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Stochastic Green functions in elasticity of composites

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THIS PAPER deals with stochastic Green functions in elasticity of composites. On the basis of the general equation of the Green function for displacements in an infinite medium with random multiphase structure, a stochastic operator is derived. The solution is presented in the form of a Neumann type series. The correlation moment functions of arbitrary order for the Green function are given, enabling us to determine the desired statistical characteristics of the displacement field, and, consequently, of strains, stresses, etc. under prescribed external loads.

Praca zajmuje się stochastycznymi funkcjami Greena w sprężystości ośrodków złożonych. Na podstawie ogólnego równania funkcji Greena dla przemieszczeń w ośrodku nieograniczonym o strukturze losowej wielofazowej wyprowadzono operator stochastyczny i przedstawiono rozwiązanie w postaci szeregu typu Neumanna. Wyznaczono funkcje korelacyjne dowolnego rzędu dla funkcji Greena, co umożliwia obliczenie odpowiednich charakterystyk probabilistycznych pola przemieszczeń, a stąd odkształceń, naprężeń itd. przy zadanych obciążeniach zewnętrznych.

Работа посвящена определению стохастических функций Грина в теории упругости композитных сред. Исходя из общего уравнения функции Грина для перемещений в неограниченной среде с вероятностной многофазной структурой, выведен стохастический оператор и дано его решение в виде ряда Неймана. Для функции Грина выведены корреляционные функции произвольного порядка, что даёт возможность рассчитать соответствующие вероятностные характеристики для поля перемещений, а отсюда и для полей деформаций, напряжений и т. п. при заданных внешних нагрузках.

THE USUAL object of classical elastic calculations is to determine stresses and strains at any internal point of the body under prescribed external loads. This can be achieved provided that all data, in particular elastic properties of the material, are known in a deterministic sense. Unfortunately, this does not occur in composite (grain-composed, multiphase) materials where the internal geometry is random. Consequently, we may hope at most to determine statistic quantities characterizing stress and strain fields (e.g. mean values, variances etc.) at a given point or, more generally, intercorrelations of arbitrary order at different points. In what follows, we assume that external loads as well as geometric (overall) form and boundary conditions are given deterministically while the material is random inhomogeneous. Our direct objective will consist in deriving an inverse stochastic operator which takes the form of a stochastic Green function, and in obtaining, by means of the latter, statistical (averaged) quantities, mentioned above, under arbitrary loads.

The literature on the subject is fairly extensive, it is for the most part concerned, however, with the problem of macroscopic material constants (elastic tensor), where special solution techniques have been developed (e.g. variational principles and bounding inequalities). Twofold restrain is inherent in those results: (1) the elastic macro-tensor cannot be defined as independent of position unless the medium is infinite and stochastically homogeneous; (2) by means of the elastic macro-tensor we are able to determine mean stress and strain only, which will not always be found sufficient. It is obvious that a general solution must provide methods of calculation of correlation functions of any order for the displacement field under prescribed external loads. By the well-known procedure of taking derivatives of these functions, we arrive at the correlation functions for strains, stresses, and so on. The correlation method in the elastic theory of composites was used first by LIFSHITZ and ROZENZWEIG [5] and presented in an elegant form by BROWN [2] (for the second-order tensor field appearing in electric permittivity); some generalizations for elastic problems, similarly making use of polarisation fields, have been given by the present author — e.g. [4]. Much has been done in this field by the Russian school (cf. [6]). Stochastic Green functions appear as a part of the theory of stochastic equations (for an essential review of methods cf. ADOMIAN [1]). Elasticity of composites is, of course, only one of the possible applications.

