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Visco-plasticity solution by finite element process

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ELASTO-VISCO-PLASTIC constitutive relations are used as the basis for a finite element formulation. The process leads to a relatively simple computational algorithm and may be used for solution of truly visco-plastic situations or simply as an artifice for a convenient solution of plasticity problems. Several numerical examples illustrate the paper.

Za podstawę sformułowania metody elementów skończonych przyjęto związki konstytutywne sprężysto-lepkoplastyczności. Postępowanie daje się sprowadzić do względnie prostego algorytmu obliczeniowego i można je zastosować do rozwiązywania zagadnień istotnie lepkoplastycznych lub też po prostu jako pewien dogodny sposób rozwiązywania problemów plastycznych. Podano szereg przykładów liczbowych.

При формулировке зависимостей метода конечных элементов приняты определяющие уравнения упруго-вязкопластичности. Способ решения сводится к относительно простому расчётному алгоритму, который можно применить для решения существенно вязкопластических задач или рассматривать как некоторый способ решения пластических задач. Даётся ряд численных примеров.

1. Introduction

The application of the finite element process to the solution of both creep (visco-elastic) and plasticity problems is well documented [1–12]. Many different approaches have been used in plasticity to overcome the difficulties associated with

(a) the incremental form of the strain-stress relations,

(b) the necessity of adjusting the stresses to the yield condition,

(c) the indeterminacy of "tangential" stiffness matrices when collapse situation is reached and

(d) the difficulties associated with strain softening [13].

In contrast to this, creep and visco-elastic problems have almost universally been dealt with by a relatively simple computational scheme in which time is the basic parameter and "initial strain" is introduced [1].

It is surprising therefore that problems of visco-plasticity to which the simpler concepts can be applied have not been dealt with so far numerically. It is the objective of this paper to show that

(i) the visco-plastic problems can be dealt with by relatively straightforward numerical formulation and

(ii) that it can be effectively used not only to obtain real time-dependent solutions but, by treating the viscous effects as a purely computational artifice, leads to elasto-

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plastic solutions overcoming some of the difficulties inherent in the classical plastic formulation.

It could be argued that the visco-plastic model of the materials is indeed physically more acceptable than any purely plastic one in which implication of instantaneous plastic flow is made. It is perhaps this physical feature which makes the numerical finite element process successful and which opens the door to a host of new applications. Whilst plasticity and visco-plasticity are well documented in the context of metal behaviour, extension of the former to a wide range of problems introduced for instance in soil and rock mechanics have met only with limited success as time dependence is pronounced and strain softening frequently present. It is anticipated that by combining both effects in one comprehensive model more realistic solutions will be obtained. Such problems as the progressive failure of soil masses or time-dependent transfer of loads from a rock mass to a tunnel lining present possible applications of the visco-plastic model.

2. The visco-plastic model

In the visco-plastic model of the material we assume that the only "instantaneous" strains which can be produced by stresses are the elastic ones. To these is added a time-dependent strain whose rate depends on the excess by which some function of the stresses exceeds a "threshold" or "yield" value [14–21].

To illustrate the situation conceptually we introduce a uniaxial model of Fig. 1. Here the slider (plastic component) can only become active if $\sigma > Y$ (in which σ is the total

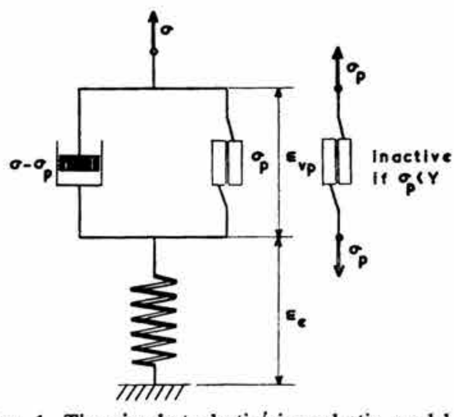

FIG. 1. The simplest elastic/visco-plastic model.

actual "stress" applied and Y is some yield value). On instantaneous load application only elastic straining of the spring takes place, the excess $\sigma - Y$ being taken by the "dashpot" which results in a time variable strain.

In such a model the value of Y may well be made strain dependant and also nonlinear characteristics introduced into the dashpot and spring without difficulty.

While the model can degenerate into a simple plastic one by a simple omission of the dashpot component, a fundamental difference arises when this is present: the total stress can exceed the yield value instantaneously by any desired amount — a fact noted physically in many experiments [16].

The simple uniaxial, conceptual situation can be generalised to a multiaxial case. A full discussion and validation of the process can be found in the excellent surveys by OLSZAK, PERZYNA and others [14–17]. Only the essentials will be given here using vector and matrix notation for computational convenience.

(a) First, as in the uniaxial model, the total strain $\boldsymbol{\epsilon}$ will be considered as a sum of the elastic $\boldsymbol{\epsilon}_e$ and visco-plastic $\boldsymbol{\epsilon}_{vp}$ phases together with initial, prescribed strains. Thus, with $\boldsymbol{\epsilon}_\theta$ standing for specified initial strains say of thermal origin, we have

$$\boldsymbol{\epsilon} \equiv \{\varepsilon_x \varepsilon_y \varepsilon_z \gamma_{yz} \gamma_{zx} \gamma_{xy}\}^T = \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_{vp} + \boldsymbol{\epsilon}_\theta. \tag{2.1}$$

(b) Second, in the usual manner, the total stress σ will be related to elastic strains through a nonlinear reversible law as

$$d\boldsymbol{\sigma} = \mathbf{D}_T d\boldsymbol{\epsilon}_e \tag{2.2}$$

in which $\mathbf{D}_T$ is the tangential elasticity matrix and which is generally dependent only on $\boldsymbol{\epsilon}_e$.

Alternatively, differentiating with respect to time, we have

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}_T \dot{\boldsymbol{\epsilon}}_e \quad \text{with} \quad \frac{d}{dt}\boldsymbol{\sigma} \equiv \dot{\boldsymbol{\sigma}} \quad \text{etc.} \tag{2.3}$$

(c) Third, defining the scalar yield condition as

$$F = F(\boldsymbol{\sigma}, \boldsymbol{\epsilon}_{vp}, Y) = 0, \tag{2.4}$$

where Y is some yield stress value and $\boldsymbol{\epsilon}_{vp}$ is the visco-plastic strain, we shall assume that the visco-plastic creep phenomenon only occurs when

$$\mathrm{F} > 0. \tag{2.5}$$

When $F < 0$ only elastic strains are possible, while $F > 0$ exhibits both elastic and visco-plastic effects. Again the reader should contrast this with ideal plasticity which does not permit states $F > 0$, $F = 0$ being a necessary condition for plastic flow.

