Archives of Mechanics • Archiwum Mechaniki Stosowanej • 24, 5–6, pp. 793–801, Warszawa 1972

Acoustic wave in finitely deformed elastic material

Z. WESOŁOWSKI (WARSZAWA)

THE PAPER is devoted to the propagation of a weak discontinuity wave in the reference configuration of a nonlinear elastic medium. Having determined the jumps of the second derivatives we construct the acoustic tensor. On the basis of the linearized equations of motion we determine the equations of bicharacteristics and rays and, furthermore, the equation for the jump of the second derivatives of the displacement.

Rozważa się propagację fali słabej nieciągłości w konfiguracji odniesienia nieliniowego ośrodka sprężystego. Po wyznaczeniu skoku drugich pochodnych buduje się tensor akustyczny. Opierając się o zlinearyzowane równania ruchu znajduje się równania bicharakterystyk i promieni, jak również równanie skoku drugich pochodnych przemieszczenia.

Рассматривается распространение волны слабого разрыва относительно исходной конфигурации нелинейной упругой среды. После определения скачка вторых производных построен акустический тензор. На основе линеаризированного уравнения движения получены уравнения бихарактеристик и лучей, а также уравнения для разрывов вторых производных от перемещения.

In one of his elegant papers, TRUESDELL [1] stated a condition for propagation of an acoustic wave in an elastic material. In this paper we shall present the law of variation of the amplitude of such a wave for a material with arbitrary symmetry properties. This law was derived in an entirely different way and a different form by CHEN [2, 3], CHADWICK and OGDEN [4] and BOWEN and WANG [5].

Our considerations concern the propagation of a discontinuity surface in the reference configuration.

1. The fundamental relations

Let us introduce two systems of coordinates $\{x^i\}$ and $\{X^\alpha\}$. If X^α are the coordinates in the reference configuration B_R and x^i those in the current configuration B, then the motion of body is described by the relations

$$x^i = x^i(X^\alpha, t), \tag{1.1}$$

where t is the time. The elastic energy σ is a function of the displacement gradient $x^i_{,\alpha}$

$$\sigma = \sigma(x^i_{,\alpha}), \tag{1.2}$$

and the equation of motion has the form

$$A_i{}^\alpha{}_k{}^\beta x^k\|_{\alpha\beta} + q_i + \varrho_R b_i = \varrho_R \ddot{x}_i, \tag{1.3}$$

where ϱ_R is the density in B_R and b_i the body force; the functions $A_i{}^\alpha{}_k{}^\beta$ and q_i are given by the relations

$$A_i{}^\alpha{}_k{}^\beta = A_k{}^\beta{}_i{}^\alpha = \varrho_R \frac{\partial^2\sigma}{\partial x^i{}_{,\alpha}\partial x^k{}_{,\beta}}, \tag{1.4}$$

$$q_i = \varrho_R \frac{\partial^2\sigma}{\partial x^i{}_{,\alpha}\partial X^\alpha}.$$

The double vertical line denotes the total covariant differentiation

$$(\cdot)\|_\alpha = (\cdot)|_\alpha + (\cdot)|_r x^r{}_{,\alpha}.$$

Suppose that

$$\bar{x}^i = x^i(X^\alpha, t) + u^i(X^\alpha, t), \tag{1.5}$$

is a motion slightly different from the motion (1.1). TOUPIN and BERSTEIN proved [6] that the displacement u^i statisfies the equation

$$\mathscr{L}_{ir}u^r = (A_i{}^\alpha{}_k{}^\beta u^k\|_\beta)\|_\alpha - \varrho_R \ddot{u}_i = 0. \tag{1.6}$$

The functions $A_i{}^\alpha{}_k{}^\beta$ are calculated for the basic motion (1.1) and are therefore independent of u_i. The Toupin-Berstein equation is an almost linear equation with variable coefficients.

