Archives of Mechanics • Archiwum Mechaniki Stosowanej • 24, 5-6, pp. 819-825, Warszawa 1972

Thermoelastic equations for ferromagnetic bodies

H. PARKUS (VIENNA)

THE BASIC equations of an elastic, ferromagnetic, heat and current conducting body are derived directly from the first and second law of thermodynamics.

Podstawowe równania ośrodków sprężystych, ferromagnetycznych, przewodzących ciepło i prąd elektryczny, wyprowadzono bezpośrednio z pierwszej i drugiej zasady termodynamiki.

Из первого и второго принципов термодинамики выводятся основные уравнения упругих ферромагнитных материалов, обладающих свойствами тепло и электропроводности.

1. Introduction

THE GENERAL equations for electrically nonconducting ferromagnetic bodies have been given by TIERSTEN [1] and, for the isothermal case, by BROWN [2]. Tiersten's equations are presented in an extremely complicated form, mainly for the reason that the principle of frame indifference is incorporated right from the start. It is the purpose of the present paper to give a different derivation of the equations which at the same time, generalizes them to include electric conduction. The results appear in a form similar to that given by BROWN which is considered more appropriate for engineering applications.

Throughout the paper the international MSK system of units will be used.

2. The first and second law

The first law of thermodynamics, i.e., the energy balance for a ferromagnetic body of instantaneous volume V may be written as

$$(2.1)\qquad \frac{d}{dt}\int\limits_V \left[\varrho\left(\frac{v^2}{2}+U\right)+U_e\right]dV = \int\limits_V (\varrho r+f_i v_i)\,dV + \oint\limits_{\partial V}\left[\tau^*_{ij}v_j + a_{ij}\varrho\frac{d\mathcal{M}_j}{dt} - Q_i - (\mathbf{E}\times\mathbf{H})_i + U_e v_i\right]n_i\,dA.$$

The left-hand side represents the time rate of the total energy (kinetic, internal and electromagnetic) enclosed in V. The terms on the right-hand side are: heat production by the heat source distribution, rate of work of volume forces f_i, of surface forces $\tau^*_{ij}n_i$ and of "exchange forces" $a_{ij}n_i$, transport of heat $-Q_i n_i$ and of electromagnetic energy $-(\mathbf{E}\times\mathbf{H})_i n_i$ through the surface into the body (n_i positive outwards) and, finally, the influx of electromagnetic energy $U_e v_n$ due to the motion of the body through the external electromagnetic field.

The magnetization vector density $\mathcal{M}_i$ is introduced here with reference to the unit of mass

$$\varrho \mathcal{M}_i = M_i. \tag{2.2}$$

The, as yet unknown, stress tensor τ^*_{ij} contains the mechanical stress tensor τ_{ij} plus additional magnetic effects. The exchange tensor a_{ij} covers the exchange forces between the mechanical continuum and the electronic spin continuum. It, too, is unknown.

The motion of a particle in an inertial frame will be described by its spatial coordinates (Eulerian coordinates)

$$x_i = x_i(\mathbf{X}_A, t), \quad i = 1, 2, 3, \tag{2.3}$$

where $X_A, A = 1, 2, 3$ represents the material coordinates (Lagrangian coordinates) which initially coincide with x_i,

$$x_i(\mathbf{X}_A, 0) = X_i. \tag{2.4}$$

The deformation gradient

$$f_{iA} := x_{i,A} \tag{2.5}$$

serves as a strain measure. The particle velocity v_i is given by

$$v_i = dx_i/dt. \tag{2.6}$$

The second law of thermodynamics is assumed in the form of the *Clausius-Duhem* inequality as

$$\frac{d}{dt}\int\limits_m S\,dm \geqslant \int\limits_m \frac{r}{T}dm - \oint\limits_{\partial V} \frac{Q_i n_i}{T} dA, \tag{2.7}$$

where S denotes entropy per unit mass and T is absolute temperature.

