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On the propagation of waves in elastic-plastic semi-space, due to a moving and variable loading along the boundary

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WE SOLVE the problem of propagation of nonlinear waves in a semi-space when on its boundary there acts a stepwise and moving with a supersonic velocity loading. It is assumed that the soil obeys the deformation theory of plasticity. The existence is proved of two nonlinear waves analogous to the longitudinal and transverse waves in the case of elastic strain, which for the stepwise loading are propagated with constant velocities depending on the magnitude of the loading and the properties of the medium. We present the solution of the problem when the transverse wave is an unloading wave. It turns out that the nonlinearity of the material of the semi-space, in contrast to the elastic problem, leads to a jump variation of the normal stress on the transverse wave. Results of computations are given in the form of Tables.

Rozwiązano zagadnienie propagacji fal nieliniowych w półprzestrzeni obciążonej na powierzchni skokowo zmiennym obciążeniem, poruszającym się z prędkością ponaddźwiękową. Zakłada się, że grunt spełnia warunki odkształcenia teorii plastyczności. Wykazuje się istnienie dwóch nieliniowych fal analogicznych do fal podłużnych i poprzecznych w teorii odkształceń sprężystych, które przy obciążeniu skokowym przemieszczają się z prędkościami zależnymi od wielkości obciążenia i od własności ośrodka. Przedstawia się rozwiązanie zagadnienia dla przypadku, gdy fala poprzeczna jest falą odciążenia. Okazuje się, że nieliniowość materiału półprzestrzeni — w odróżnieniu od przypadku sprężystego — prowadzi do skokowej zmienności naprężeń normalnych na fali poprzecznej. Wyniki obliczeń podano w postaci stabelaryzowanej.

Решена задача о распространении нелинейных волн в полупространстве, когда на его границу действует ступенчатая и бегущая со сверхзвуковой скоростью нагрузка. Предполагается, что грунт подчиняется деформационной теории пластичности. При этом показано существование двух нелинейных волн, аналогичных продольной и поперечной волны при упругих деформациях, которые для ступенчатой нагрузки распространяются с постоянными скоростями, зависящими от величины нагрузки и от свойства среды. Приведено решение задачи, когда поперечная волна является волной разгрузки. Оказывается, что учет нелинейности свойств материала полупространства, в отличие от упругой постановки задачи, приводит к скачкообразному изменению нормального напряжения на поперечной волне. Результаты расчетов приводятся в виде таблиц.

The problem of propagation of pressure into a semi-space occupied by an elastic-plastic, viscoelastic and liquid medium or by "the plastic gas", was investigated by many authors [1-5].

The author of the paper [1] solved the problem of motion of the soil occupying the elastic-plastic semi-space, when on its surface there occurs a strong explosion. The solution was based on an approximation and a mathematical model of the soil described in [6].

The solution of an analogous problem for a stepwise loading on the basis of the Mises yield condition is presented in [2], while the papers [3, 4] took into account the visco-elastic and visco-elastic-plastic properties of the soil. In [5], the semi-space consists of a perfect medium and the loading occurs on the front only, the compression is linear, while behind the wave front unloading takes place and the velocity of the unloading wave is greater than that of the loading moving over the free surface, or is equal to infinity.

We present here an analytic solution of the problem of propagation of nonlinear waves in a semi-space, when on its boundary there acts a stepwise loading moving with a supersonic velocity. It is assumed that the soil obeys the deformation theory [8] and the following experimentally determined laws of deformation are investigated:

$$\text{(a)} \quad \sigma = \sigma(\varepsilon), \quad \sigma_i = \sigma_i(\varepsilon_i),$$

$$\text{(b)} \ \sigma = \sigma(\varepsilon), \quad \sigma_i = \sigma_i(\varepsilon, \varepsilon_i), \quad \text{(c)} \ \sigma = \sigma(\varepsilon, \varepsilon_i), \quad \sigma_i = \sigma_i(\varepsilon, \varepsilon_i),$$

where ε, ε_i, σ, σ_i are the first and the second invariants of strain and stress, respectively.

