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On the mechanical behaviour of quasi-linear bodies

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CONSIDERED here are quasi-linear materials, having only two possible forms of equations of state (constitutive equations): the reversible (elastic) and the irreversible (viscous). They may also have two limit stresses: the yield limit and the strength of material. The symbols of the rheological properties are discussed, together with the rules for composing the complicated bodies by means of simple elements. An examination is made of the elastic-plastic and linear hardening bodies, the linear visco-plastic and brittle materials. It is shown, that even simple rheological models of materials make it possible to describe certain complicated properties of matter.

Rozpatrzono materiały quasi-liniowe, mające tylko dwie możliwe postacie równań reologicznych stanu (równań konstytutywnych): odwracalną (sprężystą) i nieodwracalną (lepką). Mogą one też mieć dwie wartości granicznych naprężeń (dwie granice reologiczne): granicę plastyczności i granicę wytrzymałości. Omówione zostały symbole i reguły tworzenia modeli reologicznych z elementów. Rozpatrzono materiały elastoplastyczne i z liniowym wzmocnieniem, jak również materiały odznaczające się płynięciem plastycznym i materiały kruche. Pokazano, że nawet użycie bardzo prostych modeli pozwala na opis niektórych dość złożonych własności materiału.

Рассматриваются квази-линейные материалы, для которых возможны только две формы реологических уравнений состояния, определяющих уравнений): обратимая (упругая) и необратимая (вязкая). Эти материалы могут также иметь два реологических предела, выраженных в напряжениях: предел текучести и предельное сопротивление. Описаны применяемые в дальнейшем символы реологических свойств материала и представлены правила конструирования реологических моделей из этих символов. Рассмотрены упруго-пластические материалы и материалы линейно-упрочняющиеся. Кроме того рассмотрены материалы, характеризующиеся пластическим течением, а также хрупкие материалы. Показано, что применение даже простейших моделей позволяет описать некоторые довольно сложные свойства рассмотренных материалов.

1. Notations

LET US deal with the quasi-linear materials. From thermodynamic considerations we obtain only two possible forms of equations of state (or constitutive equations):

linear elasticity

$$\sigma_{ik} = G_{ijkl}\gamma_{jl}, \tag{1.1}$$

linear viscosity

$$\sigma_{ik} = \eta_{ijkl} s\gamma_{jl}, \tag{1.2}$$

in which: σ are the stresses, γ are the strains, G and η are the moduli of elasticity and viscosity, respectively, $s = d/dt$. The moduli G and η are called the rheological moduli of the material.

The following connections are valid:

$$\sigma_{ik} = \sigma_0\delta_{ik} + \tau_{ik}, \quad \sigma_0 = \frac{1}{3}\sigma_{ii}, \tag{1.3}$$

$$\gamma_{ik} = \frac{1}{2}(u_{i,k}+u_{k,i}) = \gamma_0\delta_{ik} + \vartheta_{ik}, \quad \gamma_0 = \frac{1}{3}\gamma_{ii}, \tag{1.4}$$

σ_{ik} and σ_{ik} being the stress and strain tensors and τ_{ik} and ϑ_{ik} the stress and strain deviators, respectively, δ_{ik} is the Cronecker's delta; the Einstein summation convention is used.

Some materials change their mechanical behaviour by exceeding the specified value of load. Each material shows a limited ability to carry load. The first limiting value of load we call the yield limit of material, the second value is denoted as strength of material. We shall denote the yield limit by simple tension using the character θ, and the strength by simple tension using the character R. The stress θ and R are the rheological limits of the material under consideration. On the illustration, we shall denote the rheological

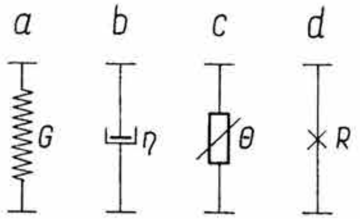

FIG. 1.

moduli G and η by the usual symbols of spring and dashpot, respectively (Figs. 1a and 1b). For the rheological limits θ and R we propose the denotations shown on Figs. 1c and 1d, respectively.

