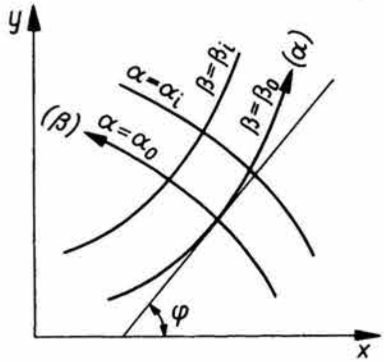

FIG. 2.

represent two families of lines on the plane xOy. They may be written in the form of a vector equation

$$\bar{r} = x\bar{i}+y\bar{j}. \tag{2.2}$$

Differentiation of Eq. (2.2) with respect to the parameters α and β yields the vectors

$$\bar{e}_\alpha = \frac{\partial x}{\partial \alpha}\bar{i}+\frac{\partial y}{\partial \alpha}\bar{j} \tag{2.3}$$

tangent to the line $\beta = \text{const}$, and

$$\bar{e}_\beta = \frac{\partial x}{\partial \beta}\bar{i} + \frac{\partial y}{\partial \beta}\bar{j} \tag{2.4}$$

tangent to the line $\alpha = \text{const}$.

The families of lines (2.1) or (2.2) are orthogonal provided that

$$\frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \beta} = 0. \tag{2.5}$$

The lengths of the tangent vectors (2.3) and (2.4), called the Lamé parameters

$$h_\alpha^2 = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2, \quad h_\beta^2 = \left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 \tag{2.6}$$

enable the determination of the differentials of the curve are in the form:

$$ds_\alpha = h_\alpha d\alpha, \quad ds_\beta = h_\beta d\beta. \tag{2.7}$$

The angle φ of inclination of the tangent to the curve $\beta = \text{const}$ may be determined on the basis of Eqs. (2.3) or (2.4),

$$\begin{aligned} \cos\varphi &= \frac{1}{h_\alpha}\frac{\partial x}{\partial \alpha}, \quad & \cos\varphi &= \frac{1}{h_\beta}\frac{\partial y}{\partial \beta}, \\ \sin\varphi &= \frac{1}{h_\alpha}\frac{\partial y}{\partial \alpha}, \quad & \sin\varphi &= -\frac{1}{h_\beta}\frac{\partial x}{\partial \beta}. \end{aligned} \tag{2.8}$$

From these relations may be calculated

$$\frac{\partial \varphi}{\partial \alpha} = -\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}, \quad \frac{\partial \varphi}{\partial \beta} = \frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}, \tag{2.9}$$

and the curvatures are

$$\frac{1}{\varrho_\alpha} = -\frac{1}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial \beta}, \quad \frac{1}{\varrho_\beta} = -\frac{1}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial \alpha}. \tag{2.10}$$

Assuming the curves (2.1) as coordinates, we can write the equations of equilibrium of the plate

$$\begin{aligned} h_\alpha h_\beta T_\alpha &= \frac{\partial(h_\beta M_\alpha)}{\partial \alpha} + \frac{\partial(h_\alpha M_{\alpha\beta})}{\partial \beta} - M_\beta\frac{\partial h_\beta}{\partial \alpha} + M_{\alpha\beta}\frac{\partial h_\alpha}{\partial \beta}, \\ h_\alpha h_\beta T_\beta &= \frac{\partial(h_\beta M_{\alpha\beta})}{\partial \alpha} + \frac{\partial(h_\alpha M_\beta)}{\partial \beta} - M_\alpha\frac{\partial h_\alpha}{\partial \beta} + M_{\alpha\beta}\frac{\partial h_\beta}{\partial \alpha}, \\ &\frac{\partial(h_\beta T_\alpha)}{\partial \alpha} + \frac{\partial(h_\alpha T_\beta)}{\partial \beta} + h_\alpha h_\beta p(\alpha, \beta) = 0. \end{aligned} \tag{2.11}$$

The assumption that one of the families (2.1) is the family of straight lines enables the Eqs. (2.1) to be written in the form

$$x = x_0(\varphi) + \varrho\cos\varphi, \quad y = y_0(\varphi) + \varrho\sin\varphi, \tag{2.12}$$

with the orthogonality condition

$$\operatorname{tg}\varphi = -\frac{dx_0}{dy_0}, \tag{2.13}$$

provided that for $\varrho = 0$ one of the curves of the family is obtained.

We can, however, assume that the set of coordinates is formed by involutes of a certain curve and their curvature radii. In that case, for $\varrho = 0$, the evolute equation of all curves of the family (Fig. 3) is obtained, and instead of Eq. (2.13) — another equation holds:

$$\operatorname{tg}\varphi = \frac{dy_0}{dx_0}. \tag{2.14}$$

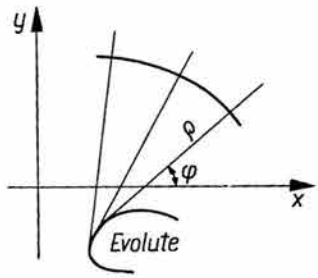

FIG. 3.