Take an inhomogeneous infinite (in a deterministic sense, for the time being), elastic body and form the condition of equilibrium

(1) $$\nabla \cdot \sigma = f$$

and the generalised Hooke's law

(2) $$\sigma = C\varepsilon = C\nabla u$$

deduce generalised Lamé equation

(3) $$(\nabla \cdot C\nabla)u = Lu = f,$$

where σ, ε are stress and strain tensors, u, f displacement and body force vectors, C elastic tensor (of the 4-th rank), and multiplication is of order 2($C\varepsilon$ may also be regarded as a matrix-upon-vector operation in the 6-dimensional E-space). In Cartesian coordinates (3) takes the form

(4) $$\partial_j C_{ijkl} \partial_l u_k = L_{ik} u_k = f_i.$$

Consequently, the displacement Green function G of (4) is the solution of the equation

(5) $$\partial_j C_{ijkl} \partial_l G_{km} = \delta_{im} \delta(x-\xi),$$

where δ_{ik} and $\delta(x-\xi)$ are Kronecker and Dirac deltas. On multiplying both sides of (5) by $f_m(\xi)$ and integrating with regard to ξ, we obtain, since L_{ik} works on x only,

$$L_{ik} \int G_{km}(x, \xi) f_m(\xi) d\xi = \int \delta_{im} \delta(x-\xi) f_m(\xi) d\xi = f_i(x),$$

where $d\xi = d\xi^1 d\xi^2 d\xi^2$; hence

(6) $$u_k(x) = \int G_{km}(x, \xi) f_m(\xi) d\xi.$$

For the infinite inhomogeneous medium, $G_{km}(x, \xi) = G_{km}(x-\xi)$ is the fundamental solution, to be found from the system of 3 Eqs. (5) (respective to index i).

In order to find G, we decompose the tensor C and, consequently, the operator L in the equation (5), as well as the function G in two parts, following the scheme:

$$(L_{ik}^0 + L'_{ik})(G_{km}^0 + G'_{km}) = \delta_{im} \delta$$

or, explicitly:

(7) $$L_{ik}^0 G_{km}^0 + L_{ik}^0 G'_{km} + L'_{ik} G_{km}^0 + L'_{ik} G'_{km} = \delta_{im} \delta$$

and assign following interpretation to the above quantities. L_0 is the operator for a homogeneous reference medium and G^0 the solution of the equation

$$L^0_{ik} G^0_{km} = \delta_{im} \delta \tag{8}$$

— i.e., G^0 is the Green tensor of classical elasticity which in the sequel will be assumed as known. Often, we shall choose $L^0 = \overline{L}$, the latter being the operator for the mean tensor C; thus, L might be regarded as a fluctuation operator. In argument, however, we sometimes prefer to make it correspond to a given (for instance, the "weakest") phase and to avoid negative components of C. Taking (8) into account, we obtain from (7):

$$L^0_{ik} G'_{km} = -L'_{ik} G^0_{km} - L'_{ik} G'_{km}, \tag{9}$$

where $L'_{ik} G^0_{km}$ is a known function, since G^0_{km} is known and, consequently, (9) is a differential equation for G'. If we assume, for the time being, the right-hand side of the equation to be known, we can, formally, write the solution in the form:

$$G'_{km}(x, \xi) = -\int G^0_{kp}(x, \xi) L'^{\eta}_{pq} G^0_{qm}(\eta, \xi) d\eta - \int G^0_{kp}(x, \eta) L'_{pq} G'_{qm}(\eta, \xi) d\eta, \tag{10}$$

L' operating in η. This is an integro-differential equation respective to G'. For the sake of brevity, we introduce the notation:

$$\Gamma_{kq} = -\left(\int d\eta\, G^0_{kp} L'_{pq}\right) = -\left(\int d\eta\, G^0_{kp}(x, \eta)\, \partial_s C'_{psqt}(\eta) \partial_t\right),$$

where $\partial_s = \partial/\partial\eta^s$, $\partial_t = \partial/\partial\eta^t$, $C' = C - C^\circ$, and rewrite (10) as follows:

$$G'_{km} = \Gamma_{kq} G^0_{qm} + \Gamma_{kq} G'_{qm}.$$

Presupposing that the solution of this equation may be expanded in a Neumann type series (mathematical assumptions will be discussed later), we obtain:

$$G'_{km} = (\Gamma_{kq} + \Gamma_{kp_1}\Gamma_{p_1 q} + \Gamma_{kp_1}\Gamma_{p_1 p_2}\Gamma_{p_2 q} + \ldots) G^0_{qm} \tag{11}$$