(d) A further assumption regarding the direction of visco-plastic straining must now be made. In common with plasticity of non-associated type we shall assume this to be defined by the gradients of a "plastic potential", $Q(\boldsymbol{\sigma}, \boldsymbol{\epsilon}_{vp}, Y)$. Thus we have

$$\dot{\boldsymbol{\epsilon}}_{vp} = \lambda \frac{\partial Q}{\partial \boldsymbol{\sigma}} \tag{2.6}$$

in which

$$\frac{\partial Q}{\partial \boldsymbol{\sigma}} \equiv \left\{ \frac{\partial Q}{\partial \sigma_x} \frac{\partial Q}{\partial \sigma_y} \frac{\partial Q}{\partial \sigma_z} \frac{\partial Q}{\partial \sigma_{yz}} \frac{\partial Q}{\partial \sigma_{zx}} \frac{\partial Q}{\partial \sigma_{xy}} \right\}^T. \tag{2.7}$$

For special cases, an associated form of visco-plasticity $Q \equiv F$ may be used and here philosophical arguments of the least work principle can be brought in to justify its validity. In general, we are simply introducing Eq. (2.6) and (2.7) as a commonly made assumption

(e) The magnitude of the visco-plastic strain rate is governed by a flow rule giving the coefficient λ of (2.6) as a function of stresses and visco-plastic strains

$$\lambda = \gamma\left\langle\phi\left(\frac{F}{F_0}\right)\right\rangle \tag{2.8}$$

with F being the value of the yield function, F_0 introduced as a reference fixed value making F/F_0 dimensionless, and γ a "fluidity" coefficient with dimension s^{-1}.

In above, $\langle\rangle$ denotes a zero term if $F < 0$ and ϕ is a scalar function such that

$$\begin{aligned} \phi(x) &> 0, \quad x > 0, \\ \phi(x) &= 0, \quad x = 0. \end{aligned} \tag{2.9}$$

Different functions ϕ have been proposed [16].

A linear relationship of the type

$$\phi(x) = x \tag{2.10$_1$}$$

is obviously the simplest and probably quite adequate to describe the behaviour of many materials. Relationships such as

$$\phi(x) = x^n, \tag{2.10$_2$}$$

or

$$\phi(x) = (e^{nx}-1) \tag{2.10$_3$}$$

have been suggested and indeed are incorporated in the program described later. For computational purposes, a numerical definition of $\phi(x)$ is most convenient and can be made to fit any experimental data.

Summarising we note that quite generally we can write the constitutive relations simply as

$$\dot{\boldsymbol{\epsilon}} = \mathbf{D}_T^{-1}\dot{\boldsymbol{\sigma}} + \gamma\left\langle\phi\left(\frac{F}{F_0}\right)\right\rangle\frac{\partial Q}{\partial \boldsymbol{\sigma}}. \tag{2.11}$$

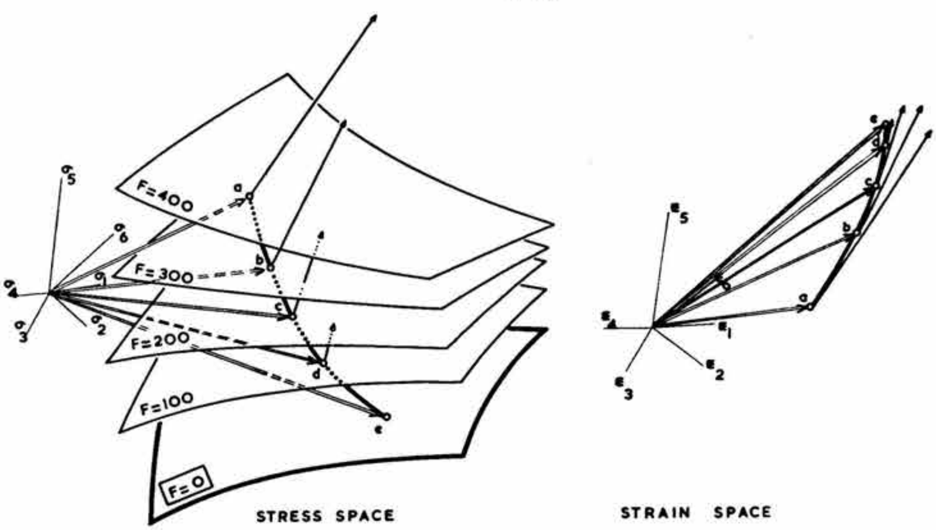

FIG. 2a. Associated visco-plastic constitutive relations.

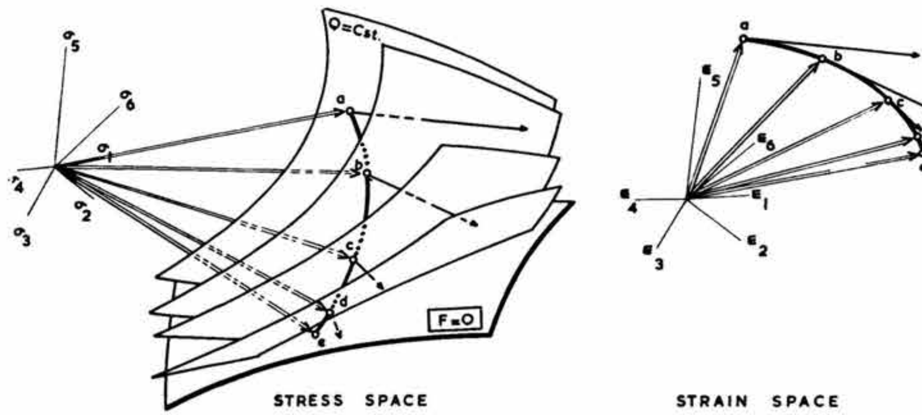

FIG. 2b. Non-associated visco-plastic constitutive relations.

This together with the initial conditions of the problem must suffice to obtain numerical solutions.