Applying now Gauß'theorem to Eq. (2.1) and remembering that

$$\frac{d}{dt}\int\limits_V U_e\,dV - \oint\limits_{\partial V} U_e v_n\,dA = \int\limits_V \frac{\partial U_e}{\partial t} dV, \tag{2.8}$$

one obtains the differential equation form of the first law as

$$\varrho \frac{d}{dt}\left(\frac{v^2}{2} + U\right) + \frac{\partial U_e}{\partial t} = \varrho r + f_i v_i + \frac{\partial}{\partial x_j}\left[\tau^*_{ji} v_i + a_{ji}\varrho \frac{d\mathcal{M}_i}{dt} - Q_j - (\mathbf{E}\times\mathbf{H})_j\right]. \tag{2.9}$$

Similarly, for the second law from Eq. (2.7),

$$\varrho T \frac{dS}{dt} \geqslant \varrho r - Q_{i,i} + \frac{Q_i}{T} T_{,i}. \tag{2.10}$$

The free energy F per unit of mass, defined by

$$F = U - TS \tag{2.11}$$

of the elastic, ferromagnetic body, is assumed as a function of strain, magnetization vector and its gradient, and of temperature:

$$F = F(x_{i,A}, \mathcal{M}_i, \mathcal{M}_{i,j}, T). \tag{2.12}$$

We have then

$$(2.13)\qquad \varrho\frac{dF}{dt}=\varrho\left(\frac{\partial F}{\partial x_{i,A}}\frac{dx_{i,A}}{dt}+\frac{\partial F}{\partial \mathcal{M}_i}\frac{d\mathcal{M}_i}{dt}+\frac{\partial F}{\partial \mathcal{M}_{i,j}}\frac{d\mathcal{M}_{i,j}}{dt}+\frac{\partial F}{\partial T}\frac{dT}{dt}\right)$$

and

$$(2.14)\qquad \varrho\frac{\partial F}{\partial x_{i,A}}\frac{dx_{i,A}}{dt}=\varrho\frac{\partial F}{\partial x_{i,A}}v_{i,A}=\varrho\frac{\partial F}{\partial x_{i,A}}v_{i,j}x_{j,A}$$

$$=\frac{\partial}{\partial x_j}\left(\varrho\frac{\partial F}{\partial x_{i,A}}x_{j,A}v_i\right)-\frac{\partial}{\partial x_j}\left(\varrho\frac{\partial F}{\partial x_{i,A}}x_{j,A}\right)v_i,$$

$$(2.15)\qquad \varrho\frac{\partial F}{\partial \mathcal{M}_{i,j}}\frac{d\mathcal{M}_{i,j}}{dt}=\frac{\partial}{\partial x_j}\left(\varrho\frac{\partial F}{\partial \mathcal{M}_{i,j}}\frac{d\mathcal{M}_i}{dt}\right)-\frac{\partial}{\partial x_j}\left(\varrho\frac{\partial F}{\partial \mathcal{M}_{i,j}}\right)\frac{d\mathcal{M}_i}{dt}.$$

For the electromagnetic energy, we introduce the expression

$$(2.16)\qquad U_e=\frac{1}{2}(\varepsilon_0E^2+\mu_0H^2),$$

where $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic field intensity, respectively, and ε_0 and μ_0 are dielectric constant and permeability in vacuum, respectively. Then, making use of Maxwell's equations,

$$(2.17)\qquad \nabla\times\mathbf{H}=\mathbf{j}+\frac{\partial\mathbf{D}}{\partial t},\qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

and of the constitutive relations for a moving, non-polarized body,

$$(2.18)\qquad \mathbf{D}=\varepsilon_0\mathbf{E},\qquad \mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}),$$

we find

$$(2.19)\qquad \frac{\partial U_e}{\partial t}=\varepsilon_0E_i\frac{\partial E_i}{\partial t}+\mu_0H_i\frac{\partial H_i}{\partial t}=-\nabla\cdot(\mathbf{E}\times\mathbf{H})-j_iE_i-\mu_0H_i\frac{\partial M_i}{\partial t}$$

$$=-\nabla\cdot(\mathbf{E}\times\mathbf{H})-j_iE_i-\mu_0H_i\varrho\frac{d\mathcal{M}_i}{dt}+\mu_0(\varrho H_i\mathcal{M}_iv_k)_{,k}-\mu_0\varrho\mathcal{M}_iH_{i,k}v_k.$$

