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Some remarks on plane flow of granular media

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BASING on the known static solutions of plane strain problems of granular media, kinematic solutions of the following particular problems are considered in the paper: (a) the retaining wall problem, (b) punch indentation, (c) flow through a vertical channel. The kinematic fields are determined for the models introduced by (i) Drucker and Prager, (ii) Jenike and Shield, (iii) Genev. Correctness of the solutions was investigated from the point of view of non-negative values of the dissipated power. A number of solutions corresponding to models (ii) and (iii) is shown to be incorrect. The results are of interest in the experimental verification of the models.

W oparciu o znane rozwiązania statyczne dla płaskiego stanu odkształcenia ośrodka rozdrobnionego rozpatrzono rozwiązania kinematyczne dla problemu: a) odporu ścianki, b) wciskania stempla, c) przepływu przez kanał pionowy. Pola kinematyczne znaleziono dla modelu 1) Druckera i Pragera, 2) Jenike i Shielda, 3) Geniewa. Zbadano poprawność rozwiązań z uwagi na nieujemność mocy dysypowanej. Wykazano, że szereg otrzymanych rozwiązań dla modeli 2) i 3) jest niepoprawnych zarówno dla ośrodka ważkiego jak i nieważkiego. Rozpatrzone problemy brzegowe należą do klasy zadań o niedopuszczalnych kinematycznych warunkach brzegowych. Otrzymane wyniki mają istotne znaczenie przy weryfikacji eksperymentalnej omawianych modeli na podstawie porównania doświadczalnych i teoretycznych pól prędkości w zadaniach brzegowych. Wskazują one na konieczność uprzedniego sprawdzania poprawności rozwiązań teoretycznych dla realizowanego doświadczenia.

Исходя из известных статических решений для плоского деформированного состояния размельчённой среды, построены кинематические решения для следующих задач: а) подпор стенки; б) вдавливание штампа; в) течение в вертикальном канале. Кинематические поля построены для моделей: 1) Друкера—Прагера; 2) Енике—Шилда; 3) Гениева. Исследована корректность решений в отношении неотрицательной мощности диссипируемой энергии. Показано, что многие решения для моделей 2) и 3) некорректны, как для весомой, так и для невесомой сред. Рассмотренные краевые задачи принадлежат к классу задач с недопустимыми кинематическими краевыми условиями. Полученные результаты имеют существенное значение при экспериментальной проверке рассматриваемых моделей, состоящей в сравнении теоретических полей скоростей со скоростями, полученными в опытах. Обнаруженные результаты указывают на необходимость предварительной проверки корректности теоретических решений при проведении поверочных опытов.

1. Introduction

THE BOUNDARY value problem of plane, quasi-static flow of a granular medium, described by the model of a rigid-perfectly plastic body and the Coulomb-Mohr yield condition, is rather well known as regards static solutions are concerned (cf. e.g. [12]). Much less attention has been paid thus far to kinematic solutions, one of the reasons being the controversial propositions concerning the kinematics of granular media. In numerous technological processes (e.g. in earthmoving [14]), however, the knowledge of the static solution itself is insufficient.

In the present paper is investigated the possibility of constructing kinematic counterparts to the known static solutions of the following three problems: (a) shifting of a retaining wall, (b) punch indentation, (c) flow through a parallel-walled channel with a narrow opening. Cohesionless media are considered and the kinematic models proposed by DRUCKER and PRAGER [4], JENIKE and SHIELD [6] and GENEV [5] are used.

The conditions to be satisfied by the kinematic solutions based on the static solutions are: (a) feasibility of motion of the material, b) fulfilment of kinematic boundary conditions, c) compatibility of deformations, d) non-negativeness of the dissipation power in plastic domain elements, along the discontinuity lines of the velocity field and in the regions of contact with the tool or the structure. The correctness of the velocity fields in a granular medium from the point of view of satisfaction of all four conditions has been investigated in a very few papers only (e.g. [3, 7]); in the majority of publications, the problem is reduced to verifying the first three conditions.