The equations of equilibrium of the plate written in the reference frame (2.12) are immediately obtained from Eq. (2.11) by assuming

$$\alpha = r, \quad \beta = \varphi$$

and calculating

$$h_\alpha = h_\varrho = 1, \quad h_\beta = h_\varphi = \varrho.$$

The equations are finally written in the form

$$\begin{aligned}
&\frac{\partial(\varrho M_r)}{\partial\varrho} + \frac{\partial M_{r\varphi}}{\partial\varphi} - M_\varphi = \varrho T_r,\\
&\frac{\partial(\varrho M_{r\varphi})}{\partial\varrho} + \frac{\partial M_\varphi}{\partial\varphi} + M_{r\varphi} = \varrho T_\varphi,\\
&\frac{\partial(\varrho T_r)}{\partial\varrho} + \frac{\partial T_\varphi}{\partial\varphi} + \varrho p(\varrho, \varphi) = 0.
\end{aligned} \tag{2.15}$$

A particular case of the coordinates (2.12) are polar coordinates whose evolute is a point.

2.2. Integration of equations of a plastic plate

In the case of plates made of the K. W. Johansen material, the yield condition is given by Eq. (1.1), its geometric image being the square $ABCD$ shown in Fig. 1. In connection with the convention concerning the model of the body and the physical relations, it may be assumed, that the curvature vectors of the deformed plate surface are perpendicular to the square $ABCD$. Except for the states corresponding to the vertices of the square, a ruled surface is always obtained. In an isotropic plate the directions of principal curvatures of the deformed surface coincide with the directions of principal moments or, more precisely, with the directions of trajectories of the principal moments. It may be concluded that the aquilibrium equation should be written in the form (2.15).

In the case of stresses corresponding to the vertices of the square $ABCD$ (Fig. 1), two basic cases should be distinguished:

(a) The state corresponding to the vertex occurs on a certain line of the plate, and

(b) The state corresponding to the vertex occurs on a certain area of the plate.

In the first case, a yield line is formed along the curve. In the second case, we may substitute into Eqs. (2.11)

$$M_\alpha = M_0, \quad M_\beta = M_0 \tag{2.16}$$

or

$$M_\alpha = -M_0, \quad M_\beta = M_0, \tag{2.17}$$

which corresponds to the vertices A and B. Corresponding considerations may also be applied to the remaining vertices C and D.

Let us simultaneously assume that the lines (2.1) are the principal moment trajectories, which implies the equality

$$M_{\alpha\beta} = 0. \tag{2.18}$$

After substitution of Eqs. (2.16), (2.18) in Eq. (2.11), it is easily found that

$$T_\alpha = 0, \quad T_\beta = 0, \quad p = 0,$$

which means that this state of pure bending in a certain region may occur only if $p = 0$ — i.e., the only load consists of moments continuously distributed along the contour of the region. If the plates were loaded continuously on the entire region, such a state could not exist.

In the case of the state (2.17), we obtain for the bisector sections of the principal directions:

$$M_1 = M_2 = 0, \quad M_{12} = M_0.$$

It is easily observed that here also a continuous load perpendicular to the middle surface of the plate cannot occur. In view of the problem under consideration, of a plate loaded on its entire surface by vertical forces directed downwards, that case will not be considered here.

It is evident from the above considerations that we may confine our attention to the Eqs. (2.15). If it is additionaly assumed that the coordinates (2.12) coincide with the trajectories of principal moments, the torque connected with that system vanishes.

Taking this observation into account and complementing the equations of equilibrium by the yield condition according to Eq. (1.1), we obtain

$$\frac{\partial(\varrho M_r)}{\partial\varrho} - M_\varphi = \varrho T_r, \tag{2.19}$$

(2.19) [cont.]
$$\frac{\partial(\varrho T_r)}{\partial\varrho}+\varrho p(\varrho,\varphi)=0,$$
$$T_\varphi=0,\quad M_\varphi=M_0.$$

Owing to the assumption that Eqs. (2.19) are referred to the system of trajectories of principal moments, the system of equations is determinate, the number of unknowns being equal to the number of equations. The additional equation determining the position of the trajectory will be derived later.