It is proved that there exist two nonlinear waves analogous to the longitudinal and transverse waves for elastic strain, which in the case of the stepwise loading are propagated with constant velocities depending on the magnitude of the loading and the properties of the medium. Numerical results are presented in the form of Tables.

Assume that there acts on the elastic-plastic semi-space a stepwise loading moving along its surface with a velocity D (Fig. 1). Since the loading acting on the boundary of

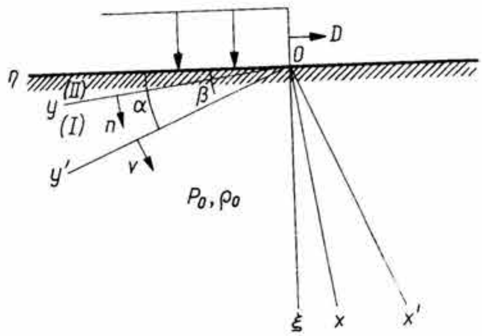

FIG. 1.

the semi-plane is stepwise and as mentioned before, the solution of the problem is based on the deformation model [8] of the soil, there occur in the semi-plane the regions of disturbance (I) and (II) where the strain, stress and the velocity of the particles are different constant quantities. The existence of regions of constant flow is obvious; an analogous phenomenon was also observed in [2].

On the basis of [8] we have the stress-strain relations

$$\sigma_{\xi\xi} = \lambda\varepsilon + 2G \cdot \varepsilon_{\xi\xi}, \quad \sigma_{\eta\eta} = \lambda\varepsilon + 2G \cdot \varepsilon_{\eta\eta}, \quad \tau_{\xi\eta} = G \cdot \varepsilon_{\xi\eta}, \tag{1}$$

where the Lamé parameters λ and G have the form

$$\lambda = \frac{\sigma(\varepsilon)}{\varepsilon} - \frac{2}{9}\frac{\sigma_i(\varepsilon_i)}{\varepsilon_i}, \quad G = \frac{1}{3}\frac{\sigma_i(\varepsilon_i)}{\varepsilon_i}, \tag{2}$$

i.e. they depend on the functions $\sigma(\varepsilon)$ and $\sigma_i(\varepsilon_i)$; the equations of motion have the form

$$\varrho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_{\xi\xi}}{\partial \xi} + \frac{\partial \tau_{\xi\eta}}{\partial \eta}, \quad \varrho_0 \frac{\partial^2 v}{\partial t^2} = \frac{\partial \sigma_{\eta\eta}}{\partial \eta} + \frac{\partial \tau_{\xi\eta}}{\partial \xi}, \tag{3}$$

where u and v are the displacements.

For definiteness we consider the case (a). To determine the parameters of the constant flow in the regions (I) and (II) it is sufficient to use the kinematic and dynamic conditions [since the equations of motion (3) are identically satisfied] on the longitudinal and transverse waves

$$\tag{4} \begin{aligned} \frac{\partial u_1/\partial x'}{\cos(\nu, x')} &= \frac{\partial u_1/\partial y'}{\cos(\nu, y')} = -\frac{1}{a}\frac{\partial u_1}{\partial t'}, \\ \frac{\partial v_1/\partial x'}{\cos(\nu, x')} &= \frac{\partial v_1/\partial y'}{\cos(\nu, y')} = -\frac{1}{a}\frac{\partial v_1}{\partial t'}, \end{aligned}$$

$$\varrho_0 a\frac{\partial u_1}{\partial t'} = -\sigma_{1x'x'}, \quad \varrho_0 a\frac{\partial v_1}{\partial t'} = -\tau_{1x'y'},$$