Consider the reference configuration B_R. To simplify the calculation we assume that both systems $\{x^i\}$ and $\{X^\alpha\}$ are Cartesian. Consider a moving surface $\mathscr{S}_R$ described by one of the relations

$$X^\alpha = X^\alpha(M^1, M^2, t), \tag{1.7}$$

$$t = \Psi(X^\alpha), \tag{1.8}$$

where M^K, $K = 1, 2$ are the parameters of the surface $\mathscr{S}_R$. The vectors $X^\alpha{}_{,K}$ are tangent to $\mathscr{S}_R$. If the unit normal to $\mathscr{S}_R$ is N_α, then

$$N_\alpha = \Psi_{,\alpha}(\Psi_{,\rho}\Psi_{,\rho})^{-1/2}, \quad N_\alpha X^\alpha{}_{,K} = 0. \tag{1.9}$$

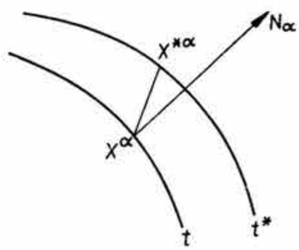

FIG. 1.

Figure 1 represents two successive position of $\mathscr{S}_R$ at the instants t and t^*. In accordance with (1.8), we have

$$t^* - t = (X^{*\alpha} - X^\alpha)\Psi_{,\alpha} + o(X^{*\alpha} - X^\alpha),$$

and, passing to the limit.

$$\Psi_{,\alpha}X^\alpha{}_{,t} = 1. \tag{1.10}$$

Substituting now for N_α from (1.9), we obtain:

$$N_\alpha X^\alpha{}_{,t} = U, \tag{1.11}$$
$$U = (\Psi_{,\rho}\Psi_{,\rho})^{-1/2}, \quad N_\alpha = U\Psi_{,\alpha}.$$

By definition U, is the speed of the surface $\mathscr{S}_R$. The derivatives $X^\alpha{}_{,t}$ constitute the velocity of the point $M^K = \text{const}$ depending, however, on the parametrization of the surface. The projection of $X^\alpha{}_{,t}$ on the normal is independent of the parametrization.

Locally, the surface $\mathscr{S}_R$ divides the space into two regions. Let us denote the quantities referring to the points in front of (behind) the surface $\mathscr{S}_R$ by the index $F(B)$ and consider an arbitrary field $H = H(X^\alpha, t)$. The field itself or its derivatives may suffer discontinuities on $\mathscr{S}_R$. On the sides F and B of the surface $\mathscr{S}_R$ the field H can be expressed in terms of (M^K, t),

$$\begin{aligned} H &= H_F(M^K, t) \quad \text{on} \quad \mathscr{S}_{RF}, \\ H &= H_B(M^K, t) \quad \text{on} \quad \mathscr{S}_{RB}. \end{aligned} \tag{1.12}$$

The derivatives of H_F and H_B are

$$\frac{\partial H_F}{\partial M^K} = \left(\frac{\partial H}{\partial X^\alpha}\right)_F X^\alpha{}_{,K},$$

$$\frac{\partial H_B}{\partial M^K} = \left(\frac{\partial H}{\partial X^\alpha}\right)_B X^\alpha{}_{,K}.$$

The first terms in the right-hand sides are calculated in three dimensions. Subtracting, we obtain

$$[\![H]\!]_{,M} = [\![H_{,\alpha}]\!] X^\alpha{}_{,K}, \tag{1.13}$$

where the double bracket denotes the jump

$$[\![(\cdot)]\!] = (\cdot)_F - (\cdot)_B. \tag{1.14}$$

Assume that H is continuous but its first derivatives suffer a discontinuity. Then the left-hand side of (1.13) vanishes, whence

$$[\![H_{,\alpha}]\!] X^\alpha{}_{,K} = 0, \tag{1.15}$$

and, consequently, in accordance with (1.9),

$$[\![H_{,\alpha}]\!] = AN_\alpha, \tag{1.16}$$

where A is arbitrary parameter depending on X^α.