The relationships of visco-plasticity are illustrated in a "six-dimensional" stress space in Fig. 2. In Fig. 2a the associated rule is shown where as in Fig. 2b a non-associated situation is given. Regions shown are those in which the visco-plastic flow occurs and the direction of strain rate implied by the normality requirements is indicated.

Strain hardening or softening is implied in the form of surface F which can alter with ϵ_{vp}, the total visco-plastic strain reached.

Further, the position of F may itself be made dependent on the strain rate [18] but we shall not consider such a situation in the following.

3. Extension of the visco-plastic model

It is easy to visualise (and not complex to implement computationally) an extension of the visco-plastic model to include visco-elastic behaviour or to represent more complex behaviour by placing several models in series as shown in Fig. 3. Details of evaluating the strain rates associated with the Kelvin elements of the visco-elastic model are given in Ref. [6] and will not be considered here except to note that in computation the stress and strain rate of each model must be separately evaluated and stored.

A further possibility of extension of the model to cover almost any material behaviour is to use overlay models [23].

In metals it has been customary to treat creep and plasticity phenomena as separate features. To account for the creep rate increase with stress, power laws have been extensively used with indices as high as 6 to 8. It is interesting to speculate that in fact the combined behaviour of creep and plasticity can be obtained by stipulating two or three series of visco-

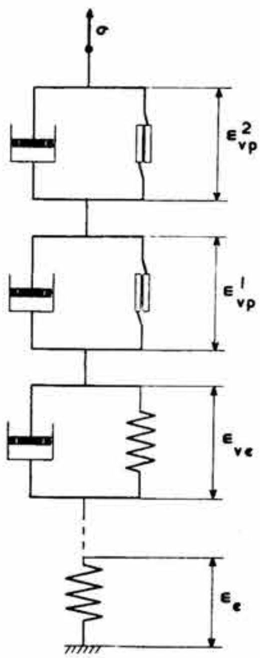

FIG. 3. A composite visco-elastic visco-plastic model.

plastic models with different "yield values". Indeed the commonly made assumption in creep strains is to impose normality of strain components to the effective, von Mises, stress. This clearly is implied directly and with more justification in the visco-plastic model.

4. Computational procedure for quasi-static visco-plastic model

4.1. General considerations

We shall formulate the problem in the displacement finite element form, using most general nonlinear elastic behaviour and visco-plastic strain model, so that the general structure of the resulting equations can be appreciated. Later we shall specialise in describing a computational scheme of particular physical simplicity using the so-called "initial strain" procedure.

The full details of finite element methodology is given elsewhere [1] and the general notation of that text will be used here.

In the formulation we shall deal with the so called quasi-static situation in which inertia terms are negligible. Similarly the effects of any thermal coupling will be neglected.

Both assumptions are consistent with slow phenomena but in principle there is no difficulty of extending the formulation to include them.

4.2. Finite element discretisation and associated matrix equations

(a) We prescribe a displacement field in a piecewise (element by element) manner by means of appropriate shape functions

(4.1) $$\mathbf{f} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \mathbf{N}(x, y, z)\boldsymbol{\delta},$$

where $\boldsymbol{\delta}$ is a set of nodal displacements.

(b) Strains are derived from displacements by differentiation([1])

(4.2) $$\boldsymbol{\epsilon} = \mathbf{B}(x, y, z)\boldsymbol{\delta}.$$

(c) By virtual work we obtain equilibrium relationships

(4.3) $$\int\limits_V \mathbf{B}^T \boldsymbol{\sigma} dV - \mathbf{R}_l = 0$$

in which $\boldsymbol{\sigma}$ are the stresses associated to the strains $\boldsymbol{\epsilon}$ and $\mathbf{R}_l$ the total body and boundary loads reduced to the nodes of the structure.

(d) We introduce now the constitutive relations and combining them with Eq. (4.3) we fully define the problem. As rate relationships are involved, it is convenient to rewrite Eq. (4.3) in a differential form

(4.4) $$\int\limits_V \mathbf{B}^T \frac{d\boldsymbol{\sigma}}{d\boldsymbol{\epsilon}_e} \dot{\boldsymbol{\epsilon}}_e dV - \dot{\mathbf{R}}_l = 0$$

or, using (2.1) and (2.3)

(4.5) $$\int\limits_V \mathbf{B}^T \mathbf{D}_T (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_{vp} - \dot{\boldsymbol{\epsilon}}_\theta) dV - \dot{\mathbf{R}}_l = 0$$

and introducing (4.2)

(4.6) $$\mathbf{K}_T \dot{\boldsymbol{\delta}} - \int\limits_V \mathbf{B}^T \mathbf{D}_T \dot{\boldsymbol{\epsilon}}_{vp} dV - \dot{\mathbf{R}} = 0$$

in which $\mathbf{K}_T$ is the tangential stiffness matrix

(4.7) $$\mathbf{K}_T = \int\limits_V \mathbf{B}^T \mathbf{D}_T \mathbf{B} dV$$

and

(4.8) $$\dot{\mathbf{R}} = \dot{\mathbf{R}}_l + \int\limits_V \mathbf{B}^T \mathbf{D}_T \dot{\boldsymbol{\epsilon}}_\theta dV.$$

Equation (4.6) completed by the constitutive relations

(4.9) $$\dot{\boldsymbol{\epsilon}}_{vp} = \boldsymbol{\epsilon}_{vp}(\boldsymbol{\sigma})$$

and

(4.10) $$\dot{\boldsymbol{\sigma}} = \mathbf{D}_T (\mathbf{B}\dot{\boldsymbol{\delta}} - \dot{\boldsymbol{\epsilon}}_{vp} - \dot{\boldsymbol{\epsilon}}_\theta)$$

([1]) We restrict ourselves to infinitesimal strains and small displacements here.

defines a system of three simultaneous matrix ordinary differential equations. Their solution can be obtained by several marching schemes such as Euler extrapolation, Runge Kutta or other predictor-corrector systems, starting from initial conditions simply given by an instantaneous purely elastic response under loads and prescribed strains application.