Let us consider a plate which is symmetric with respect to the axis Ox (Fig. 4) and has a sagging yield line on this axis. The straight line trajectory B_0B_1 passes through the

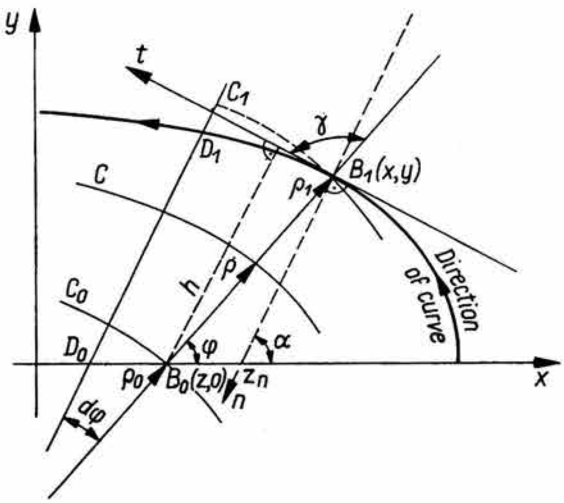

FIG. 4.

point $B_1(x, y)$ at the contour of the plate. In the figure are also drawn the curvilinear trajectories: C_1 — passing through the point B_1, and C_0 — passing through B_0, as also an arbitrary trajectory C. The curvature radii of the trajectories at the points of intersections with the straight line B_0B_1 are respectively denoted by ϱ_1, ϱ_0, ϱ.

With the assumption p = const, the Eqs. (2.19) can be integrated. Taking into account the condition at the point B_0 — i.e. for $\varrho=\varrho_0$ — we obtain $T_{r0}=0$, $M_{r0}=M_0$ and

(2.20)
$$T_r=-\frac{1}{2}p\frac{\varrho^2-\varrho_0^2}{\varrho},$$
$$M_r=M_0-\frac{1}{6}p(\varrho-\varrho_0)^2\left(1+2\frac{\varrho_0}{\varrho}\right),\quad M_\varphi=M_0.$$

The moments and distributions of shearing forces along the trajectory B_0B_1 are known; they are given by the Eqs. (2.20). Still the moment M_0 is not determined as a function of loading, of the form and dimensions of the plate, and of the system of trajectories. The limit moment M_0 or the limit load p may be assumed as the unknown value.

2.3. Limit moment. Limit load

The limit load or moment is determined from the boundary conditions at the contour of the plate. Assume the plate to be simply supported at the entire boundary. Using the transform formulae for the moments connected with the reference frame (ϱ, φ) and (t, n) (t — tangent, n — normal to the boundary), we can easily determine:

$$M_\varphi = M_0, \quad M_r = -M_0 \operatorname{ctg}\gamma. \tag{2.21}$$

and

$$\begin{aligned} M_n &= 0, \\ M_t &= -M_0(1-\operatorname{ctg}^2\gamma), \\ M_{nt} &= -M_0 \operatorname{ctg}\gamma. \end{aligned} \tag{2.22}$$

On substituting into the second of Eqs. (2.20) the value

$$\varrho = \varrho_1, \quad M_r = -M_0 \operatorname{ctg}\gamma,$$

we obtain

$$M_0 = \frac{1}{6} p \sin^2\gamma (\varrho_1 - \varrho_0)^2 \left(1 + 2\frac{\varrho_0}{\varrho_1}\right). \tag{2.23}$$

Equation (2.23) makes it possible to calculate the limit load p or the limit moment M_0 and may serve, at the same time, for establishing the system of trajectories.

The same equation may be represented in an alternative form. In Fig. 4 are shown two rectilinear trajectories B_0B_1 and D_0D_1 lying very close to each other. These trajectories, together with the edge of the plate and the sagging yield line located on the horizontal axis, determine the area $B_0B_1D_1D_0$.

Let us calculate the static moment dS_p of the load acting in the region of that quadrangle, taken with respect to the tangent t

$$dS_p = \int_{\varrho_0}^{\varrho_1} (\varrho_1 - \varrho) \sin\gamma p \varrho d\varrho d\varphi.$$

The differential of the arc of the curve bounding the plate is expressed by

$$ds = \frac{\varrho_1 d\varphi}{\sin\gamma}.$$

Once the integral is calculated, the ratio

$$\frac{dS_p}{ds} = \frac{1}{6} p \sin^2\gamma (\varrho_1 - \varrho_0)^2 \left(1 + 2\frac{\varrho_0}{\varrho_1}\right) \tag{2.24}$$

can be determined. On comparing Eqs. (2.23) with (2.24), we observe that

$$M_0 = \frac{dS_p}{ds}. \tag{2.25}$$

Hence the ratio of the statical moment of the load occurring in the region determined by two trajectories lying infinitely close to each other, by the contour of the plate and by the sagging yield line, calculated with respect to the contour arc differential tangent to the boundary of the plate, is a constant value in the entire plastic region of the plate.