$$\varrho_0 a = \varrho_1\left(a - \frac{\partial u_1}{\partial t'}\right),$$

$$\tag{5} \begin{aligned} \frac{(\partial u_1/\partial x)-(\partial u_2/\partial x)}{\cos(n, x)} &= \frac{(\partial u_1/\partial y)-(\partial u_2/\partial y)}{\cos(n, y)} = -\frac{1}{b}\left(\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}\right), \\ \frac{(\partial v_1/\partial x)-(\partial v_2/\partial x)}{\cos(n, x)} &= \frac{(\partial v_1/\partial y)-(\partial v_2/\partial y)}{\cos(n, y)} = -\frac{1}{b}\left(\frac{\partial v_1}{\partial t} - \frac{\partial v_2}{\partial t}\right), \\ \varrho_1\left(b-\frac{\partial u_1}{\partial t}\right)&\left(\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}\right) = -\sigma_{1xx}+\sigma_{2xx}, \\ \varrho_1\left(b-\frac{\partial u_1}{\partial t}\right)&\left(\frac{\partial v_1}{\partial t} - \frac{\partial v_2}{\partial t}\right) = -\tau_{1xy}+\tau_{2xy}, \\ \varrho_1\left(b-\frac{\partial u_1}{\partial t}\right) &= \varrho_2\left(b-\frac{\partial u_2}{\partial t}\right) \end{aligned}$$

and the boundary conditions for $\xi = 0$, $\eta > 0$:

$$\tag{6} \sigma_{\xi\xi} = P_0, \quad \tau_{\xi\eta} = 0.$$

Here a and b are the velocities of the longitudinal and transverse waves; the indices 1 and 2 refer to the parameters of the soil in front of and behind the wave front, in the appropriate coordinate system. To determine the angles α and β of the fronts of the longitudinal and transverse waves with the axis, we have the relations

$$\tag{7} a = D\sin\alpha, \quad b = D\sin\beta.$$

Taking into account that $\cos(n, y) = \cos(\nu, y') = 0$, $\cos(n, x) = \cos(\nu, x') = 1$, an investigation of the system (4)–(7) proves that the problem is reduced to the solution of two nonlinear algebraic equations with respect to $\varepsilon_{1x'x'}$ and β. An analysis of the system (5) shows that there occur various possibilities of deformation of the medium as the transverse wave passes thorugh it, namely

Case 1. In the dynamic Eqs. (5) we neglect the variations of the density and the normal component of the velocity of the medium as compared with the velocity of the wave. This assumption is admissible for water saturated soil and for sand under rather small pressure. In this case the third and fourth Eqs. (5) take the form

$$\varrho_0 b\left(\frac{\partial u_1}{\partial t}-\frac{\partial u_2}{\partial t}\right) = -\sigma_{1xx}+\sigma_{2xx}, \tag{8}$$

$$\varrho_0 b\left(\frac{\partial v_1}{\partial t}-\frac{\partial v_2}{\partial t}\right) = -\tau_{1xy}+\tau_{2xy}.$$

Thus the normal stress on the transverse wave, in contrast to the elastic problem [7], suffers a discontinuity due to the nonlinearity of the medium; in other words the transverse wave is a compression-shear wave. Equations (4), (5) and (8) yield the relations for the velocity of the wave fronts

$$a = \sqrt{\frac{\sigma_{1x'x'}}{\varrho_0 \varepsilon_{1x'x'}}}, \quad b = \sqrt{\frac{\sigma_{1xx}-\sigma_{2xx}}{\varrho_0(\varepsilon_{1xx}-\varepsilon_{2xx})}}, \quad a > b. \tag{9}$$

Case 2. Here $\varrho_1 = \varrho_2$, i.e. the density of the medium does not change as the transverse wave passes through it. Then the last and the third Eqs. (5) lead to the relations

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t}, \quad \sigma_{1xx} = \sigma_{2xx}. \tag{10}$$

Consequently, as in the elastic problem, in this case the normal stress on the transverse wave is continuous.