Consider now the time derivatives H_F and H_B

$$\frac{dH_F}{dt} = \left(\frac{\partial H}{\partial t}\right)_F + \left(\frac{\partial H}{\partial X^\alpha}\right)_F X^\alpha{}_{,t},$$

$$\frac{dH_B}{dt} = \left(\frac{\partial H}{\partial t}\right)_B + \left(\frac{\partial H}{\partial X_\alpha}\right)_B X^\alpha{}_{,t}.$$

The left-hand sides represent the total time rate of H observed in moving in front of (behind) $\mathcal{S}_R$. The first term constitutes the change at a fixed space point, while the second describes the convection. Subtracting, we obtain

$$\frac{d}{dt}[\![H]\!] = \left[\!\!\left[\frac{\partial H}{\partial t}\right]\!\!\right] + [\![H_{,\alpha}]\!] X^{\alpha}{}_{,t}. \tag{1.17}$$

For a continuous H, the left-hand side vanishes; therefore, substituting from (1.16) and taking into account (1.11), we obtain:

$$\left[\!\!\left[\frac{\partial H}{\partial t}\right]\!\!\right] = -AU. \tag{1.18}$$

2. The acoustic wave

Consider the case when on $\mathcal{S}_R$ the function x^i and its first derivatives $x^i{}_{,\alpha}$, $x^i{}_{,t}$ are continuous, but derivatives of higher orders are discontinuous. The set of phenomena on $\mathcal{S}_R$ is called the acoustic wave or the weak discontinuity wave, in contrast to a shock wave, i.e. the strong discontinuity wave for which already the first derivatives are discontinuous. In accordance with (1.16) and (1.18),

$$\begin{aligned} [\![x^i{}_{,\alpha\beta}]\!] &= A^i{}_{\alpha} N_{\beta}, \\ [\![x^i{}_{,\alpha t}]\!] &= -A^i{}_{\alpha} U, \\ [\![x^i{}_{,\alpha t}]\!] &= B^i N_{\alpha}, \\ [\![x^i{}_{,tt}]\!] &= -B^i U, \end{aligned} \tag{2.1}$$

where $A^i{}_{\alpha}$ and B^i are sets of parameters. Since $[\![x^i{}_{,\alpha\beta}]\!] = [\![x^i{}_{,\beta\alpha}]\!]$, we obtain:

$$A^i{}_{\alpha} = A^i N_{\alpha}.$$

The relation $[\![x^i{}_{,\alpha t}]\!] = [\![x^i{}_{,t\alpha}]\!]$ implies that

$$B^i = -UA^i.$$

Hence, finally

$$\begin{aligned} [\![x^i{}_{,\alpha\beta}]\!] &= A^i N_{\alpha} N_{\beta}, \\ [\![x^i{}_{,\alpha t}]\!] &= -UA^i N_{\alpha}, \\ [\![x^i{}_{,tt}]\!] &= U^2 A^i. \end{aligned} \tag{2.2}$$

We agree now to regard as the independent variables X^{α} rather than x^i. Since the coordinate systems are Cartesian, the total covariant differentiation is reduced to the partial differentiation. Equation (1.3) is satisfied on both sides of the discontinuity surface $\mathcal{S}_R$. Here $A_i{}^{\alpha}{}_k{}^{\beta}$ and q_i are continuous, since they depend on the gradient $x^i{}_{,\alpha}$ which is continuous by assumption. Thus we have

$$A_i{}^{\alpha}{}_k{}^{\beta} [\![x^k{}_{,\alpha\beta}]\!] = \varrho_R [\![x_{i,tt}]\!], \tag{2.3}$$