4.3. Particular case of linear elastic/nonlinear visco-plastic model

To clarify the process we make the assumption of linear elasticity and here $\mathbf{K}_T = \mathbf{K}$ and $\mathbf{D}_T = \mathbf{D}$ giving a set as follows:

$$\dot{\boldsymbol{\epsilon}}_{vp} = \gamma\left\langle\phi\left(\frac{F}{F_0}\right)\right\rangle\frac{\partial Q}{\partial \boldsymbol{\sigma}},$$

$$\mathbf{K}\dot{\boldsymbol{\delta}} - \int_V \mathbf{B}^T\mathbf{D}\dot{\boldsymbol{\epsilon}}_{vp}dV - \dot{\mathbf{R}} = 0, \tag{4.11}$$

$$\boldsymbol{\sigma} = \mathbf{D}(\mathbf{B}\boldsymbol{\delta} - \boldsymbol{\epsilon}_{vp} - \boldsymbol{\epsilon}_\theta).$$

Starting from initial conditions ar time $t = 0$, known as the solution of the standard elastic problem

$$\begin{aligned} \mathbf{K}\boldsymbol{\delta}^0 &= \mathbf{R}^0, \\ \boldsymbol{\epsilon}^0 &= \mathbf{B}\boldsymbol{\delta}^0 + \boldsymbol{\epsilon}_\theta^0, \\ \boldsymbol{\sigma}^0 &= \mathbf{D}(\boldsymbol{\epsilon}^0 - \boldsymbol{\epsilon}_\theta^0), \\ \boldsymbol{\epsilon}_{vp}^0 &= 0, \end{aligned} \tag{4.12}$$

we find the increment of viscoplastic strains estimated from $(4.11)_1$

$$\Delta\boldsymbol{\epsilon}_{vp} = \dot{\boldsymbol{\epsilon}}_{vp}\Delta t, \tag{4.13}$$

then the change in displacements

$$\Delta\boldsymbol{\delta} = \mathbf{K}^{-1}\left(\int_V \mathbf{B}^T\mathbf{D}\Delta\boldsymbol{\epsilon}_{vp}dV + \dot{\mathbf{R}}\Delta t\right) \tag{4.14}$$

and finally the change in stress

$$\Delta\boldsymbol{\sigma} = \mathbf{D}(\mathbf{B}\Delta\boldsymbol{\delta} - \Delta\boldsymbol{\epsilon}_{vp} - \Delta\boldsymbol{\epsilon}_\theta). \tag{4.15}$$

Steps (4.13) to (4.15) are computationally standard elastic solutions with $\Delta\boldsymbol{\epsilon}_{vp}$ being treated as "initial strain", hence the process is often known by this name.

4.4. Automatic time stepping

A step-by-step integration of matrix equations in time is obviously a further approximation which adds new errors to the usual space discretisation errors. This is due to the fact that $\Delta\boldsymbol{\epsilon}_{vp}$ in each time interval is calculated from a state of stress assumed to remain constant during the time increment when it is actually varying.

Errors may become excessive and results meaningless if the time increments used are too large; it is difficult to estimate a priori what time intervals should be chosen because these depend on stresses and viscous properties of the material and may vary from one element to another.

It is also desirable to keep the time increments small when the rates of stresses and strains are significant and to enlarge them as an asymptotic state in time is approached.

A way of overcoming these difficulties is as follows: we define norms of total strains and strain rates

$$(4.16)\qquad \begin{aligned} \|\boldsymbol{\epsilon}\| &= \sqrt{\varepsilon_{\mathrm{I}}^2+\varepsilon_{\mathrm{II}}^2+\varepsilon_{\mathrm{III}}^2},\\ \|\dot{\boldsymbol{\epsilon}}_{vp}\| &= \sqrt{\dot{\varepsilon}_{vp,\mathrm{I}}^2+\dot{\varepsilon}_{vp,\mathrm{II}}^2+\dot{\varepsilon}_{vp,\mathrm{III}}^2}, \end{aligned}$$

where subscripts I, II and III refer to principal tensor components. We compute a time increment such that the change of strain is a fraction of the total strain accumulated before

$$(4.17)\qquad \|\dot{\boldsymbol{\epsilon}}_{vp}\|\,\Delta t = \tau\|\boldsymbol{\epsilon}\|,$$

where τ is an increment parameter chosen a priori and dependent on the desired accuracy so that

$$(4.18)\qquad \Delta t = \tau\frac{\|\boldsymbol{\epsilon}\|}{\|\dot{\varepsilon}_{vp}\|}.$$

It has been found that $0.1 < \tau < 0.2$ leads to good and not too expensive results.

4.5. Improvement of accuracy

Various possible algorithms to improve accuracy and stabilize the solution can be introduced. In the solution described we follow a predictor corrector process as follows.

Starting from $\boldsymbol{\sigma}^{(t)}$ known at beginning of interval, we

(a) calculate $\Delta\boldsymbol{\epsilon}_{vp}$ as a function of $\boldsymbol{\sigma}^{(t)}$,

(b) solve (4.11) and find an iterated approximation $\boldsymbol{\sigma}_i^{(t+\Delta t)}$ $\quad i = 1, 2, \ldots,$

(c) find the mean stress $\boldsymbol{\sigma}_i^{(t+\frac{1}{2}\Delta t)} = \frac{1}{2}(\boldsymbol{\sigma}^{(t)}+\boldsymbol{\sigma}_i^{(t+\Delta t)})$,

(d) compute $\Delta\boldsymbol{\epsilon}_{vp}$ as a function of $\boldsymbol{\sigma}_i^{(t+\frac{1}{2}\Delta t)}$,

(e) find new $\boldsymbol{\sigma}_i^{(t+\Delta t)}$ by re-solution of (4.11) and cycle until convergence is achieved or for a given specified number of cycles. We found that two cycles are generally sufficient.

4.6. Extension to include visco-elastic effects

As stated in Sec. 3, it is easy to extend the formulation and program to composite elasto-viscoelastic-viscoplastic models.[1] Only the expression of the strain rate $\dot{\boldsymbol{\epsilon}}_{vp}$ has to be modified, Eq. $(4.11)_1$, to include the additional effects. As shown in Ref. [6], visco-elastic strain components have to be stored for each additional Kelvin unit of Fig. 3. This is purely computational and very complex materials can be incorporated in the algorithm already described.

4.7. Simplex and isoparametric elements

Any element form can clearly be used in the actual implementation. We have here concentrated on simple two-dimensional problems and the well known triangular element.