Case 3. Now $\varrho_1 \neq \varrho_2$, i.e. the soil is substantially compressed when both the transverse and longitudinal waves pass throught it. We now have two possibilities, namely $\varrho_1 < \varrho_2$ and $\varrho_1 > \varrho_2$. In the first case the transverse wave is a loading wave, while in the second it is an unloading wave. Now, as in Case 1, the normal stress on the transverse wave suffers a discontinuity. Let us consider in detail Case 1. In view of (4) and the kinematic conditions (5), we obtain, taking into account (9),

$$\frac{\partial v_1}{\partial t'} = \frac{\partial v_1}{\partial x'} = \frac{\partial v_1}{\partial y'} = \frac{\partial u_1}{\partial y'} = 0, \quad \frac{\partial u_1}{\partial t'}+a\varepsilon_{1x'x'} = 0, \tag{11}$$

$$\varepsilon_{1yy} = \varepsilon_{2yy}, \quad \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y}, \quad \frac{\partial v_1}{\partial x}-\frac{\partial v_2}{\partial x}+\frac{1}{b}\left(\frac{\partial v_1}{\partial t}-\frac{\partial v_2}{\partial t}\right) = 0. \tag{12}$$

Making use of the transformation formulae [9] for the components of stress to the axes x, y, after some transformations, we obtain from (1) and the system of Eqs. (6), (8):

$$\varepsilon_{2xy} = 2(\varepsilon_{2xx}-\varepsilon_{2yy})/(\mathrm{tg}\beta-\mathrm{ctg}\beta), \tag{13}$$

$$\varepsilon_{2xx} = \frac{\dfrac{P_0-\sigma_{1xx}-\mathrm{tg}\beta\cdot\tau_{1xy}}{\varrho_0 D^2\sin^2\beta}+\varepsilon_{1xx}-\dfrac{2\,\mathrm{tg}\beta\cdot\varepsilon_{2yy}}{\mathrm{tg}\beta-\mathrm{ctg}\beta}+\mathrm{tg}\beta\cdot\varepsilon_{1xy}}{1-2\,\mathrm{tg}\beta/(\mathrm{tg}\beta-\mathrm{ctg}\beta)};$$

$$\varepsilon_{1x'x'} = \Phi_1(\varepsilon_{1x'x'};\beta) = \varepsilon_{1x'x'}+P_0-\mathrm{tg}\beta\cdot\tau_{2xy}-\sigma_{2xx}, \tag{14}$$

$$\varepsilon_{1x'x'} = \Phi_2(\varepsilon_{1x'x'};\beta) = \frac{\varepsilon_{2xx}-\dfrac{P_0-\sigma_{1xx}-\mathrm{tg}\beta\cdot\tau_{2xy}}{\varrho_0 D^2\sin^2\beta}}{\cos^2(\alpha-\beta)}.$$

If we take into account that eventually the parameters λ_i, G_i ($i = 1, 2$), α, ε_{1yy} are functions of $\varepsilon_{1x'x'}$ and β, then in view of (13), the system of Eqs. (14) constitutes the basic system of nonlinear algebraic equations for the determination of $\varepsilon_{1x'x'}$ and β. This system was solved numerically on a computer.

Observe that similar methods were employed to investigate the cases 2 and 3 and the deformation laws (c) and (b).

Let us now present some results of the computations and discuss them. To carry out the calculations of the parameters of the soil of a definite structure (e.g. sand, clay, etc.) we made use of the experimental curves (Fig. 2) of the papers [10, 11, 12] and approximated them by the relations

(a) $\sigma = \alpha_1\varepsilon+\alpha_2\varepsilon^2, \quad \sigma_i = \beta_1\varepsilon_i+\beta_2\varepsilon_i^2;$

(b) $\sigma = a_1\varepsilon^2, \quad \sigma_i = G_0\cdot \mathrm{tg}\,\psi_0\cdot\varepsilon_i/(\mathrm{tg}\,\psi_0+G_0\cdot\varepsilon_i)+\sigma(\varepsilon)(a_0\varepsilon_i-b_0\varepsilon_i^2);$