and, in view of the relations (2.2),

$$(Q_{ik} - \varrho_R U^2 \delta_{ik}) A^k = 0, \tag{2.4}$$

$$Q_{ik} = A_i{}^{\alpha}{}_k{}^{\beta} N_{\alpha} N_{\beta}. \tag{2.5}$$

The tensor Q_{ik} is the acoustic tensor. It depends on $X^{\alpha}{}_{,t}$ and N_{α}. The vector A^k will be called hereafter the amplitude; it determines the jumps of the second derivatives and is the eigenvector, while the product $\varrho_R U^2$ is the eigenvalue of the acoustic tensor Q_{ik}. In view of the symmetry of the tensor $A_i{}^{\alpha}{}_k{}^{\beta}$ $(1.4)_2$, the tensor Q_{ik} is symmetric, i.e. $Q_{ik} = Q_{ki}$. For each direction of propagation N_{α}, there exist therefore orthogonal amplitudes A^k and the product $\varrho_R U^2$ is a real number.

The propagation condition (2.4) was derived in a different form by TRUESDELL [1] who examined the discontinuity surface moving in B rather than B_R. To derive Truesdell's propagation condition denote by dS_{α} and ds_i elements of the material surfaces and by dV and dv elements of the material volumes in B_R and B, respectively. Now, we have

$$dS_{\alpha} = N_{\alpha} dS = \varepsilon_{\alpha\beta\gamma} X^{\beta}{}_{,1} X^{\gamma}{}_{,2} dM^1 dM^2, \tag{2.6}$$
$$ds_i = n_i ds = \varepsilon_{ijk} x^j{}_{,\beta} X^{\beta}{}_{,1} x^k{}_{,\gamma} X^{\gamma}{}_{,2} dM^1 dM^2.$$

Since

$$\varepsilon_{ijk} x^j{}_{,\beta} x^k{}_{,\gamma} = \frac{1}{J} \varepsilon_{\alpha\beta\gamma} X^{\alpha}{}_{,i}, \quad J = \det x^i{}_{,\alpha},$$

we have

$$dS_{\alpha} = \frac{1}{J} x^i{}_{,\alpha} ds_i, \quad N_{\alpha} = \frac{1}{J} x^i{}_{,\alpha} n_i \frac{ds}{dS}. \tag{2.7}$$

The material volumes dV and dv satisfy the relations

$$dv/dV = J = \varrho_R/\varrho. \tag{2.8}$$

The function (1.1) maps the discountinuity surface $\mathscr{S}_R$ from the configuration B_R to the configuration B. The equation of the surface $\mathscr{S}$ and its speed u are the following:

$$t = \psi(x^i), \quad u = (\psi_{,r}\psi_{,r})^{-1/2}. \tag{2.9}$$

Set $dv = uds$, $dV = UdS$. Since both dv and dV are material volumes,

$$uds = JUdS. \tag{2.10}$$

Introducing (2.10) into the propagation condition (2.4), we obtain:

$$\left(A_i{}^{\alpha}{}_j{}^{\beta} \frac{U^2}{u^2} x^r{}_{,\alpha} n_r x^s{}_{,\beta} n_s - \varrho_R U^2 \delta_{ij}\right) A^j = 0,$$

and, finally, we arrive at Truesdell's propagation condition

$$(q_{ij} - \varrho u^2 \delta_{ij}) a^j = 0, \tag{2.11}$$

where

$$q_{ij} = \frac{\varrho}{\varrho_R} A_i{}^{\alpha}{}_j{}^{\beta} x^r{}_{,\alpha} x^s{}_{,\beta} n_r n_s, \quad a^j = \frac{U^2}{u^2} A^j. \tag{2.12}$$

The wave is longitudinal if $n_i || a_i$ and transverse when $a^i n_i = 0$. A typical wave is neither longitudinal nor transverse.