Condition (d) is expressed by the relations

$$D_v = \sigma_{ij}\dot{\varepsilon}_{ij} \geqslant 0, \quad D_l = T_i[V_i] \geqslant 0, \tag{1.1}$$

the first of which, however, does not hold in the model by DRUCKER and PRAGER and in a cohesionless medium. In the latter case the dissipation power is equal to zero in a body element both in the case of plastic loading and in the case of unloading (cf. e.g. [8]). For the case of a Drucker-Prager model, the kinematic solution should make the vector $\dot{\varepsilon}_{ij}$ directed (at every point) outwards the yield surface. This condition is expressed, in the case of plane strain, by the following inequality written in terms of the principal directions of stress:

$$\sigma_1\dot{\varepsilon}_1 < 0 \quad \text{or} \quad \sigma_2\dot{\varepsilon}_2 > 0, \quad \sigma_1 > \sigma_2 \tag{1.2}$$

what is equivalent to the positive increment of the voluminal strain rates (dilatation),

$$\dot{\varepsilon}_{ii} > 0. \tag{1.3}$$

2. Physical relations

(i) *Drucker-Prager model*. The physical assumptions of the model are reduced to the requirements of isotropy and orthogonality of the $\dot{\varepsilon}_{ij}$-vector to the yield surface. This leads to the following system of kinematic equations:

$$\begin{aligned} &(V_{x,x}-V_{y,y})\sin 2\psi-(V_{y,x}+V_{x,y})\cos 2\psi = 0,\\ &V_{x,x}(\sin\varphi-\cos 2\psi)-V_{y,y}(\sin\varphi+\cos 2\psi) = 0, \end{aligned} \tag{2.1}$$

$\tan\psi$ being the inclination of the algebraically greatest principal stress to the x-axis, and φ — the angle of internal friction. The characteristics of Eqs. (2.1) are given by the equations

$$\frac{dy}{dx} = \tan\left[\psi \mp \left(\frac{\pi}{4}+\frac{2}{\varphi}\right)\right], \quad \alpha, \beta. \tag{2.2}$$

Along the characteristics the relations

$$\begin{aligned} &dV_\alpha-(V_\alpha\tan\varphi+V_\beta\sec\varphi)d\psi = 0 \quad \text{along } \alpha,\\ &dV_\beta+(V_\alpha\sec\varphi+V_\beta\tan\varphi)d\psi = 0 \quad \text{along } \beta \end{aligned} \tag{2.3}$$

hold true.

Equations (2.2) mean that the stress and velocity characteristics coincide. Any of the velocity characteristics may be a line of discontinuity of the velocities; the tangential and normal components of the velocity jump are described by the relations

$$\begin{aligned} &[V^t] = [V_0^t]\exp[\tan\varphi(\psi-\psi_0)],\\ &[V^n] = [V^t]\tan\varphi. \end{aligned} \tag{2.4}$$

(ii) *Jenike and Shield model.* In this model, the condition of isotropy and incompressibility are assumed, which leads to the equations

$$(V_{x,x}-V_{y,y})\sin 2\psi-(V_{y,x}+V_{x,y})\cos 2\psi = 0,\quad V_{x,x}+V_{y,y} = 0 \tag{2.5}$$

whose characteristics are the lines

$$\frac{dy}{dx} = \tan\left(\psi \mp \frac{\pi}{4}\right), \quad \alpha, \beta. \tag{2.6}$$

Along the lines, the following relations are satisfied

$$dV_\alpha - V_\beta d\psi = 0 \quad \text{along } \alpha,\qquad dV_\beta + V_\alpha d\psi = 0 \quad \text{along } \beta. \tag{2.7}$$

From Eqs. (2.6) it follows that the velocity characteristics are orthogonal and do not coincide with the stress characteristics. The discontinuity line may be any of the velocity characteristics along which the velocity jump remains constant and is tangent to that line.

(iii) *The Genev model.* In this model, the deviation of principal directions of the tensors σ_{ij}, $\dot{\varepsilon}_{ij}$ is assumed constant and equal to $\pm\varphi/2$, and the material is incompressible. The kinematic relations hence have the form

$$(V_{x,x}-V_{y,y})\sin(2\psi\pm\varphi)-(V_{y,x}+V_{x,y})\cos(2\psi\pm\varphi) = 0,\quad V_{x,x}+V_{y,y} = 0. \tag{2.8}$$