(c) $\sigma = k_1(\varepsilon+\nu\varepsilon_i)+k_2(\varepsilon+\nu\varepsilon_i)^2, \quad \sigma_i/\sigma = h\cdot\mathrm{tg}\,\psi\cdot\varepsilon_i/(\mathrm{tg}\,\psi+h\cdot\varepsilon_i);$

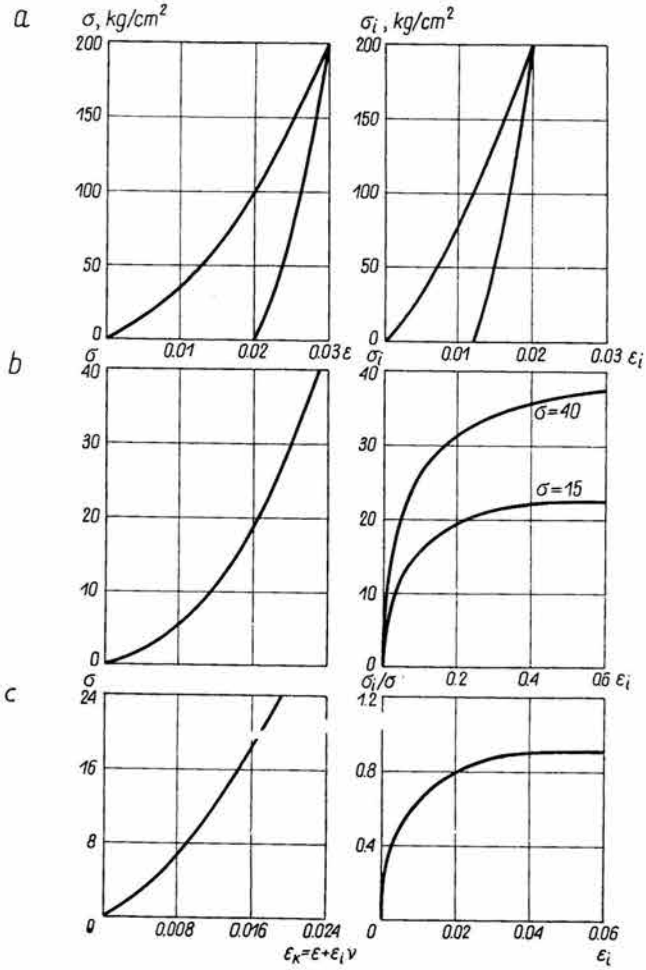

FIG. 2.

respectively. Here, the coefficients α_i, β_i, k_i $(i = 1, 2)$, a_1, G_0 and $\operatorname{tg}\psi_0$ have the dimension kg/cm², ν is the dilatation coefficient equal to 0.45. Observe that the curves a, b and c refer to the sand with 10% of clay, dense sandy soil and clay, respectively. The results of

Table 1

Soil	Sand with 10% of clay			
P_0	150 kg/cm²			
Theory	nonlinear		linear	
Parameters	Reg. I	Reg. II	Reg. I	Reg. II
$\sigma_{\xi\xi}$	144.7680	150	149.9600	150
$\sigma_{\eta\eta}$	48.4807	46.6172	37.6700	35.7000
$\tau_{\xi\eta}$	−15.1215	0	−12.9000	0
$\varepsilon_{\xi\xi}$	$2.1547\cdot10^{-2}$	$2.2721\cdot10^{-2}$	$4.0625\cdot10^{-2}$	$4.1273\cdot10^{-2}$
$\varepsilon_{\eta\eta}$	$0.0508\cdot10^{-2}$	$-0.0201\cdot10^{-2}$	$0.0523\cdot10^{-2}$	$-0.0121\cdot10^{-2}$
$\varepsilon_{\xi\eta}$	$-0.6620\cdot10^{-2}$	0	$-0.9216\cdot10^{-2}$	0
ε	$2.2054\cdot10^{-2}$	$2.2519\cdot10^{-2}$	$4.1148\cdot10^{-2}$	$4.1152\cdot10^{-2}$
ε_i	$1.4703\cdot10^{-2}$	$1.5076\cdot10^{-2}$	$2.7433\cdot10^{-2}$	$2.7495\cdot10^{-2}$
V_ξ	−12.6728	−13.1270	−17.5428	−17.7161
V_η	1.9582	−0.7394	1.9900	−0.4453
α	0.1533		0.1128	
β	0.1047		0.0698	