The following considerations will be based on the propagation condition in the form (2.4). Substituting into it (1.11), we obtain:

$$(A_i{}^{\alpha}{}_j{}^{\beta} \Psi_{,\alpha} \Psi_{,\beta} - \varrho_R \delta_{ij}) A^j = 0. \tag{2.13}$$

There exists a non-trivial solution of this system of equations if the characteristic determinant vanishes, i.e.

$$\det(A_{ij}^{\alpha\beta}\Psi_{,\alpha}\Psi_{\beta}-\varrho_R\delta_{ij}) = 0. \tag{2.14}$$

This is a differential equation the solution of which is $\Psi(X^\alpha)$ determining the wave front. In what follows we assume that $\Psi(X^\alpha)$ has been found. The vector A^j is determined to within its length. A^j will denote one fixed solution of Eq. (2.13).

3. Bicharacteristics

We proceed to determine the rays along which the acoustic wave is propagated. We mainly base on Courant's monograph [7].

Introduce the notation

$$\Phi = \Psi(X^\alpha)-t, \tag{3.1}$$

then on the front of the wave $\Phi = 0$ [cf. (1.18)]. The region F (in front of the surface $\mathscr{S}_R$) corresponds to the inequality $\Phi < 0$, while the region B to $\Phi > 0$. We can now represent the displacement $u^k(X^\alpha, t)$ as a function of X^α and Φ, $u^k = m^k(X^\alpha, \Phi)$; hence

$$\begin{aligned} u^k{}_{,\alpha} &= m^k{}_{,\alpha}+m^k{}_{,\Phi}\Phi_{,\alpha}, \\ u^k{}_{,\alpha\beta} &= m^k{}_{,\alpha\beta}+m^k{}_{,\alpha\Phi}\Phi_{,\beta}+m^k{}_{,\beta\Phi}\Phi_{,\alpha}+m^k{}_{,\Phi\Phi}\Phi_{,\alpha}\Phi_{,\beta} \;+m^k{}_{,\Phi}\Phi_{,\alpha\beta}, \\ u^k{}_{,t} &= -m^k{}_{,\Phi}, u^k{}_{,tt} = m^k{}_{,\Phi\Phi}, \end{aligned} \tag{3.2}$$

and substituting the above result into (1.6), we obtain

$$\begin{aligned}(A_i{}^\alpha{}_k{}^\beta\Phi_{,\alpha}\Phi_{,\beta}-\varrho_R\delta_{ik})\,m^k{}_{,\Phi\Phi}+A_{ik}^{\alpha\beta}[m^k{}_{,\alpha\beta}+m^k{}_{,\alpha\Phi}\Phi_{\beta}+m^k{}_{,\beta\Phi}\Phi_{,\alpha}+m^k{}_{,\Phi}\Phi_{,\alpha\beta}] \\ +A_i{}^\alpha{}_k{}^\beta{}_{,\alpha}[m^k{}_{,\beta}+m^k{}_{,\Phi}\Phi_{,\beta}] = 0.\end{aligned} \tag{3.3}$$

Thus on the surface $\mathscr{S}_R$ the function $m^k{}_{,\Phi\Phi}$ may have a discontinuity. Consequently, we shall represent the displacements u in the form

$$u^i(X^\alpha, t) = S'(\Phi)g^i(X^\alpha, t)+S(\Phi)h^i(X^\alpha, t)+k^i(X^\alpha, t), \tag{3.4}$$

where $S(\Phi)$ is a scalar function of Φ, such that its third derivative is the Heaviside function

$$\begin{aligned} &S'''(\Phi) = \eta(\Phi), \\ &\eta(\Phi) = 1 \quad \text{for} \quad \Phi > 0, \quad \eta(\Phi) = 0 \quad \text{for} \quad \Phi < 0. \end{aligned}$$

The function g^i determines the magnitude of the jump of the second derivatives on $\mathscr{S}_R$. Having introduced the function $S(\Phi)$, we may assume that the functions g^i, h^i and k^i have continuous second derivatives.