The Nabla operator is defined as $\nabla_i(\cdot)=(\cdot)_{,i}$, and $\mathbf{j}$ represents the electric current density. In writing Eq. (2.19), the continuity equation

$$(2.20)\qquad \frac{\partial\varrho}{\partial t}+\frac{\partial}{\partial x_i}(\varrho v_i)=0$$

as well as the relation

$$(2.21)\qquad \frac{d\mathcal{M}_i}{dt}=\frac{\partial\mathcal{M}_i}{\partial t}+v_j\mathcal{M}_{i,j}$$

have been utilized.

Another constitutive equation, *Ohms law*, will also be needed. For an electrically isotropic body this law reads

$$(2.22)\qquad \mathbf{j}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}-\varkappa\nabla T),$$

where σ represents the conductivity. For an anisotropic medium, σ generalizes to a symmetric tensor.

Multiplication of both sides of Eq. (2.22) by $\mathbf{j}/\sigma$ yields

$$\frac{j^2}{\sigma} = \mathbf{j}\cdot\mathbf{E} - (\mathbf{j}\times\mathbf{B})\cdot\mathbf{v} - \varkappa\mathbf{j}\cdot\nabla T. \tag{2.23}$$

Substitution of Eqs. (2.19) and (2.23), together with Eqs. (2.11), (2.13), (2.14) and (2.15) into Eq. (2.9) and inequality (2.10) renders

$$\begin{aligned}\left\{\varrho\frac{dv_i}{dt} - f_i - \frac{\partial}{\partial x_j}\left(\varrho\frac{\partial F}{\partial x_{i,A}}x_{j,A}\right) - (\mathbf{j}\times\mathbf{B})_i - \mu_0\varrho\mathscr{M}_k H_{k,i}\right\}v_i \\ + \varrho\left\{\frac{\partial F}{\partial\mathscr{M}_i} - \frac{1}{\varrho}\frac{\partial}{\partial x_j}\left(\varrho\frac{\partial F}{\partial\mathscr{M}_{i,j}}\right) - \mu_0 H_i\right\}\frac{d\mathscr{M}_i}{dt} + \varrho\left\{\frac{\partial F}{\partial T} + S\right\}\frac{dT}{dt} \\ + \frac{\partial}{\partial x_j}\left\{\left(\varrho\frac{\partial F}{\partial x_{i,A}}x_{j,A} - \tau^*_{ji} + \mu_0\varrho\mathscr{M}_s H_s\delta_{ij}\right)v_i + \varrho\left(\frac{\partial F}{\partial\mathscr{M}_{i,j}} - a_{ji}\right)\frac{d\mathscr{M}_i}{dt} + Q_j\right\} \\ + \varrho T\frac{dS}{dt} - \frac{j^2}{\sigma} - \varkappa\mathbf{j}\cdot\nabla T - \varrho r = 0\end{aligned} \tag{2.24}$$

and

$$\{\ldots\}v_i + \varrho\{\ldots\}\frac{d\mathscr{M}_i}{dt} + \varrho\{\ldots\}\frac{dT}{dt} + \frac{\partial}{\partial x_j}\{\ldots\} - \frac{j^2}{\sigma} - \varkappa\mathbf{j}\cdot\nabla T + \frac{Q_i}{T}T_{,i} \leqslant 0. \tag{2.25}$$

3. The basic equations

A number of conclusions may now be drawn from the first and second law in the form of relations (2.24) and (2.25). First, we note that the coefficient of the temperature rate dT/dt must vanish. This yields the well-known thermodynamic relation

$$S = -\frac{\partial F}{\partial T}. \tag{3.1}$$

We now apply the "principle of frame indifference" [4] by first replacing v_i by $v_i + c_i$ (rigid translation) and then $v_{i,j}$ by $v_{i,j} + \omega_{ij}$ (rigid rotation). It follows that those terms which have v_i as a factor must vanish. This renders the two equations

$$\varrho\frac{dv_i}{dt} = f_i + \tau_{ji,j} + (\mathbf{j}\times B)_i + \mu_0 M_k H_{k,i} \tag{3.2}$$

and

$$\tau^*_{ij} = \tau_{ij} + \mu_0 M_s H_s\delta_{ij}, \tag{3.3}$$

where τ_{ik}, defined by

$$\tau_{ik} = \varrho x_{i,A}\frac{\partial F}{\partial x_{k,A}} \tag{3.4}$$

represents the Cauchy stress tensor. Equation (3.2) represents the *equation of motion*, while Eq. (3.3) determines τ^*_{ij} in the energy Eq. (2.1).