The form of Eq. (2.25) is more convenient than the Eq. (2.23) for determining the system of trajectories of principal moments in the plate.

2.4. Mechanism of collapse of the plate

In order to determine the mechanism of collapse of the plate, let us apply the kinematic approach to the load capacity of the plate. The mechanism should be compatible with the geometric boundary conditions. Let us assume the mechanism to be dependent on the parameters being the angles of rotation of the individual panels θ_i. Denoting the lengths

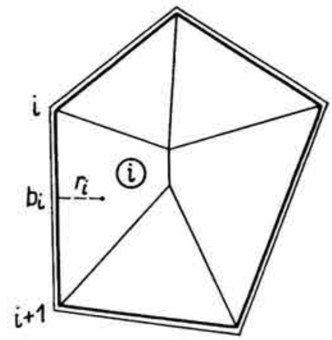

FIG. 5.

of the sides of the polygon by b_i (Fig. 5) we are able to evaluate the work done by internal forces:

$$\delta V = M_0 \sum_{i=1}^{n} b_i \theta_i$$

and the work done by external forces

$$\delta L = \sum_{i=1}^{n} S_{pi} \theta_i,$$

where

$$S_{pi} = \int_{A_i} qr dA$$

denotes the statical moment of the load on the panel calculated with respect to the axis of rotation of the panel.

The condition of extremum of M_0 applied to the equation

$$\delta V = \delta L$$

under the assumption of small angles θ_i yields the equality

$$M_0 = \frac{S_{pi}}{b_i} \tag{2.26}$$

which was derived by RZHANITSYN [14]. It should be noted that the condition (2.26) results from the kinematic analysis and simply expresses the equation of equilibrium of the panel; it may be used in that form only in the case in which the nodal forces are absent [16]. If they are present, their moments should be considered in calculating the statical moment of the load S_{pi}. The corresponding solutions for plates containing holes are given in [19, 20].

In the case of a plate with a curvilinear contour, Eq. (2.26) is transformed into Eq. (2.25). The equations describing the collapse mechanism and the set of trajectories of principal moments are identical. The same values of the limit moment and limit load are obtained from the static and kinematic solutions.

In determining the set of trajectories and the collapse pattern, certain limitations exist following from the boundary value of M_r (2.21). In the case of perfectly plastic plates, the condition

$$|M_r| \leqslant M_0 \tag{2.27}$$

holds.

From the Eq. (2.21) it follows that the angle at which the trajectory intersects the plate boundary must lie in the interval:

$$45° \leqslant \gamma \leqslant 135° \tag{2.28}$$

— i.e., the straight line trajectory (yield line) and the normal to the boundary cannot make an angle greater than 45°.

In the case of a curvilinear contour, the deflection surface is described by the straight line whose projection on the middle surface is the B_0B_1 trajectory, Fig. 6.

If we assume that the deflection of points B_0 lying on the horizontal axis is equal to w_0, then it may easily be observed that the line describing the surface lies in the plane

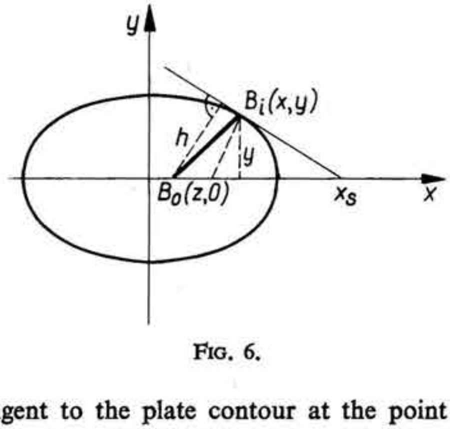

FIG. 6.

passing through the tangent to the plate contour at the point $B_1(x, y)$ and the point $B(z, 0, w_0)$ whose projection is $B_0(z, 0, 0)$.

The line of intersection of that plane with the plane xOw makes with the Ox-axis the angle whose tangent equals:

$$\frac{dw_0}{dz} = -\frac{w_0}{x_s - z} \tag{2.29}$$

and in addition we have

$$x_s = x - y\frac{dy}{dx}. \tag{2.30}$$

Equation (2.29) determines the curve of intersection of the deflected plate surface and the symmetry plane xOw.

If the coordinates of a point lying on the straight line which describes the surface are denoted by $P(X, Y, Z)$, then its equation may be written as

$$\frac{X-x}{x-z} = \frac{Y-y}{y} = -\frac{W}{w_0}. \tag{2.31}$$

In this equation appear the coordinates of the point $B_1(x, y)$ lying on the contour of the plate, the integral w_0 from Eq. (2.29), and z — the abscissa of the point of intersection of the straight line trajectory with the axis. The abscissa has to be found from Eq. (2.26), which will be done further in this paper.