or, more extensively,

$$\begin{aligned} G'_{km}(x, \xi) = {} & \int G^0_{kr}(x, \xi)\, \partial_s C'_{rsqt}(\eta) \partial_t G^0_{qm}(\eta, \xi) d\eta \\ & + \iint G^0_{kr}(x, \eta_1)\, \partial_s C'_{rsp_1t_1}(\eta_1)\, \partial_{t_1} G^0_{p_1 r_1}(\eta_1, \eta) \partial_{s_1} C'_{r_1 s_1 qt}(\eta) \partial_t G^0_{qm}(\eta, \xi) d\eta_1 d\eta \\ & + \iiint G^0_{kr}(x, \eta_1)\, \partial_s C'_{rsp_1t_1}(\eta_1)\, \partial_{t_1} G^0_{p_1 r_1}(\eta_1, \eta_2)\, \partial_{s_1} C'_{r_1 s_1 p_2 t_2}(\eta_2)\, \partial_{t_2} G^0_{p_2 r_2}(\eta_2, \eta) \times \\ & \times \partial_{s_2} C'_{r_2 s_2 qt}(\eta)\, \partial_t G^0_{qm}(\eta, \xi) d\eta_1 d\eta_2 d\eta + \ldots, \end{aligned} \tag{12}$$

where $d\eta_i = d\eta^1_i d\eta^2_i d\eta^3_i$.

For further use, we rewrite this result once more in absolute notation with slightly different numeration of points:

$$\begin{aligned} G' = {} & \int (g_{01} \otimes g_{12}) * (\nabla \cdot C'_1) d\eta_1 + \\ & + \iint (g_{01} \otimes g_{12} \otimes g_{23}) * \nabla_1 \cdot \nabla_2 \cdot (C'_1 \otimes C'_2) d\eta_1 d\eta_2 \\ & + \iiint (g_{01} \otimes g_{12} \otimes g_{23} \otimes g_{34}) * \nabla_1 \cdot \nabla_2 \cdot \nabla_3 \cdot (C'_1 \otimes C'_2 \otimes C'_3) d\eta_1 d\eta_2 d\eta_3 \\ & + \ldots + \int \ldots \int (g_{01} \otimes \ldots \otimes g_{n-1,n}) * \nabla_1 \cdot \ldots \cdot \nabla_{n-1} \cdot (C'_1 \otimes \ldots \otimes C'_{n-1}) d\eta_1 \ldots d\eta_{n-1} + \ldots, \end{aligned} \tag{13}$$

where $g = \nabla G^0$, that is, $g_{prt} = \partial_t G^0_{pr}$, except for $g_{01} = G^0(\eta_0, \eta_1)$ and indices stand for numbers of points (with $\eta_0 = x, \eta_{n-1} = \eta_1 \eta_n = \xi$ in each term, according to notation in (12)); finally, symbol $*$ denotes multiplication of the two terms under integral over all tensor indices except the first and the last — that is, for all points whose numbers repeat, while simple dot — multiplication over one index respective to the variable.

Now, it is seen that operators Γ depend on position since they contain multiplicative terms C' and, if internal geometry is random, form stochastic tensor fields; thus we obtain the stochastic Green functions G' and

$$G_{km} = G^0_{km} + G'_{km} = (I_{kq} + \Gamma_{kq} + \Gamma_{kq_1}\Gamma_{p_1 q} + \ldots) G^0_{km}, \tag{14}$$

I_{kq} being the identity operator. We may stress the fact of randomness by including in (6) the argument ω (denoting an elementary event in the probability space)

$$u_k(x; \omega) = \int G_{km}(x, \xi; \omega) f_m(\xi) d\xi. \tag{15}$$

The princiapl characteristic of a random function is its mean value at a given point; hence we determine first the mean displacement:

$$\bar{u}_k(x) = \int \overline{G}_{km}(x, \xi; \omega) f_m(\xi) d\xi. \tag{16}$$