Next we consider the terms containing $d\mathcal{M}_i/dt$ as a factor. They must vanish. Now, the magnetic equation of angular momentum(1) for the magnetic moment $\mathcal{M}$ per unit mass reads

$$\frac{d\mathcal{M}}{dt} = \gamma\mathcal{M}\times\mathbf{H}_{\rm eff}, \tag{3.5}$$

where γ is a constant and $\mathbf{H}_{\rm eff}$ represents the "effective" magnetic field(2). From a comparison of this equation with the second term of (2.24) anb (2.25) we conclude that(3)

$$(H_{\rm eff})_i = H_i - \frac{1}{\mu_0}\left[\frac{\partial F}{\partial \mathcal{M}_i} - \frac{1}{\varrho}\frac{\partial}{\partial x_j}\left(\varrho\frac{\partial F}{\partial \mathcal{M}_{i,j}}\right)\right]. \tag{3.6}$$

Finally, if we put

$$a_{ji} = \frac{\partial F}{\partial \mathcal{M}_{i,j}}, \tag{3.7}$$

the second term with $d\mathcal{M}_i/dt$ as a factor will vanish. This determines the exchange tensor a_{ij}.

After collecting the remaining terms in Eq. (2.24), we arrive at the *equation of heat conduction*

$$Q_{i,i} = \varrho r + \frac{j^2}{\sigma} + \varkappa\mathbf{j}\cdot\nabla T - \varrho T\frac{dS}{dt}, \tag{3.8}$$

where expression (3.1) for the entropy has to be substituted.

To Eq. (3.8) the *law of heat conduction* has to be adjoined. If, for instance, *Fourier's* law is adopted in the form valid for a thermally isotropic body(4),

$$Q_i = -kT_{,i} + \varkappa T j_i, \tag{3.9}$$

one obtains, after substitution into Eq. (3.8), assuming $k = \text{const}$ and using $\nabla\cdot\mathbf{j} = 0$ from Maxwell's equations,

$$k\nabla^2 T = \varrho T\frac{dS}{dt} - \varrho r - \frac{j^2}{\sigma} + Tj\cdot\nabla\varkappa. \tag{3.10}$$

The term j^2/σ represents the *Joule heat production*, while the last term exhibits the *Thomson effect*. The coefficient $\varkappa$ will, in general, be temperature-dependent, $\nabla\varkappa = (d\varkappa/dT)\nabla T$.

Differential Eq. (3.2) has to be supplemented by boundary conditions. To this effect, the *Maxwell stress tensor* m_{ij} is introduced as(5)

$$m_{ji} = H_i B_j - \frac{1}{2}\mu_0 H^2\delta_{ij} \tag{3.11}$$

(1) See [2], p. 85.

(2) After multiplication of both sides of Eq. (3.5) by $\mathcal{M}$ we get $d\mathcal{M}^2/dt = 0$, and hence $\mathcal{M}^2 = \text{const}$. Eq. (3.5), therefore, implies magnetic saturation.

(3) See [2], p. 84.

(4) See [5], § 25. An additional term appears in [5] which, however, is already included here in Eq. (3.8).