Now, we have

$$\begin{aligned} u^i{}_{,\alpha} &= S''\Phi_{,\alpha}g^i+\ldots, \quad u^i{}_{,t} = -S''g^i+\ldots \\ u^i{}_{,\alpha\beta} &= S'''\Phi_{,\alpha}\Phi_{,\beta}g^i+S''(\Phi_{,\alpha\beta}g_i+\Phi_{,\alpha}g^i{}_{,\beta}+\Phi_{,\beta}g^i{}_{,\alpha}+\Phi_{,\alpha}\Phi_{,\beta}g^i)+\ldots \\ u^i{}_{,tt} &= S'''g^i+S''(-2g^i{}_{,t}+h^i)+\ldots, \end{aligned}$$

in the derivatives we omitted terms of higher regularity than $S''(\Phi)$. Introducing these relations into Eq. (1.6), we have

$$(3.5)\quad \mathscr{L}_{ir}u^r = S'''\{(A_i{}^\alpha{}_k{}^\beta \Phi_{,\alpha}\Phi_{,\beta} - \varrho_R\delta_{ik})g^k\} + S''\{(A_i{}^\alpha{}_k{}^\beta\Phi_{,\alpha}\Phi_{,\beta} - \varrho_R\delta_{ik})h^k + [A_i{}^\alpha{}_k{}^\beta\Phi_{,\alpha\beta} + A_{ik}{}^{\alpha\beta}{}_{,\alpha}\phi_\beta]g^k + [A_i{}^\alpha{}_k{}^\beta(\Phi_{,\alpha}g^k{}_{,\beta} + \Phi_{,\beta}g^k{}_{,\alpha}) + 2\varrho_R\delta_{ik}g^k{}_{,t}]\} + \ldots = 0.$$

Each coefficient of S''', S'', S' must vanish; hence

$$(3.6)\quad (A_i{}^\alpha{}_k{}^\beta\Phi_{,\alpha}\Phi_{,\beta} - \varrho_R\delta_{ik})g^k = 0,$$

$$(3.7)\quad (A_i{}^\alpha{}_k{}^\beta\Phi_{,\beta}\Phi_{,\beta} - \varrho_R\delta_{ik})h^k + A_i{}^\alpha{}_k{}^\beta(\Phi_{,\alpha}g^k{}_{,\beta} + \Phi_{,\beta}g^k{}_{,\alpha}) + 2\varrho_R\delta_{ik}g^k{}_{,t} + (A_i{}^\alpha{}_k{}^\beta\Phi_{,\alpha\beta} + A_i{}^\alpha{}_k{}^\beta{}_{,\alpha}\Phi_{,\beta})g^k = 0.$$

Comparing (3.6) with (2.12), we arrive at the relation

$$(3.8)\quad g^k = \varkappa A^k,$$

where $\varkappa$ is a scalar. We emphasize that the length of A^k has been fixed $A^k = A^k(X^\alpha, t)$. To determine $\varkappa$ we multiply (3.7) by A^i and substitute the relation $g^i = \varkappa A^i$. Since the coefficient of h^k now vanishes (the quantity in the parenthesis in (2.13) is symmetric with respect to i, k), we obtain:

$$(3.9)\quad A^i[A_i{}^\alpha{}_k{}^\beta(\Phi_{,\alpha}\varkappa_{,\beta} + \Phi_{,\beta}\varkappa_{,\alpha}) + 2\varrho_R\delta_{ik}\varkappa_{,t}]A^k + \varkappa A^i[A_i{}^\alpha{}_k{}^\beta(\Phi_{,\alpha}A^k{}_{,\beta} + \phi_{,\beta}A^k{}_{,\alpha}) + 2\varrho_R\delta_{ik}A^k{}_{,t} + A_i{}^\alpha{}_k{}^\beta\Phi_{,\alpha\beta}A^k + A_i{}^\alpha{}_k{}^\beta{}_{,\alpha}\phi_{,\beta}A^k] = 0.$$