Owing to the deviation $\pm\varphi/2$, Eqs. (2.8) yield two orthogonal sets of velocity characteristics,

$$\frac{dy}{dx} = \tan\left(\psi - \frac{\varphi}{2} \mp \frac{\pi}{4}\right), \quad \alpha', \beta',\qquad \frac{dy}{dx} = \tan\left(\psi + \frac{\varphi}{2} \mp \frac{\pi}{4}\right), \quad \alpha'', \beta'', \tag{2.9}$$

the characteristics α' and β'' coinciding with the stress characteristics. The conditions along the characteristics are identical with (2.7). The line of discontinuity of velocity in the system (2.8) may be any of the characteristics of both systems; in spite of that, Genev assumed that the discontinuity lines were only those of the velocity characteristics which coincided with the stress characteristics. Consequently, in a field of characteristics which is continuous and does not contain any singularities, two discontinuity lines cannot intersect each other. In this paper will be considered not only the model (iii) with the discontinuity lines imposed by Genev, but also the model (iv), which enables the discontinuity lines to coincide with any of the velocity characteristics. In the both cases, the jump of velocity along the discontinuity line is constant and tangent to it.

For all the models considered, the characteristics are inextensible lines, which yields the orthogonality principle in the hodograph construction.

3. Displacement of the retaining wall

Let us consider one of the classical boundary value problems of flow of granular media — namely, the problem of displacement of a vertical retaining wall (with a rough surface) at a velocity V_0, in the direction of the medium bounded by a flat ground level. A passive state of stress occurs on the medium. The angle of friction at the wall is equal to φ_w.

3.1. Weightless medium(1)

The static solution for the weightless medium, well-known from the literature, assumes the existence of a Prandtl-type singularity at the point of contact of the wall with the ground level. The solution is obtained analytically under the assumption of a uniform loading of the ground surface, which does not influence the form of the field of stress characteristics. In Fig. 1a is shown the static solution for $\varphi = 30°$ and $\varphi_w = 15°$ consisting

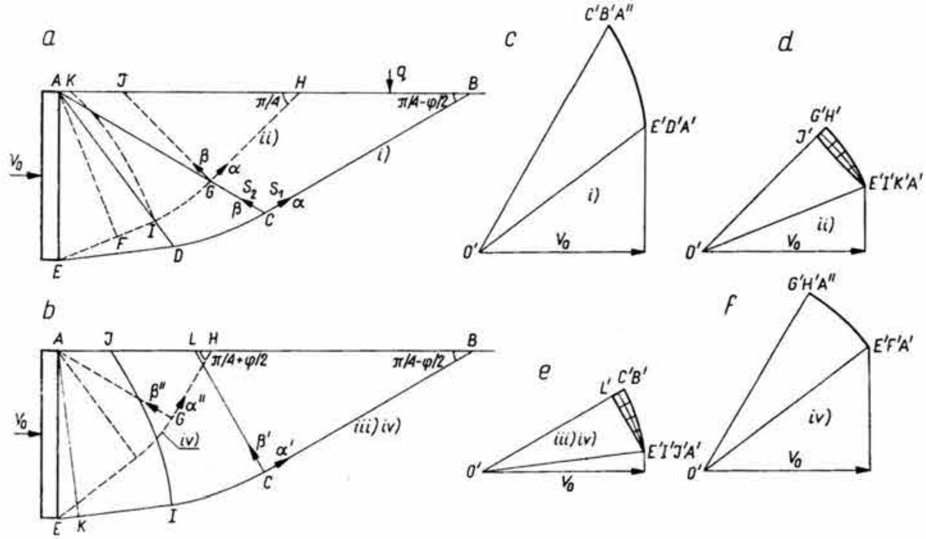

FIG. 1.

of two fields of rectilinear characteristics *ABC* and *ADE* and of the fan *ACD*. For the (i)-model, the field of velocity characteristics coincides with the stress field. In the case of other models, the velocity characteristics in *ABC* and *ADE* are rectilinear, and in the region of the fan they are determined by the following equations written in the r, θ-system:

$$r = r_0 \exp(\nu \tan\mu), \tag{3.1}$$

where

$\nu = \pm\theta,\ \mu = \frac{\varphi}{2}$ for α, β in the model (ii),

$\nu = \pm\theta,\ \mu = \varphi$ for α', β' and $\mu = 0$ for α'' and $\theta = \text{const}$ for β'' — in models (iii) and (iv).

Figs. 1a, 1b demonstrate the fields of velocity characteristics in which — under a limited wall height — the characteristics α drawn from E to the ground surface is the line of discontinuity of the velocity; the material lying below that line is at rest. Hence, in the case of model (iii), only a characteristics α' can be the discontinuity line.

(1) The problem has been considered partly in [1].