Two different states of behaviour of the material are defined as follows:

before yielding

$$\sigma_{\mathrm{extr}} < \theta, \tag{1.5}$$

at yielding

$$\sigma_{\mathrm{extr}} = \theta. \tag{1.6}$$

The connection $\sigma_{\mathrm{extr}} > \theta$ for the denotation Fig. 1c is impossible. The term "yielding" denotes, in our meaning, that the critical value of stress is in the model constant: the increment of load will cause acceleration of motion. The diminishing of load will cause an instantaneous stop of motion. Similarly, the material can carry the load only if $\sigma_{\mathrm{extr}} < R$. The connection $\sigma_{\mathrm{extr}} = R$ denotes exhaustion of the load carrying capacity of material. Therefore, the connection $\sigma_{\mathrm{extr}} > R$ is impossible.
Valid for each material is:

$$\theta \leqslant R. \tag{1.7}$$

2. Rules

We shall distinguish two parts of coupling the symbols on the illustration of the rheological model of material: the parallel and the in-series couplings, shown in Fig. 2a and 2b, respectively. The coupling in Fig. 2a shows the known Kelvin body; the coupling in Fig. 2b shows the Maxwell liquid. No difference results from coupling the symbols of moduli and of limits: each of them may be coupled with the other.

There exist two rules of coupling two different symbols — say, A and B:

(a) the kinematic rule, governing the strains and having the form:
in parallel coupling:

$$\gamma^A = \gamma^B = \gamma, \tag{2.1}$$

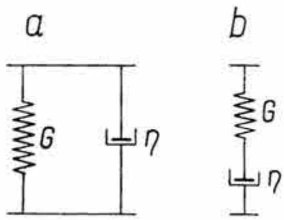

FIG. 2.

in series coupling:

$$\gamma^A + \gamma^B = \gamma; \tag{2.2}$$

(b) the dynamic rule, governing the stresses and having the form:
in parallel coupling:

$$\sigma^A + \sigma^B = \sigma, \tag{2.3}$$

in series coupling:

$$\sigma^A = \sigma^B = \sigma, \tag{2.4}$$

in which the indices A and B denote the symbols to be coupled, and the stress σ and strain γ without indices denote the values for the model as a whole.

It is possible to couple an arbitrary number of elements in either manner. It will be borne in mind that the parallel coupling of similar parallel coupled elements (e.g., the parallel coupling of Kelvin elements) can be replaced by a single element of appropriate values of elasticity and viscosity; the same applies to coupling in series of elements consisting of similar symbols coupled in series (e.g., coupling in series of Maxwell fluids).

We shall note the difference in behaviour of yield limit θ and of strength R. If the value of θ is exceeded, than the element θ still bears the stress $\tilde{\sigma} = \theta$, and the excess load will be borne by other elements coupled in parallel with the element θ. By exceeding the strength R, the element R is damaged and loses its capacity to bear. By diminution of the load the element θ is immobilized and checks further movement, if the stress acting on it is $\tilde{\sigma} < \theta$; by contrast, the element R in such case entirely loses its capacity for resistance, even with diminution of load.

3. Models of bodies

It is usual to distinguish three general and different kinds of behaviour of a material:
purely elastic,
purely viscous,
purely plastic.

The first two kinds of behaviour were described by means of the Eqs. (1.1) and (1.2). The third, we attempt to describe by means of the relation:

$$\sigma \leqslant \theta, \tag{3.1}$$

which is in fact not the equation of state, since it does not yield a connection between stress and strain (cause and consequence) in the thermodynamical meaning. Therefore, we introduce certain sophisticated explanations of what is called "plasticity", using a function of plasticity, a plastic potential and so on.

Plasticity can be quite generally explained by means of only three characteristics of material in the rheological meaning: the values of elasticity and of viscosity and the yield limit. We must bear in mind that we are considering only quasi-linear materials. Our arguments are valid, therefore, only for such materials.

We set out to find the rheological model for a linearly-hardening body. Consider the model shown in Fig. 3; it consists of three elements: the spring G_1, which is coupled in

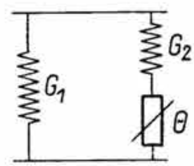

FIG. 3.

parallel with the St-Venant element $G_2-\theta$. Before reaching the yield limit θ in the St Venant body ([1]), the deviatoric load τ is borne by two springs G_1 and G_2. The deformation of the body is described by means of the Eq.

$$\vartheta_{ik} = \frac{\tau_{ik}}{2(G_1+G_2)}. \tag{3.2}$$

The yield limit in the St-Venant body will be reached if:

$$\tau^{v}_{\lim} = 2G_2\vartheta_{\max} = \frac{\theta}{2}, \tag{3.3}$$

or, substituting (3.2), if for the whole body:

$$\tau_{\max} = \Gamma_2\frac{\theta}{2} = \tau_{\lim}, \tag{3.4}$$

where

$$\Gamma_2 = \frac{G_1+G_2}{G_2}. \tag{3.5}$$

In exceeding the yield limit, the stress

$$\tau^{v}_{\lim} = \frac{\theta}{2}$$

[1]) We assume compressive loading of the body.