Assume that correlation moment functions of C of arbitrary order are known — in absolute notation,

$$K^{(n)} = \langle \bigotimes_{i=1}^{n} C'_i \rangle,$$

and, in coordinates, according to (12),

$$K^{(n)}_{rsp_1t_1 \cdots r_{n-1}s_{n-1}qt} = \langle C'_{rsp_1t_1}(\eta_1) C'_{r_1s_1p_2t_2}(\eta_2) C'_{r_2s_2p_3t_3}(\eta_3) \ldots C'_{r_{n-1}s_{n-1}qt}(\eta) \rangle,$$

$K^{(n)}$ being n-point correlation tensor of $4n$-th rank. Performing differentiation and summation over the indices (cf. (12)), we obtain

$$k^{(n)}_{rp_1t_1 \cdots r_{n-1}qt} = \partial_s \ldots \partial_{s_{n-1}} K^{(n)}_{rsp_1t_1 \cdots r_{n-1}s_n\ {}_1qt} = \frac{\partial}{\partial \eta^s_1} \cdots \frac{\partial}{\partial \eta^{s_{n-1}}} \langle C'_{rsp_1t_1}(\eta_1) \ldots C'_{r_{n-1}s_{n-}\ qt}(\eta) \rangle.$$

Since the use of tensor indices is, evidently, somewhat cumbersome, we introduce again, according to (13), the following obvious notation

$$k_{12\cdots n} = \nabla_1 \cdot \nabla_2 \cdot \ldots \cdot \nabla_n \cdot \langle C'_1 \otimes C'_2 \otimes \ldots \otimes C'_n \rangle = \nabla_1 \cdot \ldots \cdot \nabla_n \cdot K^{(n)},$$

$$g_{12\cdots n} = g_{01} \otimes g_{12} \otimes \ldots \otimes g_{n-1,n} \otimes g_{n,n+1}$$

indices denoting numbers of points $n = 1, 2, 3, \ldots$, except for the first and the last for g, for example, for $n = 1$, $g_1 = g_{01} \otimes g_{12}$, while g (without indices) stands for $g_{01} = G^0$. Making use of this notation and averaging term by term in the series, we obtain the following solution (cf. (14))

$$\overline{G}(x, \xi) = g + \int g_1 * k_1 d\eta_1 + \iint g_{12} * k_{12} d\eta_1 d\eta_2 + \iiint g_{123} * k_{123} d\eta_1 d\eta_2 d\eta_3 + \ldots. \tag{17}$$

This is an expansion respective to multi-point correlation functions of increasing order and multiplication rule the same as in (13). Then, in view of (16), we are able to determine average $\bar{u}$ at any point for every load distribution.

Let us calculate, furthermore, second correlation functions of u, providing a characteristic of dispersion. On multiplying both sides of (15), written for different arguments in either case, and averaging, we obtain

$$\langle u_k(x_1)u_l(x_2)\rangle = \iint \langle G_{km}(x_1,\xi_1)G_{ln}(x_2,\xi_2)f_m(\xi_1)f_n(\xi_2)d\xi_1 d\xi_2, \tag{18}$$

where, using (14), with an abbreviated obvious notation,

$$\begin{aligned}\langle G_1 G_2\rangle &= \langle (I_1+\Gamma_1+\Gamma_1^{(2)}+\ldots)(I_2+\Gamma_2+\Gamma_2^{(2)}+\ldots)G_2^0\rangle \\ &= \langle I+(\Gamma_1+\Gamma_2)+(\Gamma_1^{(2)}+\Gamma_1\Gamma_2+\Gamma_2^{(2)})+\ldots\rangle G_1^0 G_2^0\end{aligned} \tag{19}$$

or, making use of argument the same as for (17),

$$\begin{aligned}\langle GG'\rangle &= gg'+\Big(\int g_1 g' * k_{12}d\eta_1 d\eta_2+\int gg'_{1'} * k_{1'}d\eta_{1'}\Big)\\ &+\Big(\iint g_{12}g' * k_{12}d\eta_1 d\eta_2+\iint g_1 g'_{1'} * k_{11'}d\eta_1 d\eta'_{1'}+\iint gg'_{1'2'} * k_{1'2'}d\eta_{1'}d\eta_{2'}\Big)+\ldots,\end{aligned}$$