3. Principal moment trajectories

3.1. Geometric relations

It is seen from Fig. 4 that certain relations exist between the trajectories of principal moments and the plate contour; they have to be used. The segment B_0B_1 which equals the difference of lengths of the curvature radii of trajectories C_1 and C_0 can be calculated from two different right angle triangles,

$$\varrho_1 - \varrho_0 = \frac{h}{\sin\gamma} = \frac{y}{\sin\varphi}; \tag{3.1}$$

y being the ordinate of the point $B_1(x, y)$ lying on the plate contour, and h — the distance of point B_0 from the tangent to the plate contour at B_1. Both magnitudes are indicated in Fig. 4.

The distance h, understood as the length of a straight line sector, can easily be calculated. In the case of a curve directed according to Fig. 4,

$$h = (x-z)\frac{dy}{ds} - y\frac{dx}{ds}. \tag{3.2}$$

Under reversed direction of the curve, the value of h calculated according to the formula (3.2) is negative, hence the signs of both terms standing on the right-hand side have to be changed.

It is also easily observed that

$$\varrho_0 = -\sin\varphi\frac{dz}{d\varphi}, \quad \varrho_1 = \sin\gamma\frac{ds}{d\varphi}, \tag{3.3}$$

and the ratio of curvature radii, Eq. (3.1) being taken into account, is equal to

$$\frac{\varrho_0}{\varrho_1} = -\frac{y}{h}\frac{dz}{ds}. \tag{3.4}$$

The minus sign in the first of Eqs. (3.3) and in (3.4) results from the assumed direction of the curve.

3.2. Equations determining the position of trajectories

The basis for the determination of trajectories is Eq. (2.25), which remains true at every load [4]. Let us confine our considerations to a uniform load. Then the equation may be written in the form of Eq. (2.23) which is, however, never suitable for seeking the solutions.

The static moment of the uniform load acting on the area $B_0B_1D_1D_0$ (Fig. 4) is calculated as a product of the load and the static moment of the area

$$dS_p = pdS_t.$$

The static moment of the area is calculated directly from the geometry

$$dS_t = \frac{1}{6}(h^2 ds - 2hy\,dz). \tag{3.5}$$

After substituting Eq. (3.5) into (2.25), we obtain:

$$M_0 = \frac{1}{6}p\left(h^2 - 2hy\frac{dz}{ds}\right) \tag{3.6}$$

— therefore, another form of the Eq. (2.23). It is readily observed that Eq. (3.6) may be obtained directly from Eq. (2.23) by substituting

$$(\varrho_1 - \varrho_0)\sin\gamma = h$$

on the basis of Eqs. (3.1) and (3.4).

For a circular plate loaded uniformly, the limit moment equals

$$M_0 = \frac{1}{6}pr_0^2, \tag{3.7}$$

if the radius of the circle is denoted by r_0.

Equating both sides of Eqs. (3.6), (3.7), we obtain

$$h^2 - 2hy\frac{dz}{ds} = r_0^2. \tag{3.8}$$

This equation will be called fundamental. Solution of that equation in the case of a definite form of the contour enables a simple determination of all components of the complete solution.

3.3. Analysis of the fundamental equation

In spite of the apparent simplicity of Eq. (3.8), its integral cannot be found and written in a closed form. The magnitude h appearing in that equation is a linear function of z, and hence the Eq. (3.8) can very easily be reduced to the classical form of the Abel equation of second kind.

Equation (3.8) can be reduced to a nonlinear partial differential equation of the first order. Substituting Eq. (3.2) and the total differential of two independent variables

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy,$$

we obtain

$$\left(y^2-r_0^2+2y^2\frac{\partial z}{\partial x}\right)dx^2+\left[2y^2\frac{\partial z}{\partial y}-2y(x-z)-2y(x-z)\frac{\partial z}{\partial y}\right]dx\,dy$$
$$+\left[(x-z)^2-r_0^2-2y(x-z)\frac{\partial z}{\partial y}\right]dy^2=0.$$

Assuming now the left-hand side to be a square of the total differential of two variables, we are led to the following equation

$$(3.9)\qquad \left[y(x-z)\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}\right]^2+2r_0^2\left[y^2\frac{\partial z}{\partial x}-y(x-z)\frac{\partial z}{\partial y}\right]+r_0^2[y^2-r_0^2+(x-z)^2]=0.$$

The solution is now reduced to the determination of such an integral $z = f(x, y)$ which passes through the contour of the plate. Attempted solutions of each of the two Eqs. (3.8), (3.9) for an elliptical plate failed.

The integral of Eq. (3.8) may easily be represented by the Taylor or McLaurin series.

No serious difficulties are connected with the numerical solution. Both methods will be demonstrated in the case of an elliptical plate.