The determination of the velocity vectors starts from the point E at which the velocity V_0 is decomposed into the velocity along the discontinuity line and the relative slip velocity at the wall. Once the velocities on the discontinuity line are known, the mixed problem is formulated in the region between the line and the wall, and a characteristic problem is formulated in the remaining region. The hodographs obtained are shown in Figs. 1c, d, e, f.

All the solutions thus obtained ensure satisfaction of the first three conditions to be satisfied by the kinematic field. The fourth condition is satisfied only for the models (i) and (iv) in the system α'', β''; to prove the solutions to be incorrect in the case of the other models, it is sufficient to consider the region GHJ or BCL where the dissipation power is negative. The analysis of solutions for other values of φ and φ_w is now limited to the investigation of the influence of changes of φ_w at a constant φ.

In the case of models (i) and (iv) and the system α'', β'', the field of velocity characteristics at varying φ_w remains qualitatively similar and, as is easily verified, ensures the solution to be correct. For the remaining models, the form of the fields remains similar only in the case in which are satisfied the inequalities:

$$\varphi_w < \arctan\sin\varphi \qquad \text{for the model (ii)},$$

$$\varphi_w < \arctan\left(\frac{1+\sin^2\varphi}{\sin\varphi\cos\varphi}\right) \qquad \text{for models (iii), (iv).} \tag{3.2}$$

In the opposite case the characteristics starting from the wall do not run to the discontinuity lines (as in Figs. 1a, b) but to the boundary AB, Fig. 2. This situation leads, in the case of model (ii), to two possible velocity fields: the first may be assumed to be the field

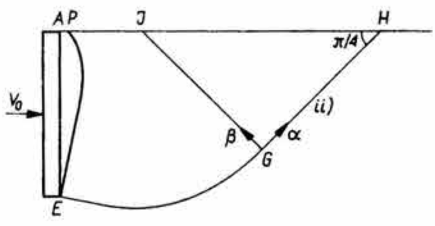

FIG. 2.

bounded by the discontinuity line EP coinciding with the characteristics β running from E to the boundary, and to the right of which the material is at rest. The second velocity field is obtained by assuming two discontinuity lines EGH and EP and a rigid motion (with velocity V_0) to the left from the EP-line. The latter case leads, however, to a negative dissipation power on the line EP. Considering the model (iii) and the non-satisfied inequality (3.2), the solution can not be found since the conditions on AE and EGH do not form any of the boundary value problems for the velocities (EP cannot be assumed to be a discontinuity line). In the case of the model (iv) and the system α', β', the remarks referring to model (ii) remain valid.

3.2. Ponderable medium

The analysis of kinematic solutions of ponderable media will be limited to the static solution, which assumes the existence of a radial stress distribution in the vicinity of the

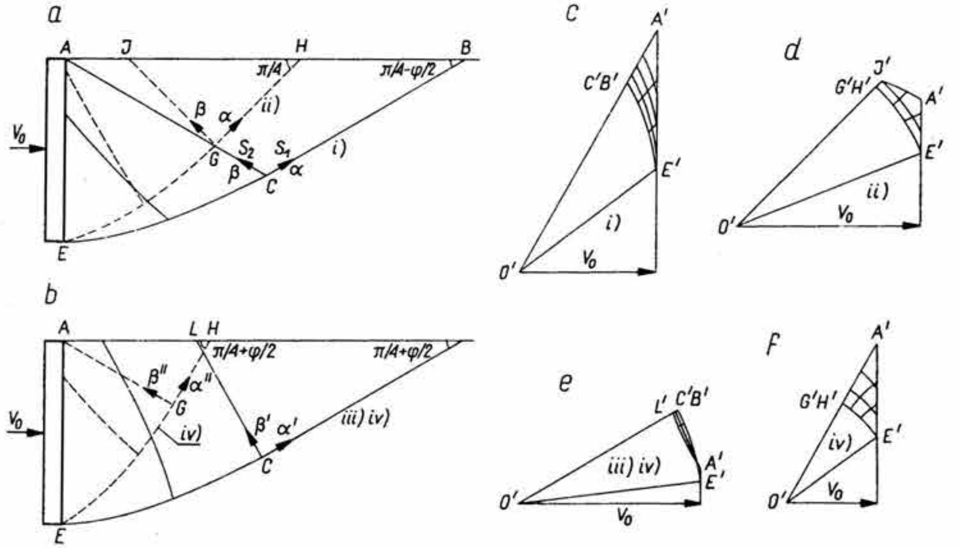

FIG. 3.