Remark. V_ξ, V_η are the velocities of the particles of the soil in m/sec, $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$, $\tau_{\xi\eta}$ are the components of stress in kg/cm²; α, β are angles in radians.

Table 2

Soil	Clay		Dense sand	
P_0	50 kg/cm²		20 kg/cm²	
Theory	nonlinear		nonlinear	
Parameters	Reg. I	Reg. II	Reg. I	Reg. II
σ_{xx}	52.0942	50.0530	15.5661	20.3086
σ_{yy}	47.2446	48.1560	9.6285	0.1753
τ_{xy}	−0.6612	0.0829	−0.5682	2.5105
ε_{xx}	$1.5343\cdot10^{-2}$	$0.5089\cdot10^{-2}$	$0.8950\cdot10^{-2}$	$1.6346\cdot10^{-2}$
ε_{yy}	$0.0275\cdot10^{-2}$	$0.0275\cdot10^{-2}$	$0.0080\cdot10^{-2}$	$0.0080\cdot10^{-2}$
ε_{xy}	$-0.4109\cdot10^{-2}$	$0.0421\cdot10^{-2}$	$-0.1697\cdot10^{-2}$	$0.4056\cdot10^{-2}$
ε	$1.5600\cdot10^{-2}$	$0.5364\cdot10^{-2}$	$0.9031\cdot10^{-2}$	$1.6427\cdot10^{-2}$
ε_i	$1.0400\cdot10^{-2}$	$0.8753\cdot10^{-2}$	$0.6021\cdot10^{-2}$	$1.0951\cdot10^{-2}$
α	0.1766		0.2167	
β	0.0436		0.1222	
a	390		307	
b	97		174	

Remark. a, b are the velocities of the longitudinal and transverse waves, respectively, in m/sec.; α, β are angles in radians and σ_{xx}, σ_{yy}, τ_{xy} are the components of stress in kg/cm².

the calculations for various loadings P_0 are presented in Tables 1 and 2 containing also the parameters of the waves and soil in the regions I and II.

An analysis of the above Tables leads to the following conclusions. For a stepwise loading in the semi-space, there exist two nonlinear waves propagated with constant velocities depending on the magnitude of the loading and the parameters of the medium. Nonlinear transverse waves can be both loading and unloading waves. In the cases (a) and (c) they are loading waves while in the case (b) the transverse wave is an unloading wave. In the latter case, when the wave passes through the medium, the first and second invariants of the strain and stress tensors decrease.

Furthermore, we investigated parameters of the unloading wave for various loadings and the corresponding solutions were derived. In the case of decreasing behind the front moving loading, the longitudinal and transverse waves are propagated with variable velocities and their magnitudes as well as the magnitudes of the stresses and strains, can be calculated by means of the method of characteristics.

For large pressure the values of stresses and strains occuring in the semi-space and calculated by means of the linear theory differ significantly from the corresponding values obtained on the basis of the nonlinear theory. For instance, for the pressure $P_0 = 150$ kg/cm^2 the difference is 30%. The differences in phenomena in these cases are not confiend to the values of the parameters, for the magnitudes of the disturbance regions are also different. In the case (a) the region of disturbance becomes wider than in [7].

The nonlinearity of the properties of the material of the semi-space leads to a jump variation of the normal pressure on the transverse wave.

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