where primed indices stand for the second point system. A more concise formula, capable of generalization, will be obtained for unique numeration of points,

$$\begin{aligned}\langle G^{(1)}G^{(2)}\rangle &= g^{(1)}g^{(2)}+\int (g_1^{(1)}g^{(2)}+g^{(1)}g_1^{(2)}) * k_1 d\eta_1\\ &+\iint (g_{12}^{(1)}g^{(2)}+g_1^{(1)}g_2^{(2)}+g^{(1)}g_{12}^{(2)}) * k_{12}d\eta_1 d\eta_2+\ldots,\end{aligned} \tag{20}$$

or in brief

$$\langle G^{(1)}G^{(2)}\rangle = \sum_{n=0}^{\infty}\underbrace{\int\ldots\int}_{n}\Big(\sum_{i=1}^{n} g_{12\ldots i}^{(1)}g_{i+1\ldots n}\Big) * k_{1\ldots n}d\eta_1\ldots d\eta_n, \tag{21}$$

if we agree that $g_{1\ldots i}^{(1)} = g^{(1)}$ for $i=0$, and $g_{i+1\ldots n}^{(2)} = g^{(2)}$ for $i=n$, whereas the sign of integral disappears for $n=0$.

Clearly, (21) may be generalized for a correlation moment of order p,

$$\langle G^{(1)}G^{(2)}\ldots G^{(p)}\rangle = \sum_{n=0}^{\infty}\underbrace{\int\ldots\int}_{n}\Big(\sum_{\substack{i_1=0\\ \cdots\cdots\\ i_{p-1}=0}}^{n} g_{12\ldots i_1}^{(1)}g_{i+1\ldots i_2}^{(1)}\ldots g_{i_{p-1}+1\ldots n}^{(p)}\Big) * k_{1\ldots n}d\eta_1\ldots d\eta_n, \tag{22}$$

where $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_{p-1}$, and for $i_1 = i_2$, $g_{i_1+1\ldots i_2}^{(1)} = g^{(2)}$ etc. (further simplifications, not discussed here, might be made in view of the symmetry properties of g and k).

Analogically to (18), we obtain

$$\langle u^{(1)}u^{(2)}\ldots u^{(p)}\rangle = \underbrace{\int\ldots\int}_{p}\langle G^{(1)}\ldots G^{(p)}\rangle f(\xi_1)\ldots f(\xi_p)d\xi_1\ldots d\xi_p, \tag{23}$$

where $\langle u^{(1)}\ldots u^{(p)}\rangle$ is a function of $(x_1,\ldots,x_p)$, and $\langle G^{(1)}\ldots G^{(2)}\rangle$ a function of $(x_1\ldots x_p;\ \xi_1\ldots\xi_p)$.

Making use of well-known formulae of correlation theory we easily arrive at central correlation functions of any order:

$$\langle u'^{(1)}u'^{(2)}\ldots u'^{(p)}\rangle = \langle (u^{(1)}-\bar{u}^{(1)})(u^{(2)}-\bar{u}^{(2)})\ldots(u^{(p)}-\bar{u}^{(p)})\rangle,$$

$$\langle G'^{(1)}G'^{(2)}\ldots G'^{(p)}\rangle = \langle (G^{(1)}-\bar{G}^{(1)})(G^{(2)}-\bar{G}^{(2)})\ldots(G^{(p)}-\bar{G}^{(p)})\rangle$$

[do not confuse the notation with (7) and subsequent], where $\overline{G}^{(i)}$ is obtained from (17). In the expanded form, we obtain

$$\langle G'^{(1)} \dots G'^{(p)} \rangle = \langle G'^{(1)} \dots G^{(p)} \rangle - \sum_{i_1=0}^{p} \overline{G}^{(i_1)} \Big\langle \prod_{k \neq i_1} G^{(k)} \Big\rangle + \sum_{i_1, i_2} \overline{G}^{(i_1)} \overline{G}^{(i_2)} \Big\langle \prod_{k \neq i_1, i_2} G^{(k)} \Big\rangle - \dots$$

in which products in $\langle\ \rangle$ are taken for $k = 1, 2, \dots, p$, except for the indices pointed out and $1 \leqslant i_1 < i_2 < \dots \leqslant p$ and similarly for $\langle u'^{(1)} \dots u'^{(p)} \rangle$. It is easy to verify that

$$(24) \qquad \langle u'^{(1)} \dots u'^{(p)} \rangle = \underbrace{\int \dots \int}_{p} \langle G'^{(1)} \dots G'^{(p)} \rangle f(\xi_1) \dots f(\xi_p) d\xi_1 \dots d\xi_p.$$