wall([2]). The velocity characteristics for the radial state are determined by the following formulae written in the polar coordinate system r, θ and the origin at A (Fig. 3a):

$$r = r_0 \exp\left(\int_0^{\theta} \cot(\psi+\omega)\, d\theta\right), \tag{3.3}$$

where

$$\omega = \pm\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \quad \text{for } \alpha, \beta \quad \text{for the model (i)},$$

$$\omega = \pm\frac{\pi}{4} \quad \text{for } \alpha, \beta \quad \text{for the model (ii)},$$

and

$$\left.\begin{aligned} \omega &= \pm\frac{\pi}{4} - \frac{\varphi}{2} \quad \text{for } \alpha', \beta' \\ \omega &= \pm\frac{\pi}{4} + \frac{\varphi}{2} \quad \text{for } \alpha'', \beta'' \end{aligned}\right\} \text{for models (iii), (iv)},$$

$\tan\psi$ being the inclination of the algebraically smallest principal stress to the radius r.

In Figs. 3a, b are demonstrated the fields of velocity characteristics for the models considered, constructed on the basis of the static solution derived by SOKOLOVSKI [12] for

([2]) Another type of solution assumes the existence of Prandtl-type singularities.

$\varphi = 30°$ and $\varphi_w = 15°$. Construction of the hodographs shown in Figs. 3c, d, e, f is similar to that in the case of a weightless medium. Among the solutions presented, only the solution referring to model (iii) does not satisfy the condition $(1.1)_1$. Generalization of that conclusion to other values of φ and φ_w requires the consideration of a number of solutions. It should be stressed, however, that the inequalities (3.2) defining the jump change of the form of the velocity field in models (ii), (iii) and (iv) still remain valid, together with the resulting conclusions concerning the correctness of solutions, presented in (3.1).

4. Indentation of a punch

For the problem of a rigid punch driven at a constant velocity V_0 into the surface of a plane granular half-space, two symmetric solution schemes are known in the static case corresponding to the schemes proposed by HILL and PRANDTL for metals. For a weightless medium loaded uniformly by a pressure applied to the surface and with no friction at the contact, they can be obtained analytically([3]). The kinematic solution for a weightless medium and the model (i) was considered by SHIELD [11].

4.1. Weightless medium

Hill's scheme ([4]). In the case ofa smooth punch, the static field consists of three elementary regions similar to the case of the retaining wall. In Fig. 4a are shown the characteristics of stress for $\varphi = 30°$, and the corresponding velocity characteristics — in Figs. 4a, b.

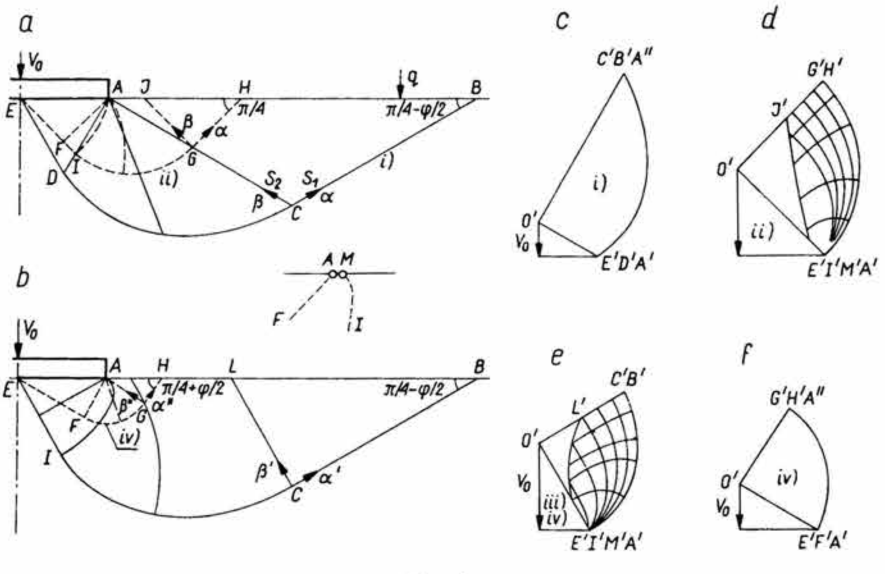

FIG. 4.

([3]) In Hill's scheme, the solution for $\varphi_w = \text{const} > 0$ is impossible.