Let

$$(3.10)\quad P_{ik} = A_i{}^\beta{}_k{}^\beta\phi_{,\alpha}\phi_{,\beta} - \varrho_R\sigma_{ik}\Phi_{,t}\Phi_{,t} = \frac{1}{U^2}A_i{}^\alpha{}_k{}^\beta N_\alpha N_\beta - \varrho_R\delta_{ik}\phi_{,t}\phi_t,$$

$$\frac{\partial P_{ik}}{\partial\phi_{,\alpha}}\varkappa_{,\alpha} = A_i{}^\alpha{}_k{}^\beta(\phi_{,\alpha}\varkappa_{,\beta} + \phi_{,\beta}\varkappa_{,\alpha}),$$

$$\frac{\partial P_{ik}}{\partial\phi_{,t}}\varkappa_{,t} = -2\varrho_R\delta_{ik}\phi_{,t} = 2\varrho_R\delta_{ik},$$

where the relation $\Phi_{,t} = -1$ has been taken into account. Let us construct a curve λ determined by the parametric equation $X^\alpha(\lambda)$, $t(\lambda)$, satisfying the differential relations

$$(3.11)\quad \frac{dX^\alpha}{d\lambda} = \frac{\partial P_{ik}}{\partial\phi_{,\alpha}}A^iA^k = A_i{}^\varkappa{}_k{}^\mu(\Phi_{,\varkappa}\delta_{\mu\alpha} + \Phi_{,\mu}\delta_{\varkappa\alpha})A^iA^k,$$

$$\frac{dt}{d\lambda} = \frac{\partial P_{ik}}{\partial\Phi_{,t}}A^iA^k = 2\varrho_R\delta_{ik}A^iA^k.$$

In accordance with (3.8), the normal to the four-dimensional surface $\Phi = 0$ has the coordinates

$$(\Phi_{,1}, \Phi_{,2}, \Phi_{,3}, -1).$$

Thus the scalar product of the vector (3.11) and the normal is

$$(3.12)\quad (2A_i{}^\varkappa{}_k{}^\mu\Phi_{,\varkappa}\Phi_{,\mu} - 2P_R)A^iA^k = 2P_{ik}A^iA^k,$$

and in view of (2.13), this expression vanishes. Therefore, the curve λ is situated on the surface $\Phi = 0$. It is a bicharacteristic of Eq. (1.6).

In accordance with (3.11), the first term of Eq. (3.9) is the derivative $dk/d\lambda$. To transform the second term we calculate the derivatives of the product

$$(3.13)\qquad \begin{aligned} (\phi A^k)_{,\alpha} &= \Phi_{,\alpha}A^k+\Phi A^k{}_{,\alpha},\\ (\Phi A^k)_{,\alpha\beta} &= \Phi_{,\alpha\beta}A^k+\Phi_{,\alpha}A^k{}_{,\beta}+\phi_{,\beta}A^k{}_{,\alpha}+\Phi A^k{}_{,\alpha\beta},\\ (\phi A^k)_{,t} &= (\phi_{,t}A^k+\Phi A^k{}_{,t})\\ (\Phi A^k)_{,tt} &= \Phi_{,tt}A^k+2\Phi_{,t}A^k{}_{,t}+\Phi A^k{}_{,tt}, \end{aligned}$$

Evidently, the second term of Eq. (3.9) is equal to $\varkappa A^i\mathscr{L}_{ir}(\Phi A^r)$ for $\Phi = 0$. Thus we finally have the differential equation

$$(3.14)\qquad \frac{d\varkappa}{d\lambda}+\varkappa R(\lambda) = 0, \qquad R(\lambda) = A^i\mathscr{L}_{ir}(\Phi A^r)/_{\phi=0}.$$