Now, some mathematical remarks will be appropriate. The basic equations [(3) and following] require that $C(x)$ be differentiable with regard to x, and subsequent equations [cf. (10)] that the derivatives be such that respective integrals do exist. However, these derivatives do not appear in the further argument, since actually we operate only correlation functions and their derivatives (cf. transition from $K^{(n)}$ to $k^{(n)}$). Consequently, the results do not depend on whether $C(x)$ is differentiable or not, provided that the correlation functions remain unchanged. In other words, if a medium with smooth $C(x)$ tends to a strict multiphase one, preserving the correlation functions, the solution remains the same — that is it converges to the same, also. More generally, if the volume measure of loci where transition from one phase to another occurs, tends to zero, correlation functions of arbitrary order converge to those obtained for the multiphase medium, as also does our solution.

Another question of prime concern is what conditions should be imposed upon correlation functions to make the series (11), (17) (and the like) absolutely convergent. It is seen that the norm of the operator Γ should be smaller than 1 (of course, we remain in Banach spaces) and, as Γ is stochastic, we may require that it be accomplished almost everywhere — i.e., with probability 1. We confine ourselves to the case of isotropy; then G° for displacements is the well-known Kelvin solution, in Cartesian coordinates,

$$(25) \qquad G^0_{kl} = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)} \left[\frac{(x_k-\xi_k)(x_l-\xi_l)}{r^3} + \frac{\lambda+3\mu}{\lambda+\mu} \frac{\delta_{kl}}{r} \right],$$

$$r = \sqrt{\sum_k (x_k-\xi_k)^2}, \qquad k, l = 1, 2, 3;$$

λ, μ denoting Lamé constants for the reference medium, or in brief

$$(26) \qquad G^0_{kl} = \frac{\alpha}{r} (n_k n_l + \beta \delta_{kl})$$

with

$$n_k = \frac{x_k-\xi_k}{r}, \qquad \alpha = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}, \qquad \beta = \frac{\lambda+3\mu}{\lambda+\mu}.$$

Performing differentiation we obtain

$$(27) \qquad g_{klm} = G^0_{kl,m} = \frac{\alpha}{r^2} (n_l \delta_{km} + n_k \delta_{lm} - \beta n_m \delta_{kl} - 3 n_k n_l n_m).$$

Evidently, the terms in the brackets in (26), (27) are bounded, and consecutive products of g in (13) decrease as much as r^{-3} under the first 3-fold integral, r^{-5} under the second 6-fold one, r^{-7} under the 9-fold one, etc., (resulting in improper integrals). The series (17) may be shown to be convergent under very general assumptions for correlation moment functions; notice, in particular, that k_1 and the respective integral in (17) disappear for a stochastically homogeneous medium.

To close the present contribution, we outline the proof of convergence of our fundamental series. In the general case, in the space L^p, the proof may be carried out by means of the known Calderon-Zygmund theorem by virtue of which Γ in (11) is a linear bounded operator. Here, we present a more simple — albeit a more restricted — approach.