([4]) The problem has been considered partly in [1].

The lines of discontinuity of velocities for each of the models considered is the characteristic running from E towards the boundary AB. From the condition that the material located below the discontinuity line remains at rest, we can determine the velocity of the material at E, and next — along the entire discontinuity line. The velocities in the deformation field are then determined by solving first the mixed problem, and next — the characteristic problem. Figs. 4c, d, e, f show the hodographs obtained in this manner.

The kinematic solutions presented above ensure the satisfaction of all conditions of the correct velocity field only in the case of models (i) or (iv) and for the system α'', β''.

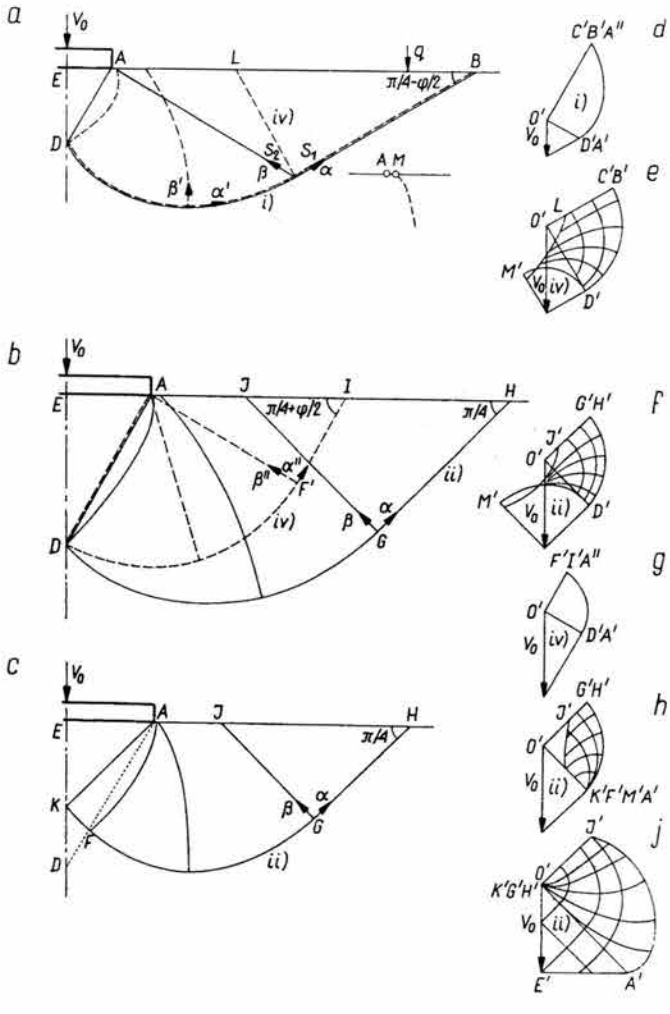

FIG. 5.

Prandtl's scheme. The static solution with Prandtl's scheme consists of two elementary regions only, ABC and CD, presented for $\varphi = 30°$ in Fig. 5a. In the region EAD a statical continuation can be found, though the region is not necessarily in a plastic state. Different velocity fields are found, depending on the assumption of plasticity or rigidity of the region EAD. Moreover, in the case of plasticity of the region EAD, the boundary condition on the line of contact with the punch, $V = V_0$, is not sufficient for a unique formulation of the boundary condition in the entire region of EAD. Similarly to the case of frictionless materials, there occurs a possibility of construction of various velocity fields (cf. [9]).

Let us first assume the region EAD to be rigid. The lines of discontinuity for the model (i) are AD and DCB. Starting from the point D, we can first determine the velocities at the discontinuity lines and then solve the characteristic problem in the region $ADCB$. The hodograph obtained is shown in Fig. 5d; the solution is correct. When the remaining models are considered, it should be observed that the velocity characteristics network is qualitatively similar to the network shown in Fig. 2. Consequently, the velocities referred to the model (iii) cannot be determined. Considering the models (ii) and (iv), various forms of the velocity field may be assumed, depending upon the choice of the lines of velocity discontinuities. In Fig. 5b is presented the field in the case of model (ii) under the assumption of two discontinuity lines MD and DGH, and Fig. 5f — the hodograph. The dissipation power on the lines MD and DGH is now positive, while in GHJ it is negative. Qualitatively similar solutions are obtained for the model (iv) by assuming the system α', β' and two discontinuity lines, Fig. 5a, e; a correct solution for the model (iv) will be obtained using the system α'', β'', Fig. 5b, g.