4. Uniformly loaded elliptical plate

4.1. Fundamental equation

Let us consider a plate bounded by the ellipse

$$(4.1)\qquad x = a\cos t, \quad y = b\sin t,$$

simply supported at the boundary and uniformly loaded.

The position of an arbitrary straight line trajectory is determined by two points: $B_1(x, y)$ lying on the boundary and $B_0(z, 0)$ — on the horizontal axis (Fig. 4). Using Eqs. (3.2), we can calculate the distance of the point B_0 from the tangent to the ellipse at B_1. By means of the dimensionless quantities

$$(4.2)\qquad \beta=\frac{b}{a}, \quad k^2=1-\beta^2, \quad \zeta=\frac{z}{a},$$

we obtain

$$(4.3)\qquad \frac{h}{a}=\frac{\beta(1-\zeta\cos t)}{\sqrt{1-k^2\cos^2 t}}.$$

From the symmetry of the system with respect to both coordinate axes it follows that with $t = 0$ the abscissa z reaches its maximum value $z_0 = a\zeta_0$, while with $t = \pi/2$, $z = 0$. For $t = 0$, we obtain:

$$h_0 = a(1-\zeta_0);$$

this magnitude measures the distance from the sagging yield line (on the Ox-axis) to the right-hand apex of the ellipse and, simultaneously, the height of the triangular element into which the quadrangle $B_0B_1D_1D_0$ is transformed in the apex. The carrying capacity of an elliptical plate is thus compared to the load capacity of a circular plate with radius r_0; it is seen that $h_0 = r_0$.

Substitution of Eqs. (4.3) and r_0 into Eqs. (3.8) yields, after simple transformations,

$$(1-\zeta\cos t)^2 - 2(1-\zeta\cos t)\sin t\frac{d\zeta}{dt} = \frac{(1-\zeta_0)^2}{\beta^2}(1-k^2\cos^2 t), \tag{4.4}$$

while

$$r_0 = a(1-\zeta_0). \tag{4.5}$$

In the differential Eq. (4.4) appears a constant ζ_0 of unknown value, though its geometrical sense is known. In order to determine the constant of integration of Eq. (4.4) and the constant ζ_0, we may use the conditions

$$t = 0, \quad \zeta = \zeta_0, \quad \frac{d\zeta}{dt} = 0 \tag{4.6}$$

and

$$t = \frac{\pi}{2}, \quad \zeta = 0, \quad \frac{d\zeta}{dt} = \frac{\beta^2-(1-\zeta_0)^2}{2\beta^2}. \tag{4.7}$$

The most important question is the solution of Eq. (4.4); unfortunately, the integral cannot be determined in a closed form. Three possible concepts are listed below.

(A) *Approximate solution.* Assume the function

$$\zeta = \zeta_0\cos t, \tag{4.8}$$

which is substituted in Eq. (4.4). This function satisfies the equation only at $t = 0$. Assuming as an additional condition that Eq. (4.4) be satisfied at $t = \pi/2$, we obtain

$$\zeta_0 = 1+\beta^2-\beta\sqrt{3+\beta^2}. \tag{4.9}$$

It appears that at the value of Eq. (4.9) so selected, the function (4.8) constitutes a rather good approximation.

(B) *Solution in the form of a series.* Introducing a new independent variable $u =$ const into the Eq. (4.4), the integral of that equation is readily represented by a Taylor series. Returning to the variable t, we obtain:

$$\begin{aligned}\zeta = \frac{\eta^2-1}{2}\frac{\cos t}{1!} &+ \left[\frac{(\eta^2-1)^2}{2} + 2(\eta^2-1^2) - k^2\eta^2\right]\frac{\cos^3 t}{3!} \\ &+ \left[4(\eta^2-1)^3 + \frac{27}{2}(\eta^2-1)^2 + 32(\eta^2-1) - 8k^2\eta^2(\eta^2-1)\right]\frac{\cos^5 t}{5!} + \dots,\end{aligned} \tag{4.10}$$

where the notation is used $\eta^2 = (1-\zeta_0)^2/\beta^2$.

(C) *Numerical solution.* For an elliptical plate with a definite ratio $\beta = b/a$, Eq. (4.4) can be solved with the conditions (4.6) and (4.7). The solution is reduced to finding such values of the parameters ζ_0 as would ensure satisfaction of the conditions already mentioned. Only one of the integrals of Eq. (4.4) satisfying Eq. (4.6) passes through the point $t = \pi/2$, $\zeta = 0$, Fig. 7.

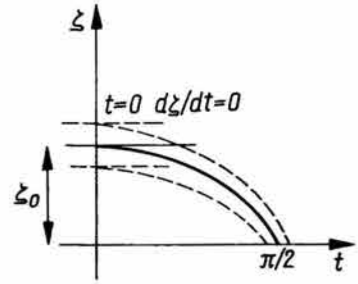

FIG. 7.