in the St Venant element cannot be exceeded and remains constant. Therefore the difference of load:

$$\tau_{\max}-\frac{\theta}{2}$$

will be borne by the element G_1 only. The strain in the body shown in Fig. 3 is, after exceeding yield limit, therefore:

(3.6) $$\vartheta_{\max}=\frac{\theta}{4G_2}+\frac{2\tau_{\max}-\theta}{4G_1}.$$

The dependence of the strain upon the stress applied is shown in Fig. 4. The line 1 shows the process of application of the load on a linearly-hardening body.

Let us now consider the process of unloading. Note that even the infinitesimal diminution of load, say, at point B in Fig. 4, will cause instantaneous joining of the spring

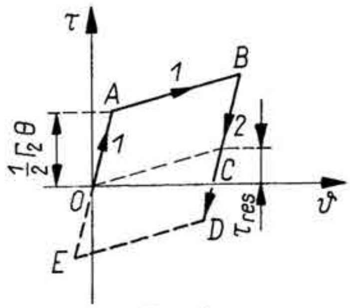

FIG. 4.

G_2. Therefore any diminution of deformation will instantly stiffen the element θ and the further process will occur according to the formula:

(3.7) $$-\vartheta_{\max}=-\frac{\tau_{\max}}{2(G_1+G_2)}.$$

When this diminution reaches the value:

(3.8) $$-\vartheta_{\max}=-\frac{\theta}{4G_2}$$

the stress in the St. Venant element will be equal to zero, but the stress in the Hookean element G_1 is not zero but is equal to:

(3.9) $$\tau_{\mathrm{res}}=\tau^{H}=\tau_{\max}-\Gamma_2\frac{\theta}{2}$$

according to formula (3.2)–(3.5). This stress will cause further deformation of our model of the body producing a pull in the St Venant element. The zero-state of external forces (the total unloading) will occur, if the pull in the St Venant element and the push in the Hookean element are equal — i.e.,:

(3.10) $$\vartheta^{v}_{\mathrm{res}}=\frac{2\tau_{\max}-\Gamma_2\theta}{4(G_1+G_2)}.$$

The residual tensile stress in the St-Venant element is therefore:

$$\tau_{\mathrm{res}}^{v} = \frac{\tau_{\max}}{\Gamma_2} - \frac{\theta}{2}. \tag{3.11}$$

If we now apply the load of the opposite sign to the material, we find that the yield limit in the direction opposite to the first loading direction will be smaller (stretch CD) than in the case of the direct application of the load to the virgin material (stretch OE). That is the well-known Baushinger's effect. The yield limit in the opposite direction will be:

$$\tau_{\mathrm{lim}}^{v} = \frac{\theta}{2} - \tau_{\mathrm{res}}^{v} = \theta - \frac{\tau_{\max}}{\Gamma_2}. \tag{3.12}$$

The material described above might be called H/V material.

Another model representing plastic flow may be called the M/V model of material. This model has been described in other publications by the present author (e.g. [1, 2, 3, 4]) and will not be discussed here.

As a third example, we shall consider a brittle material. Let us discuss the model shown in Fig. 5a: the Maxwell fluid, supplemented by an R-element coupled in series, is coupled

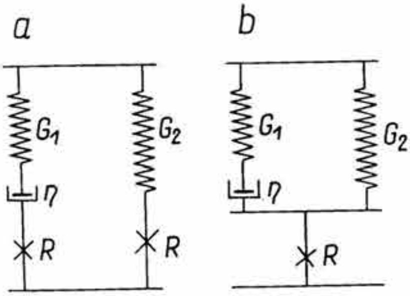

Fig. 5.

in parallel with a Hookean body connected in series with an R-element, similar to that in the Maxwell fluid. We shall call this model the S/K-model. In this case, by loading the model we observe the redistribution of load from Maxwell fluid to Hookean solid (similarly as in the case of the M/V model). If the total loading of our body exceeds the value of R, there exists a *time of failure*:

$$t_R = \Gamma_2 T \ln \frac{2\tau_{\max}}{\Gamma_1 (2\tau_{\max} - R)} \tag{3.13}$$

in which

$$\Gamma_1 = \frac{G_1 + G_2}{G_1}, \qquad T = \frac{\eta}{G_1}. \tag{3.14}$$

The derivation of this formula is entirely similar to that of the critical time of the M/V-body and will be not repeated here.