When a plastic state is assumed in the region EAD, let us consider two of the many possible velocity distributions on AE: (a) material particle velocity is constant on AE and equals V_0; (b) the vertical velocity equals V_0, while the horizontal velocity increases linearly with the distance from the symmetry axis. In both cases the network of velocity characteristics is the same, and the differences concerning the discontinuity lines are: for (a) the discontinuity lines are KA and KGH; for (b) no discontinuity line (strong discontinuity) is present. From the hodographs corresponding to the cases mentioned, Fig. 5h, j, it follows that a correct solution is furnished by the assumption (b).

4.2. Ponderable medium

As regadrs this type of medium, let us consider only the static solution corresponding to the Prandtl scheme. The form of solution, which may be established in a numerical way only, depends on the value of φ, on the specific weight γ, on the surface load q, and on the width b of the punch. Let us confine ourselves to the sole statical solution given in [2] for $\varphi = 38°$, $\gamma = 1.66$ G/cm^3 and $q/b = 0.166$ G, and to the kinematic solution for the model (ii). The velocity field for the model (i) was considered in [2]; it fulfills all the requirements of a correct solution.

The stress field presented in Fig. 6a consists of the region ABC of rectilinear characteristics and the region ACD where the both families of characteristics are curvilinear. Assuming the region EAD to be rigid, we obtain the network of velocity characteristics presented in Fig. 6b; of two possible forms of the field, we select the case in which MD and DGH are the discontinuity lines. The corresponding hodograph is presented in Fig. 6c which indicates that the dissipation power in GHJ is negative.

If we assumed a plastic state in EAD, the stress characteristics in the region would approach the contact line at various angles, corresponding to a certain distribution of φ_w. If φ_w equals the right-hand side of Eq. (3.2), then one of the velocity characteristics is tangent to the contact. Such a situation occurs in the case of the static solution consider-

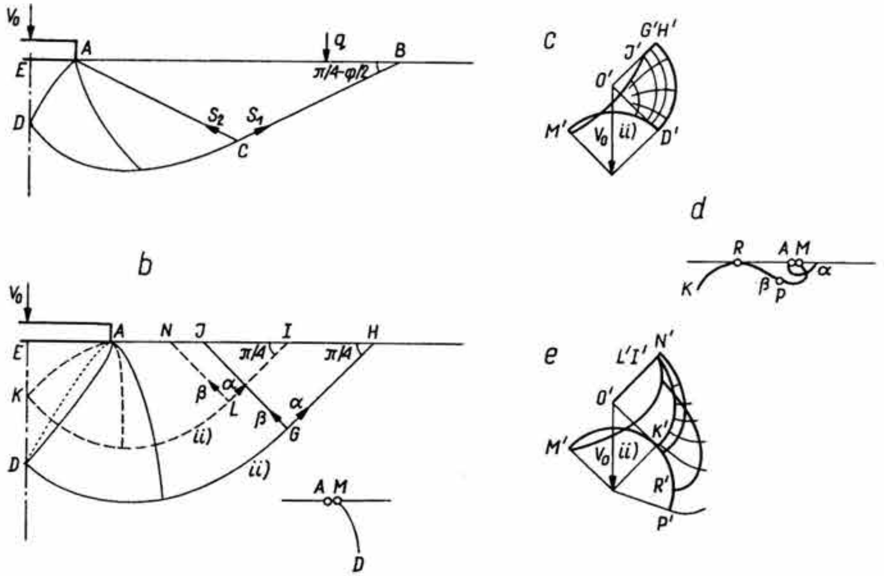

FIG. 6.

ed, yielding a very complex course of characteristics in the vicinity of point A, Fig. 6d. Assuming the existence of a rigid region under the punch, bounded by the discontinuity line coinciding with the characteristics tangent to the contact line at R, a rather complex hodograph is obtained, Fig. 6e, and a negative dissipation power in a part of the region.

A detailed analysis of other velocity fields under the assumption of a plastic state in EAD is difficult owing to the complex form of characteristics in the vicinity of point A. Estimation of the correctness of these solutions for other values of the parameters requires the construction of numerous static fields. Certain information may be furnished by the solutions presented by STUTZ [13] in which the analysis of the dissipation power indicates that — depending on the parameters of the problem — both correct and incorrect results are obtained.