Since $A^i(X^\alpha, t)$ and $\Phi(X^\alpha, t)$ are known and $X^\alpha = X^\alpha(\lambda)$, $t = t(\lambda)$, the above equation is an ordinary differential equation for the function $\varkappa(\lambda)$. Its solution has the form

$$(3.15)\qquad \varkappa(\lambda) = C\exp\Big[-\int\limits_0^\lambda R(\lambda)d\lambda\Big],$$

where C is an integration constant. Consequently, if at one point of the curve λ we have $\varkappa = 0$, then along the whole curve $\varkappa = 0$. The projection of the bicharacteristic λ on the space X^α is the ray r, the equation of which is $X^\alpha = X^\alpha(\lambda)$. If at a point of the space $\varkappa = 0$, then along the whole ray r passing through this point, $\varkappa = 0$. The discontinuities of the displacement u^i are propagated along the rays r. Equation (3.14) is liner in $\varkappa$, because the calculations were based on the linearized Eq. (1.6).

The equation of the ray is determined by $(3.11)_1$. However, in accordance with (1.11)

$$(3.16)\qquad \frac{dX^\alpha}{d\lambda} = A^iA^k\frac{1}{U}\,(A_i{}^\varkappa{}_k{}^\mu(N_\varkappa\delta_{\mu\alpha}+N_\mu\delta_{\varkappa\alpha}),$$

and therefore, in view of (2.5) and (3.11),

$$(3.17)\qquad \frac{dX^\alpha}{d\lambda} = \frac{1}{U}A^iA^k\frac{\partial Q_{ik}}{\partial N_\alpha}, \qquad \frac{dX^\alpha}{dt} = \frac{1}{2\varrho_R U}\frac{AA^k}{A^rA^r}\frac{\partial Q_{ik}}{\partial N_\alpha}.$$

The derivative dX^α/dt is the vector of the ray velocity U_r^α, the length of which we denote by U_r. Multiplying by N_α we, have

$$U_r^\alpha N_\alpha = \frac{Q_{ik}}{\varrho_R U}\frac{A^iA^k}{A^rA^r}$$

and, in accordance with (2.4),

$$(3.18)\qquad U_r^\alpha N_\alpha = U.$$

The projection of the ray velocity on the normal is therefore equal to the speed of propagation. Consequently,

$$(3.19)\qquad U_r \geqslant U.$$

All above relations can easily be written in curvilinear coordinates

To this end, all spatial partial derivatives should be replaced by the total covariant derivatives and the time partial derivatives by the material time derivatives.

Our considerations concerned the propagation of a discontinuity surface in the reference configuration B_R. With the help of the relation (1.1) we can immediately pass to the corresponding relations for the configuration B. However, for brevity, we do not quote here the transformations.

References

1. C. TRUESDEL, *General and exact theory of waves in finite elastic strain*, Arch. Rat. Mech. Anal., **8**, 1961.
2. P. J. CHEN, *The growth of acceleration waves of arbitrary from in homogeneously deformed elastic materials*, Arch. Rat. Mech. Anal., **30**, 1968.
3. P. J. CHEN, *On the growth of longitudinal waves in anisotropic elastic materials*, Arch. Rat. Mech. Anal., **36**, 1971.
4. P. CHADWICK, R. W. OGDEN, *On the definition of elastic moduli*, Arch. Rat. Mech. Anal., **44**, 1971.
5. R. M. BOWEN, C. C. WANG. *Acceleration waves in inhomogeneous isotropic elastic bodies*, Arch. Rat. Mech. Anal., 1, **38**, 1970.
6. R. A. TOUPIN, B. BERNSTEIN, *Sound waves in deformed perfectly elastic materials*, J. Acoust. Soc. Amer. 33, 1961.
7. R. COURANT, *Partial differential equations*, New York 1962.

INSTITUTE OF FUNDAMENTAL TECHNICAL RESEARCH
POLISH ACADEMY OF SCIENCES

Received April 7, 1972