(5) The Maxwell stress tensor is used here solely as an auxiliary quantity and no deeper meaning is ascribed to it.

and Eq. (3.2) is rewritten in the form

$$(3.12)\qquad (\tau_{ji}+m_{ji})_{,j}-\left(\frac{\partial \mathbf{D}}{\partial t}\times \mathbf{B}\right)_i = \varrho\frac{dv_i}{dt}.$$

As usual, the displacement current $\partial\mathbf{D}/dt$ will be neglected. Now, if v denotes the absolute velocity in the direction of its normal of a surface of discontinuity moving through the body, and if v_n is the velocity in the same direction of the corresponding body particle, the following jump condition([6]) follows from Eq. (3.12),

$$(3.13)\qquad [\tau_{ji}+m_{ji}]n_j = [\varrho(v_n-v)v_i],$$

where $[\varphi]:=\varphi^+-\varphi^-$. To Eq. (3.13) the condition of continuity has to be added,

$$(3.14)\qquad \varrho[(v_n-v)] = 0.$$

If the surface of discontinuity coincides with the *surface of the body*, we have $v = v_n$ and $\tau^+_{ji}n_j = p_i$, where p_i is the external surface load. Remembering, furthermore, that the magnetic field intensity experiences a jump acrosse the body surface([7]) of magnitude

$$(3.15)\qquad H^+_i-H^-_i = M_n n_i,$$

while the normal component B_n of $\mathbf{B}$ remains continuous, one obtains, utilizing Eq. (3.11),

$$[m_{ji}]n_j = B_n(H^+_i-H^-_i)-\frac{\mu_0}{2}(H^+_s-H^-_s)(H^+_s+H^-_s)n_i$$

$$= B_nM_nn_i-\frac{\mu_0}{2}M_nn_s(2H^+_s-M_nn_s)n_i = M_n\left(B_n-\mu_0H^+_n+\frac{\mu_0}{2}M_n\right)n_i$$

(no summation over index n!),

but $B_n = B^+_n = \mu_0H^+_n$. Hence, Eq. (3.13) finally renders the *boundary condition*

$$(3.16)\qquad \tau_{ji}n_j = p_i+\frac{\mu_0}{2}M^2_{n_l}n_i.$$

In addition to body forces $\mu_0M_jH_{j,i}$ and surface forces $\mu_0M^2_nn_i/2$, the magnetized body is also exposed to a distribution of couples as a consequence of the nonsymmetry of the stress tensor:

$$(3.17)\qquad \tau_{ij}-\tau_{ji} = \varrho\left(\frac{\partial F}{\partial\mathscr{M}_i}\mathscr{M}_j-\frac{\partial F}{\partial\mathscr{M}_j}\mathscr{M}_i+\frac{\partial F}{\partial\mathscr{M}_{i,A}}\mathscr{M}_{j,A}-\frac{\partial F}{\partial\mathscr{M}_{j,A}}\mathscr{M}_{i,A}\right).$$

A thorough discussion of these effects is given in [2].

4. Objectivity

The constitutive equations as obtained in the preceding section are not objective, i.e., they are not invariant under an orthogonal transformation of coordinates x_i. In order to make them objective the deformation gradient $x_{i,A}$ would have to be replaced by a differ-

([6]) See, for instance [3], p. 503 ff.

([7]) See [2], p. 57.

ent strain measure in the expression (2.12) for the free energy. The same holds true for the magnetization vector $\mathscr{M}_i$. Details of the procedure may, for instance, be found in [2], p. 69, and [6], p. 44.

References

1. H. F. TIERSTEN, *Coupled magnetomechanical equations for magnetically saturated insulators*, J. Math. Phys., 5, 1298, 1964.
2. W. F. BROWN, Jr., *Magnetoelastic interactions*, Springer-Verlag, Berlin-Heidelberg-New York 1966.
3. C. TRUESDELL and R. A. TOUPIN, *The classical field theories*, Encyclopedia of Physics (S. FLÜGGE, editor) 3/1, Springer-Verlag, Berlin 1960.
4. A. E. GREEN and R. S. RIVLIN, *On Cauchy's equations of motion*, Z. ang. Math. Phys., **15**, 290, 1964.
5. L. D. LANDAU und E. M. LIFSCHITZ, *Elektrodynamik der Kontinua*, Akademie-Verlag, Berlin 1967.
6. C. TRUESDELL and W. NOLL, *The non-linear field theories of mechanics*, Encyclopedia of Physics (S. FLÜGGE, editor), 3/3, Springer-Verlag, Berlin 1965.

TECHNISCHE HOCHSCHULE WIEN

Received May 12, 1972