Therefore, if

$$R < 2\tau_{\max} < \Gamma_2 R \tag{3.15}$$

there exists a time of failure — i.e., the material will be not destroyed instantaneously after application of the load, but after the passage of the time t_R. This material adequately describes the behaviour of brittle rocks, concrete, etc., if we assume, that R is the *tensile* strength of the material. An example of solving the problem of a tunnel in brittle rock was given in [5].

4. Remarks

The H/V-material described above is cited by OLSZAK, PERZYNA, SAWCZUK [6], Chapt. 4. All the principles of the solution of problems cited in the monograph referred to may be applied to the H/V-body. The importance of the introduction of the H/V-body is that this body can be treated in the same way as the elastic body without the necessity to introduce the function $m(\varepsilon)_i$ of the intensity of deformation ([6] Chapt. 4, Form. 2.3). Furthermore, the existence of Bauschinger's effects in the case of an H/V-body is a natural consequence of a single general assumption: that the yield limit is constant and is the same in both directions of loading of virgin material.

To represent what is called the visco-plastic flow of material, we can use the M/V-model (Fig. 6). It may be worthwhile to observe that in the theory of the M/V-body there is

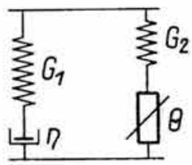

FIG. 6.

no necessity to introduce the Drucker's postulate concerning the positive sign of the work of hardening. This positive nature of the M/V-body is a simple consequence of basic laws of irreversible thermodynamics, which are the basis of the equations of state.

From the investigations of the properties of an M/V-body it follows that:

(a) the limit shear stress of a body is not a constant value. It varies between an instantaneous value $\tau_{1lm} = \frac{1}{2}\Gamma_2\theta$ and a permanent value $\tau_{1lm} = \frac{1}{2}\theta$. Its value is dependent on the time of action of load on the body: the shorter this time, the larger the limit shear stress of the M/V-body.

(b) The plastic flow of an M/V-body need not necessarily be coaxial with the stress deviator. Such a coaxiality may be assumed; but one may also assume, for instance, the validity of the following hypothesis [4]:

The directions of plastic flow of an M/V-body are determined by the value of the actual partial shear stress in the St-Venant element during flow.

That assumption gives the angle between the maximal principal stress and the direction of flow, variable with time and determined by means of the formula:

$$\sin 2\psi(t) = \frac{\theta}{\theta + (2\tau_{\max} - \theta)\exp\left(-\frac{t - t_{kr}}{T}\right)}, \tag{4.1}$$

where t_{kr} is the time of reaching the yield limit at the point under consideration, ψ is the angle mentioned above.

(c) It is obvious that the theory of plastic flow developed by [6] can be applied to the M/V-body; the simplicity of the meaning of parameters of state in this body makes it possible to introduce considerable simplifications. For instance, it is possible to apply the rules of viscous flow to the state of an M/V-body after exceeding the yield limit.

(d) It follows from (a) that plastic zones are not invariant after application of constant load. Therefore, the application of the M/V-model of a solid makes it possible to find in an elementary manner the development of plastic zones with time, even by an invariable load. No known theory of plastic flow can give such possibilities.

To explain the properties of brittle materials, the S/K-model has been proposed. This model is characterized by existence of the time of failure. Before failure, it behaves like a standard material (Zener's material) and therefore it can describe quite a wide class of materials, in particular almost all rocks and unreinforced concretes. The possibility of determining the development of cracked zones with time is entirely similar to that of the M/V-model.

An objection may be raised concerning the disregard of the influence of the plastic (respectively, cracked) zone on the reversible (viscoelastic) regions of the body under consideration. Such an objection is valid; investigations concerning the problem concerned have not yet been made. Nevertheless, there are indications that the influence referred to is considerably lower than might be assumed. In the textbook [7] are cited some numerical examples [Chapt. 18]; it can easily be seen that taking into account the interaction between cracked (resp. plastic) and reversible zones exerts only a small influence on the stress distribution around the cracked (plastic) zone. Therefore, the present author believes that by assuming quasi-linear behaviour of the material, the interaction between elastic (reversible) and cracked (plastic, irreversible) zones may be disregarded. A precise investigation of this problem is, of course, desirable.

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