The approximate value of ζ_0 may be evaluated according to Eq. (4.9). The results of approximate solutions of Eq. (4.9) and the numerical solutions are shown in Fig. 8.

From the direct comparison of the results it is seen that the approximate solutions according to Eq. (4.9) yield comparatively good results. The greatest divergence of the

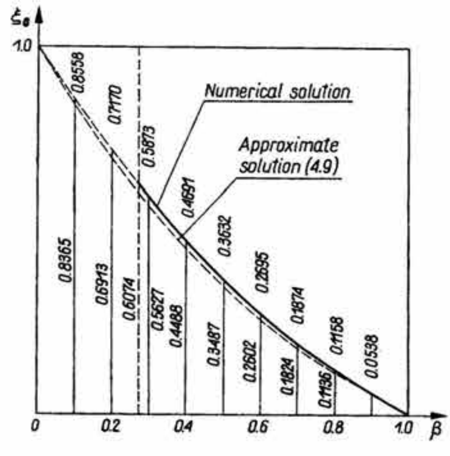

FIG. 8.

two solutions amounts to 4% of the numerical solution, which may be treated as accurate for practical purposes. The great error is found for plates in which the ratio of axes is $\beta = b/a = 0.3\text{–}0.5$.

Once the value of ζ_0 is known, r_0 is determined from Eq. (4.5), and next the limit moment or limit load — from the equality

$$M_0 = \frac{1}{6} p r_0^2 .$$

Solution of the Eq. (4.4) makes it possible to determine the values of ζ corresponding to various values of the angle t thus enabling the determination of the positions of all trajectories of the principal moments, in the entire region of the plate.

4.2. Distribution of trajectories and moments

Solution of Eq. (4.4) in the form of a determined value of ζ_0 and $\zeta = z/a$ at arbitrary points of the circumference of the plate given by the angle t makes it possible to find the

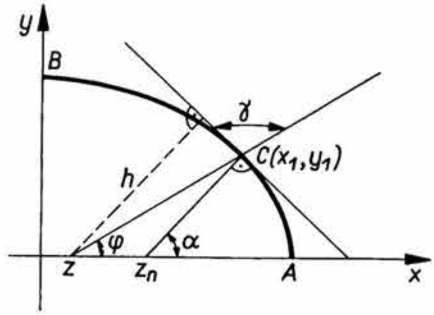

FIG. 9.

set of straight line trajectories in the entire region of the plate. This solution consequently yields the following geometrical magnitudes:

Angle of inclination of the trajectory to the horizontal axis

$$\operatorname{tg}\varphi = \frac{\beta \sin t}{\cos t - \zeta}.$$

The angle γ at which the trajectory intersects the boundary of the plate

$$\operatorname{tg}\left(\gamma - \frac{\pi}{2}\right) = \frac{\operatorname{tg}\alpha - \operatorname{tg}\varphi}{1 + \operatorname{tg}\alpha \operatorname{tg}\varphi}.$$

The distance h from the point of intersection of the trajectory with the horizontal axis to the tangent to the elipse:

$$\frac{h}{a} = \beta \frac{1 - \zeta \cos t}{\sqrt{1 - k^2 \cos^2 t}}.$$

The curvature radii of the trajectory at the points of intersection with the horizontal axis and with the ellipse

$$\frac{\varrho_0}{a} = \frac{\left(\frac{h}{a}\right)}{\cos\left(\frac{\pi}{2} - \gamma\right)} \frac{(1-\zeta_0)^2 - \left(\frac{h}{a}\right)^2}{3\left(\frac{h}{a}\right) - (1-\zeta_0)^2},$$

$$\frac{\varrho_1}{a} = \frac{\left(\frac{h}{a}\right)}{\cos\left(\gamma - \frac{\pi}{2}\right)} + \frac{\varrho_0}{a}.$$

Coordinates of the evolute of curvilinear trajectories

$$\frac{x_0}{a} = \cos t - \frac{\varrho_1}{a}\cos\varphi,$$

$$\frac{y_0}{a} = \beta\sin t - \frac{\varrho_1}{a}\sin\varphi.$$

It is also easy to determine the variation of the radial moment and the transversal force along an arbitrary trajectory

$$M_r = M_0 - \frac{1}{6}p(\varrho-\varrho_0)^2\left(1+2\frac{\varrho_0}{\varrho}\right),$$

$$T_r = -\frac{p(\varrho^2-\varrho_0^2)}{2\varrho},$$

while in the perpendicular direction

$$M_\varphi = M_0, \quad T_\varphi = 0.$$

The solution is accurate provided the condition is fulfilled

$$45° \leqslant \gamma \leqslant 135°,$$

following from the limitation:

$$|M_r| \leqslant M_0$$

valid for plates made of perfectly elastic-plastic or rigid-plastic materials.