To this end, taking account of (6), we establish the inequality

$$|u_k(x)| \leqslant \int |\varphi_{km}(x,\xi)| \frac{|f_m(\xi)|}{r_{x\xi}} d\xi,$$

where

$$\varphi_{km} = G_{km}(x,\xi) r_{x\xi}$$

in particular, for G°, according to (26),

$$\varphi^0(x,\xi) = \varphi^0(n) = \alpha(n_k n_m + \beta\delta_{km})$$

and

$$r_{x\xi} = \sqrt{\sum_i (x_i - \xi_i)^2}, \quad i = 1, 2, 3.$$

Hence, if we assume

$$||G|| = \sup|\varphi_{km}(x,\xi)| = \sup_{x,\xi;k,m} |G_{km}(x,\xi) r_{x\xi}|,$$

$$||f|| = \sup_{x;m} \int \frac{|f_m(\xi)|}{r_{x\xi}} d\xi,$$

we obtain:

(28) $$\sup_{x;k} |u_k(x)| \leqslant ||G|| \, ||f||.$$

Note that the condition for $||f||$ requires for the vector-valued function f to be integrable in the expression under the integral sign over the whole space, under any choice of the origin $x = 0$. Provided that (28) holds, the displacement u is bounded, which, theoretically, is a restrictive assumption, still justified physically. In particular, it includes finite, spatially bounded fields of loads. It is seen that continuous functions g conduce to continuous u (consider the inequality for the difference of u in neighbor points). The norm of the *operator* can be used as the norm for an *element* in the space of the functions G (to be exact, for $||G||$ to be a norm in (28) it ought to be shown that it is the smallest of bounding values, which is rather evident).

Now, consider the operator Γ and transform the following expression, taking into account the Gauss theorem (cf. (10) and next), with $F = \Gamma G'$,

$$F_{km}(x,\xi) = \int G^0_{kp}(x,\eta)\left(C'_{psqt}(\eta)\right)_{,s}\left(G'_{qm}(\eta,\xi)\right)_{,t}d\eta$$
$$= \int (G^0_{kp}C'_{psqt,s}G'_{qm})_{,t}\,d\eta - \int (G^0_{kp}C'_{psqt,s})_{,t}G'_{qm}\,d\eta = \oint G^0_{kp}C'_{psqt,s}G'_{qm}n_t\,dS$$
$$-\int (G^0_{kp}C'_{psqt,s})_{,t}G'_{qm}\,d\eta = -\int G^0_{kp,t}C'_{psqt,s}G'_{qm}\,d\eta - \int G^0_{kp}C'_{psqt,st}G'_{qm}\,d\eta$$
$$= \int \frac{\varphi^0_{kp}(x,\eta)}{r^2_{x\eta}}n_t(\eta)\left(C'_{psqt}(\eta)\right)_{,s}\frac{G'_{qm}(\eta,\xi)r_{\eta\xi}}{r_{\eta\xi}}d\eta - \int \frac{\varphi^0_{kp}(x,\eta)}{r_{x\eta}}\left(C'_{psqt}(\eta)\right)_{,st}\frac{G'_{qm}(\eta,\xi)r_{\eta\xi}}{r_{\eta\xi}}d\eta$$

since $G^0_{,t} = -\varphi^0 n_t/r^2$ and the surface integral, taken over small spheres with $r \to 0$ comprising points x and ξ and a large sphere of $r \to \infty$ (possibly, outside the range of inclusions), disappears, as may easily be verified.

On multiplying both sides of the above expression by $r_{x\xi}$, taking absolute values and using $||G^0|| = \sup|G^0 r| = \sup|\varphi^0|$, and similarly for $||G'||$ and $||\Gamma G'||$, we arrive at the inequality

$$\text{(29)} \qquad ||\Gamma G'|| \leqslant \left(||\Gamma^0|| \sup_{x,\,\xi;\,p;\,q} \int \left|\frac{n_t C'_{psqt,s}}{r_{x\eta}} - C'_{psqt,st}\right| \frac{r_{x\xi}}{r_{x\eta}r_{\xi\eta}}d\eta\right)||G'||.$$

The term in the brackets ( ) denoting the norm of the operator, $||\Gamma||$; this depends, clearly, on the function $C'(\eta)$ that should vanish in infinity, in an adequate manner. For $||\Gamma|| < 1$, the series (11) is convergent in the sense explained, for every realisation — i.e., also stochastically. For a multiphase medium, the derivatives of C' equal 0, except for grain boundaries where we can replace them by δ-functions (number of grains diminishing in infinity).

In a less restrictive approach the fields of u and C may be required to be such that the strain does exist at any point; for details cf. [4] by the present author.

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