Experience shows that much improved results in nonlinear analysis follows the use of isoparametric numerically integrated elements [12, 13].

For a production type application, these are strongly advised.

5. Application

Here, the examples included are relatively simple and designed firstly to demonstrate the accuracy of the computational scheme including the effects of varying time steps and secondly, to test the applicability in a simple plasticity case where the time aspect is introduced as an artifact. Anticipating the results we would expect here an iteration process similar in all respects to that implied in the modified Newton RAPHSON [1, 13] approach but with a parameter $(\gamma \Delta t)$ which in effect acts as an accelerator (or decelerator) and which can be adjusted to obtain convergence at all times. The results shown later bear out this assumption.

The first example is that of a thick-walled sphere solved analytically by WIERZBICKI [22]. This "one-dimensional" problem is treated as a "two-dimensional" axisymmetric one here

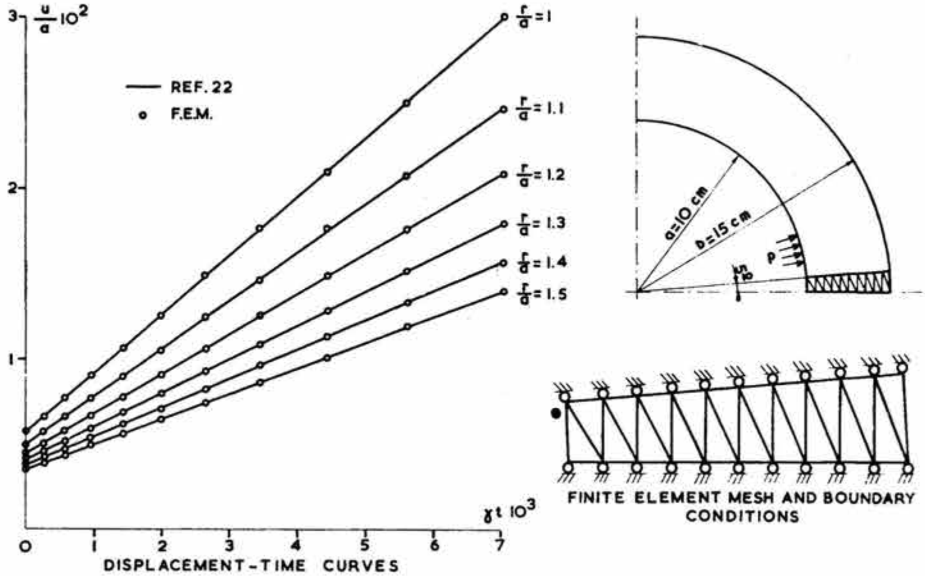

FIG. 4. Thick sphere under internal pressure exceeding the collapse value
$E = 2.1 \times 10^6$ kg/cm², $\nu = 0.3$, Von Mises yield function. Perfect plasticity with $F_0 = Y = 2800$ kg/cm². Linear creep, $\gamma = 0.01$ $\tau = 20\%$. Pressure $= 11\,200$ kg/cm².

to test the validity of the program. A segment of 5° is considered and described on Fig. 4, which also gives the comparison of numerical and exact solution for displacement-time relations. Figure 5 shows the time-dependent changes in stress distribution. In both cases the exact solution is reproduced to degree of accuracy associated with plotting.

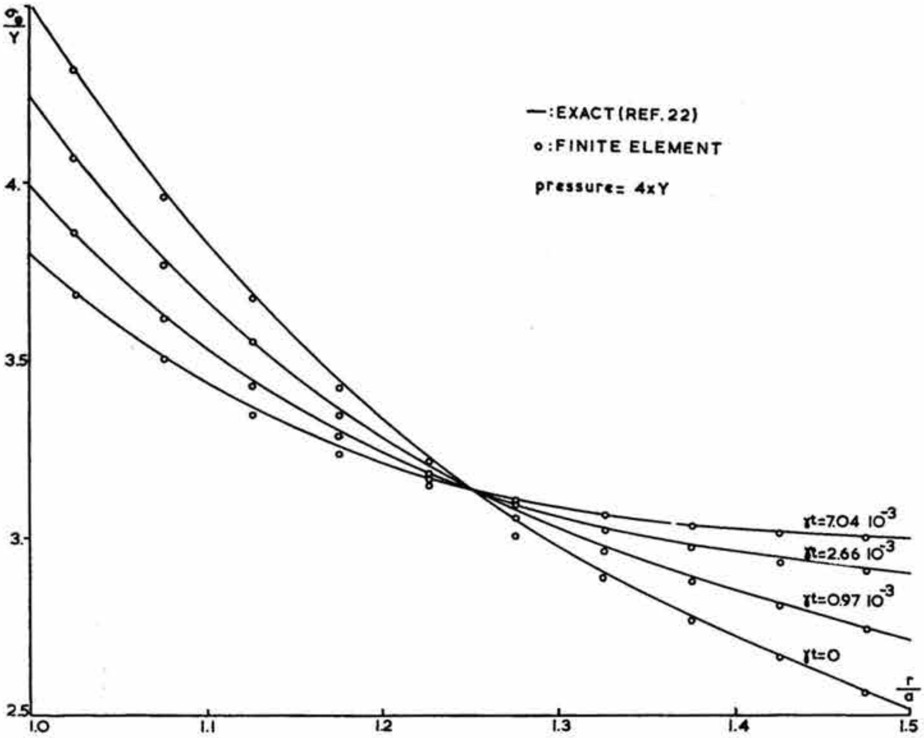

FIG. 5. Thick sphere: time-dependent circumferential stress distribution.