4.3. Plate with a hogging yield line

If the angle γ of intersection of the trajectory with the plate boundary, calculated according to the solution presented, does not fall within the interval $(\pi/4, 3\pi/4)$, the solution proves to be incorrect. On a certain sector of rectilinear trajectories which do not fulfill the condition for γ, close to the boundary, the radial moment would be smaller than $-M_0$, what is impossible. Analysis of the numerical solutions indicates that for the ellipse with axial ratio $\beta > 0.284$, the inequality $\mathrm{tg}(\gamma-\pi/2) < 1$ holds, and a hogging yield line cannot be formed.

The hogging yield line is formed for $\beta \leqslant 0.284$. On the basis of considerations presented in [4], it is established that the hogging yield line is then located on a circular arc of

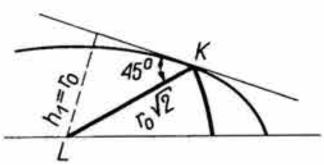

FIG. 10.

radius $r_0\sqrt{2}$. It is evident (Fig. 10) that the distance from center L of the circle to the tangent to the ellipse at point K (through which the yield line passes) is equal to $h_1 = r_0$. Inserting this value of h into Eq. (3.8), we calculate:

$$\frac{dz}{ds} = 0.$$

Let us denote by index 1 those values which correspond to the point K from which the hogging yield line starts. It is easily found on the basis of Eq. (4.3) that

$$\frac{r_0}{a} = \frac{h_1}{a} = \frac{\beta(1-\zeta_1\cos t_1)}{\sqrt{1-k^2\cos^2 t_1}};$$

the abscissa of L being

$$\zeta_1 = \frac{z_1}{a} = \frac{\beta - k^2 \sin t_1 \cos t_1}{\beta\cos t_1 - \sin t_1}.$$

Using this value, we can calculate:

$$\frac{r_0}{a} = \beta\frac{\sin t_1 \sqrt{1-k^2\cos^2 t_1}}{\sin t_1 - \beta\cos t_1}. \tag{4.11}$$

The Eq. (3.8) for a plate in which a hogging yield line is created may now be written as

$$(1-\zeta\cos t)^2 - 2(1-\zeta\cos t)\sin t\frac{d\zeta}{dt} = \frac{\sin^2 t_1(1-k^2\cos^2 t_1)}{(\sin t_1 - \beta\cos t_1)^2}\cdot(1-k^2\cos^2 t), \tag{4.12}$$

instead of the Eq. (4.4) valid for a plate without the hogging yield line.

Solution of Eq. (4.12) is then reduced to the determination of such a parameter t_1 and a function $\zeta = \zeta(t_1, t)$ which satisfy the conditions:

$$\begin{aligned} &\text{for } t = t_1, \quad d\zeta/dt = 0,\\ &\text{for } t = \pi/2, \quad \zeta = 0. \end{aligned} \tag{4.13}$$

The numerical solution for the plate with axial ratio $\beta = 0.2$ yields the values

$$t_1 = 1.343881 \text{ rad},$$
$$\zeta_1 = 0.011231,$$
$$\frac{r_0}{a} = 0.204525.$$

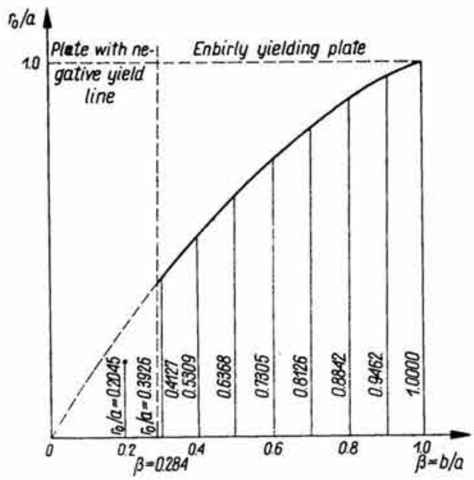

FIG. 11.

The radii r_0 of reduced circular plates having identical limit moments are presented in Fig. 11.

In the case of formation of a hogging yield line, the limit moment and limit load are less than those calculated under the assumption of full plasticity of the entire region of the plate (except $\beta = 0.284$).

The system of trajectories and moment distribution in a plate with a hogging yield line are different in two types of regions. In the region bounded by a circular arc the solution is the same as in the case of a circular plate clamped at the boundary and subjected to the conditions of axial symmetry. In the region bounded by an elliptical arc (plate boundary), all quantities may be determined analogously to the case of a plate yielding on its entire region.

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