5. Flow through a channel

Let us consider the flow of a weightless material through a parallel-walled channel closed by a bottom containing a hole, Fig. 7a. The material velocity is assumed to be equal to V_0 at a sufficiently large distance from the hole. The static solution in the vicinity of the hole depends on the ratio of dimensions of the channel and the hole, and on the friction at the walls. Let us assume that the channel is perfectly smooth and its dimensions are such that, for a given value of φ, the static solution corresponds to the characteristics

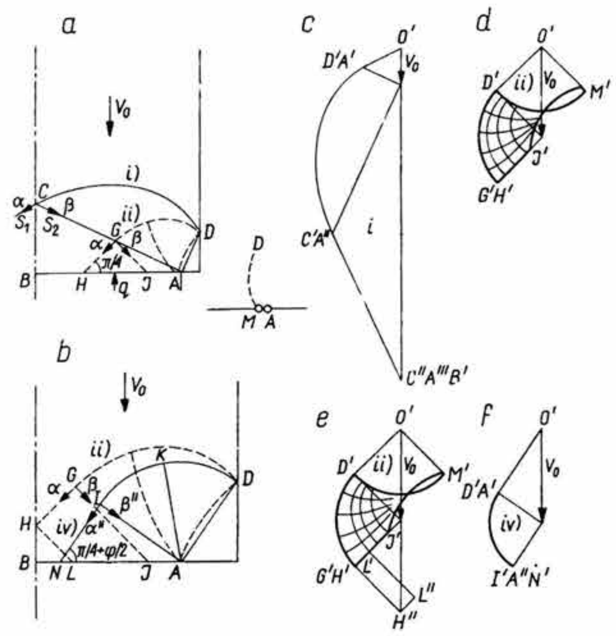

FIG. 7.

network shown(5) in Fig. 7a (in the case of $\varphi = 40°$). This solution, identical with the region $ABCD$ from the punch problem, requires the action of a certain pressure on AB in the case of cohesionless media.

In the case of model (i), for which the discontinuity lines are AD, DC, CA, the hodograph presented in Fig. 7c is obtained. The solution is correct. If the model (ii) is considered and the inequality holds

$$(1+\sin\varphi)\exp\left(\frac{\pi}{2}\tan\varphi\right) > 2, \tag{5.1}$$

the velocity field does not intersect the axis of symmetry, Fig. 7a, and MD, DGH are the discontinuity lines. In the opposite case, HL is the additional discontinuity line, Fig. 7b. The corresponding hodographs are shown in Fig. 7d, e. In the both cases the dissipation power in GJH is negative.

For the model (iii), the field of velocity characteristics cannot be constructed, since the system α'', β'' is not admissible, and the system α', β' — owing to symmetry of the problem — leads to a contradiction in selecting the characteristics in the region ABC. An analogous inconsistency is found in discussing the model (iv) and system α'', β'' if the field intersects the symmetry axis, which is true provided that

$$\left(1+\tan\frac{\varphi}{2}\right)\exp\left[\frac{\pi}{2}\left((\tan\varphi-\tan\frac{\varphi}{2}\right)\right] < 2. \tag{5.2}$$

The inequality (5.2) not satisfied, the velocity field and the hodograph presented in Fig. 7b, f are obtained; the solution is correct.

(5) A continuation into the rigid region exists in this case.

6. Conclusions

The analysis presented of the kinematic solutions based on static solutions demonstrates that for the assumed values of physical parameters, the models (ii) and (iii) yield in several problems certain thermodynamically incorrect velocity fields. The models whose velocity characteristics do not coincide with the stress characteristics may lead to very complicated (and difficult for exact numerical determination) fields of characteristics.

The remarks presented here do not yield any conclusions concerning the correctness of the models. The physical relations for the models to the velocities sought for, and hence the field of characteristics is independent of the kinematical boundary conditions. Incorrectness of the solutions is then the result of inadmissible — in view of Eq. (1.1) — kinematical boundary conditions. The problem was considered by RYCHLEWSKI [10] in the case of frictionless media and the punch indentation, where the criteria of admissible boundary conditions were formulated.

The conclusions of the present paper are of great importance for the experimental verification of the models considered, consisting in comparing the experimental and theoretical velocity fields in boundary value problems. The cases of retaining walls and of the punch indentation are the most fundamental and frequently considered. In these considerations, it is always necessary to make sure whether the theoretical solution corresponding to the physical and geometrical parameters appearing in the experiments is correct.

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