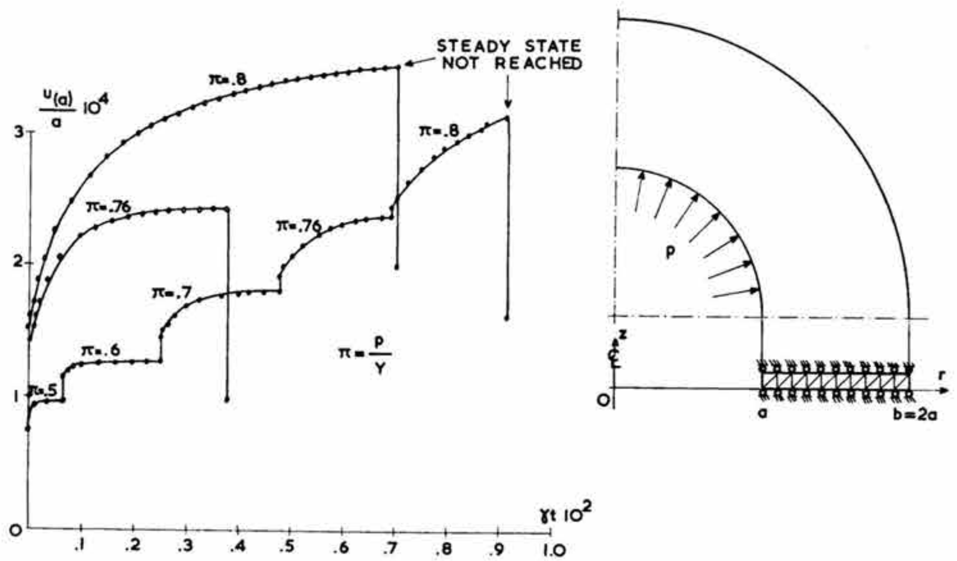

FIG. 6. Thick cylinder under internal pressure below the collapse value.

$E = 2\times 10^7$ kg/cm², $\nu = 0.3$. Von Mises yield function. Perfect plasticity with $E = Y = 2000$ kg/cm². Linear creep $\gamma = 0.0001$ $\tau = 10\%$.

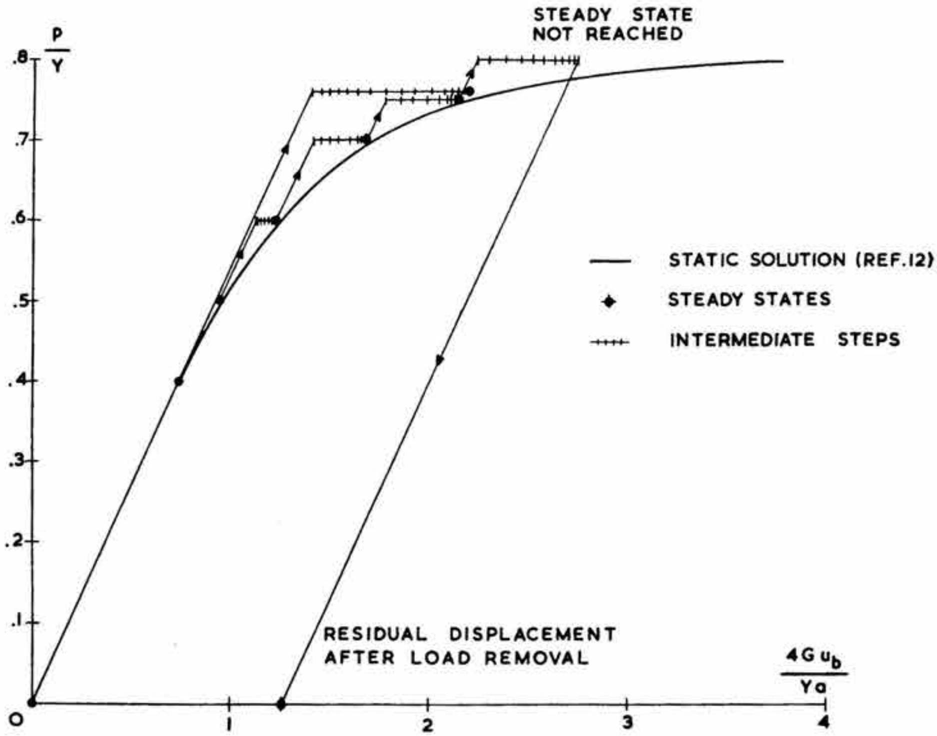

FIG. 7. Thick cylinder: pressure-displacement curve.

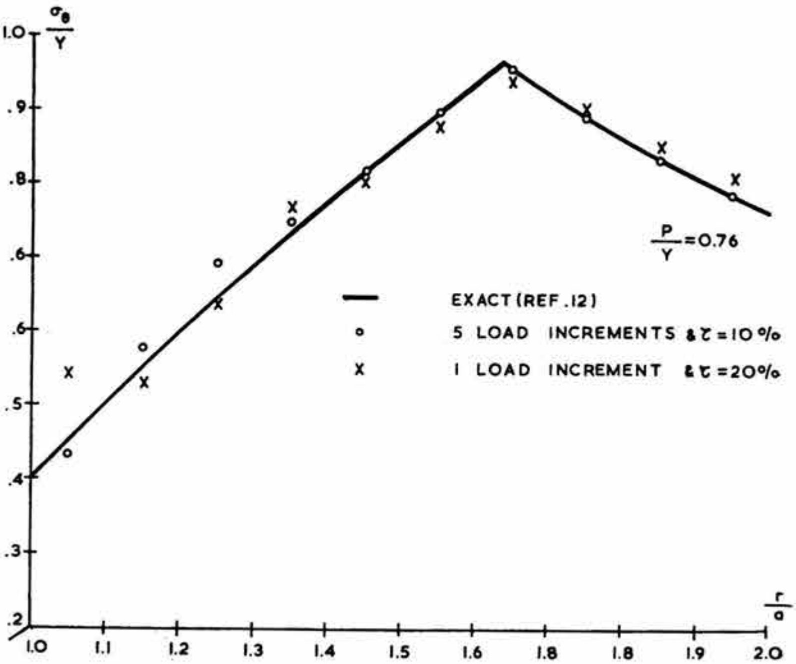

FIG. 8. Thick cylinder: hoop stress distribution accuracy.

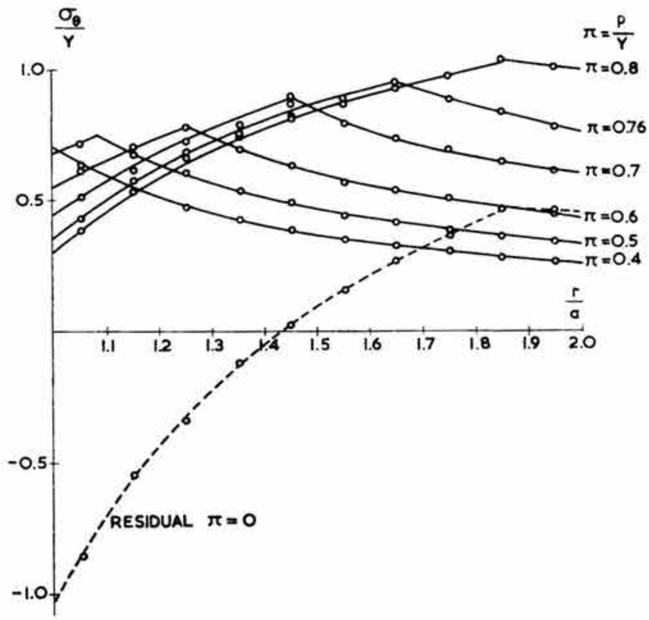

FIG. 9. Thick cylinder: steady state elasto-plastic hoop stress distribution for various applied pressures

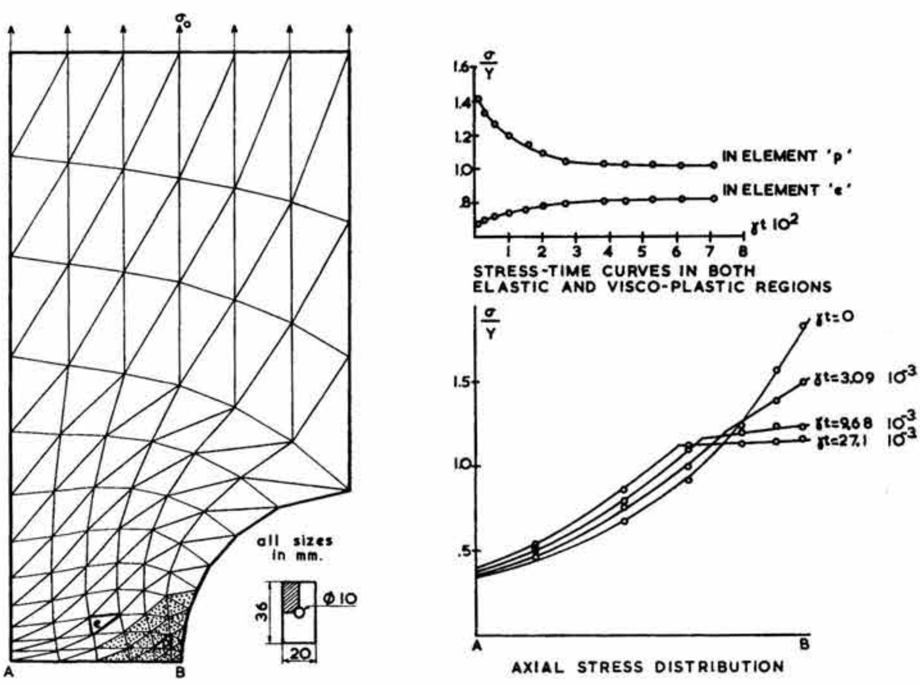

FIG. 10. Perforated tension strip.

Mesh 144 elts, 91 nodes. Shaded elements are those in which plastic flow occurred. $E = 7000\,\text{kg/mm}^2, \nu = 0.2$, Von Mises yield function. Perfect plasticity with $F = Y = 24.3\,\text{kg/mm}^2$. Linear creep $\gamma = 0.01$, $\tau = 20\%$. Applied pressure $\sigma_0 = p = 9.5\,\text{kg/mm}^2$

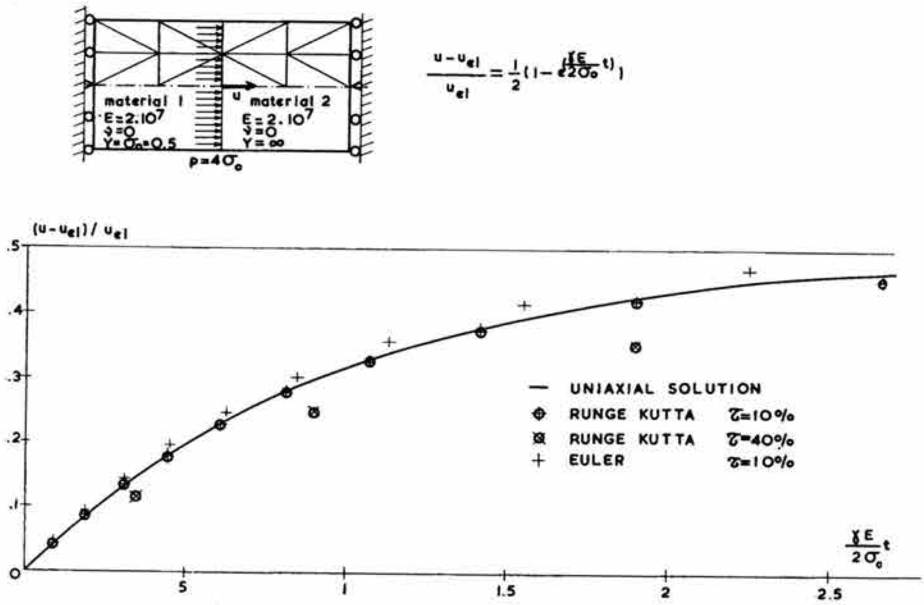

FIG. 11. Composite sheet. Perfect Mises plasticity. Displacement-time curve for various time stepping techniques.

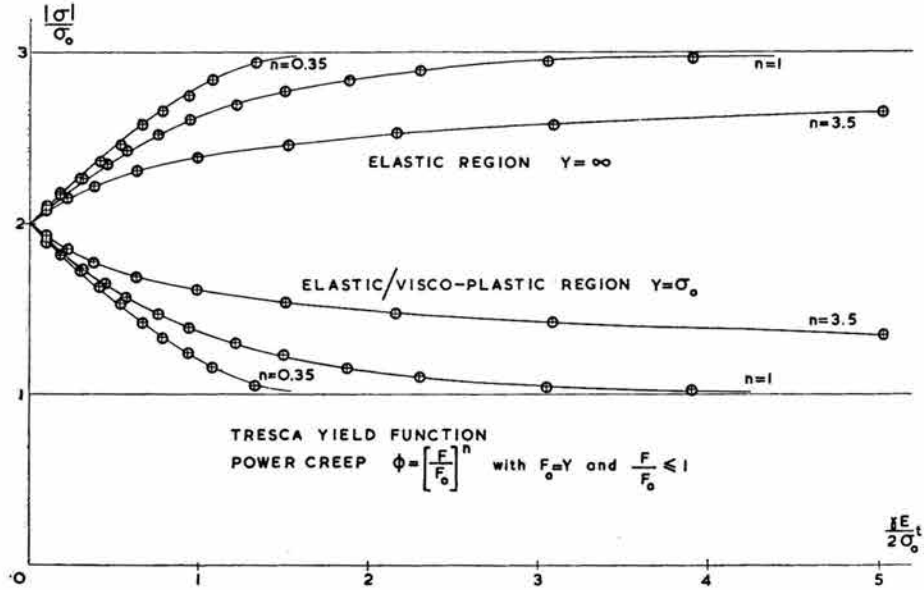

FIG. 12. Composite sheet. Perfect Tresca plasticity. Stress-time curves for various flow rules. Applied pressure $p = 4\sigma_0$.

A linear visco-plastic law was used in this example and details are given in the figure legend.

In the second and third examples, the visco-plastic effect is introduced artificially to study the progression towards steady state elasto-plastic solution. The problems are those of a simple thick cylinder under uniform pressure (Figs. 6–9) and of a perforated tension strip (Fig. 10) treated as axisymmetric and plane stress cases, respectively.

In the case of a cylinder, analytical as well as numerical solutions of the purely plastic problem are available [12, 13] for a comparative study.

Figure 6 shows displacement-time curves obtained for various sequences of load appli-

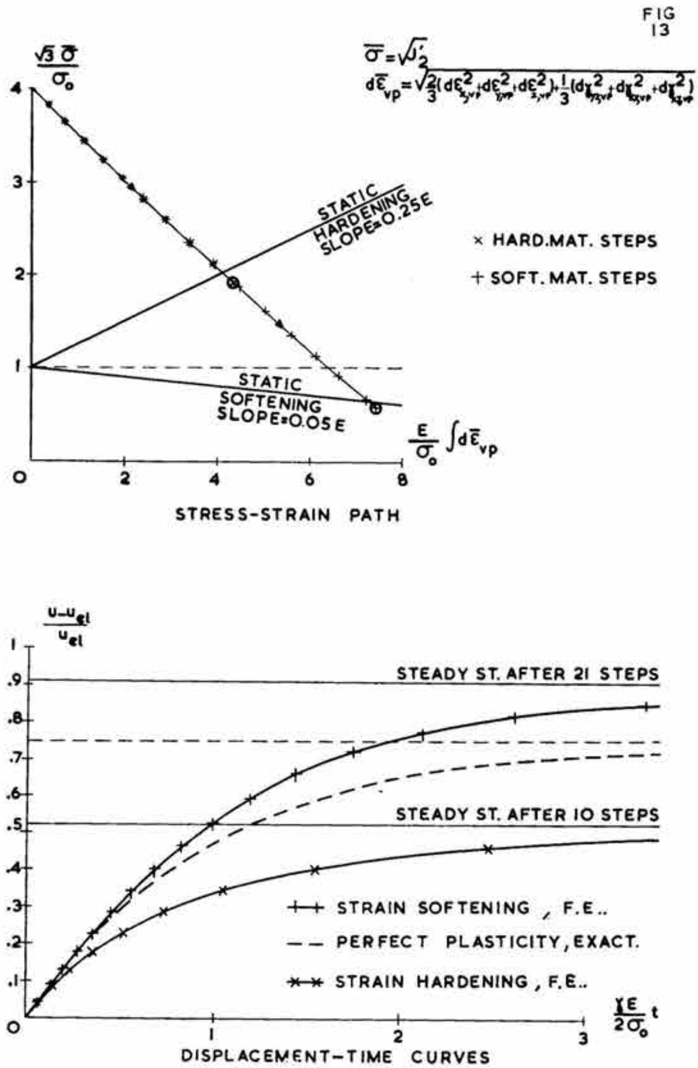

FIG. 13. Composite sheet. Isotropic strain hardening and softening using Mises yield function. Applied pressure $p = 8\sigma_0$.

cation. Each load increment was left until steady state conditions were reached with a desired accuracy. In Fig. 7 we show a plot of the load-displacement progression for an incrementally applied load and for a single loading step. In both cases the expected convergence to the exact steady state solution is recorded.

In Fig. 8 a study of stress distribution is made for these two load situations, additionally including a variation of the time increment parameter. Discrepancies are small and the shorter time intervals give a slightly better approximation. Figure 9 shows the steady state stress distribution for various loads.

While the first two examples are of rather trivial geometry, Fig. 10 illustrates a more realistic application. Here a perforated strip studied by VALLIAPPAN [5] is subject to a suddenly applied load below the collapse value. Final spread of plastic zones and time variation of stresses are shown. It is interesting to note a very good convergence of results after only 20 time increments. This approximates to the cost of the standard elasto-plastic solution and, although exact comparison of details is not available here for the two solutions, the results appear very similar.

Clearly it is desirable to investigate

(a) the most efficient time stepping scheme,

(b) the effects of nonlinear viscoplastic law,

(c) strain hardening as well as strain softening characteristics of behaviour.

Much yet remains to be done but in the last example, of somewhat trivial nature, we study some of these problems. Here a composite sheet of two materials one with an infinite yield point is subject to a line load Fig. 11. The same figure compared the effects of using the predictor corrector process with that of the simple and cheaper Euler extrapolation. Clearly, the first is more accurate but the question of cost involved in reaching more accuracy via this method or simply by taking smaller time increments needs to be studied.

In Fig. 12 we plot the results for three various types of visco-plastic laws. In none was any difficulty of convergence experienced. Finally, Fig. 13 illustrates the same problem with strain hardening and softening effect.

6. Concluding remarks

The techniques of visco-plastic analysis have been demonstrated and show that this extension not only allows a new category of problems to be dealt with but can also be effectively used as a vehicle for classic elasto-plastic solutions. Clearly, many extensions are possible and indeed desirable. The inclusion of dynamic terms and thermal coupling are but two possibilities. Even with the present limitations it is desirable to extend the applications to the study of collapse situations. Those can be characterized by the lack of asymptotic displacement convergence, and effective computational techniques are now being investigated in that context.

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