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Buckling of viscoplastic cylindrical shells loaded by radial pressure impulse

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THIS PAPER considers the problem of dynamic buckling of a viscoplastic cylindrical shell subject to the action of a radial pressure impulse. The constitutive equations assumed in the paper are referred to an incompressible viscoplastic material. The solution concerning a perfectly plastic material is also obtained as a limiting case of the viscoplastic solution. The length of the shell is taken into account according to [8]. The influence of viscosity of the material and of the geometric parameters upon the following phenomena is discussed: the beginning and course of the buckling process, final deformation magnitude, the duration of the deformation process, the critical impulse magnitude for shells made of aluminium alloys and mild steel. Theoretical results are compared with experiments.

W pracy rozpatrywany jest problem dynamicznego wyboczenia lepkoplastycznej powłoki cylindrycznej pod wpływem przyłożonego promieniowego impulsu ciśnienia. Przyjęto równania konstytutywne dla nieściśliwego materiału lepkoplastycznego. Pokazane jest również rozwiązanie w ramach teorii plastyczności jako wynik analizy granicznego przypadku równań opisujących własności materiału lepkoplastycznego. Uwzględnienie długości powłoki jest wprowadzone według pracy [8]. Przedyskutowano wpływ lepkości materiału oraz parametrów geometrycznych na powstanie i przebieg niestateczności, na wielkość końcowych odkształceń, czas trwania procesu deformacji i wartości impulsu krytycznego dla powłok wykonanych ze stopów aluminiowych i miękkich stali. Uzyskane rezultaty teoretyczne porównano z doświadczeniami.

В работе рассмотрена задача о динамической устойчивости вязкопластической цилиндрической оболочки под действием радиально прикладываемого импульса давления. Приняты определяющие уравнения несжимаемого вязкопластического материала. Получено также решение в рамках теории пластичности, являющееся результатом предельного перехода в уравнениях для вязкопластического материала. Длина оболочки учитывается согласно работе [8]. Обсуждается влияние вязкости материала и геометрических параметров оболочки на возникновение и вид потери устойчивости, величину остаточных деформаций, продолжительность процесса деформирования и величину критического импульса для оболочек, выполненных из сплавов алюминия и мягких сталей. Полученные теоретические результаты сравниваются с экспериментальными данными.

1. Introduction

THE PROBLEM of dynamic buckling of elastic-plastic or plastic cylindrical shells has been considered in numerous papers published in recent years. The majority of the papers were formulated within the framework of linearized shell theory for the model of an elastic-plastic or rigid-plastic body with linear hardening [1–8]. These solutions concern shells made of strain-rate insensitive materials.

The solutions in which the viscosity effects are taken into consideration are presented in [9–11]. It has been demonstrated that the viscosity of the material exerts a stabilizing influence upon the behaviour of the shell in the process of dynamic buckling.

In [12], the higher order terms are taken into account in kinematic relations, and the result obtained is compared with the solution derived by means of the linear theory [4].

In the recent paper [8], the length of the shell is taken into consideration on the basis of experimental results. The solution concerns shells made of rigid-plastic materials with linear hardening, and loaded by a radial impulse.

The aim of the present paper is to solve the problem of buckling of a viscoplastic cylindrical shell of an arbitrary length and loaded by a radial pressure impulse. The length of the shell is taken into account according to [8]. The possibility will be shown of obtaining plastic solutions treated as a limiting case of solutions concerning viscoplastic materials. The theoretical results obtained in the paper will be compared with results of experimental investigations.

2. Constitutive equations

Let us assume the particular case of constitutive equations for strain-rate sensitive materials introduced by PERZYNA [13]

$$\dot{\varepsilon}_{ij} = \gamma\Phi(F)\frac{\partial F}{\partial\sigma_{ij}}, \tag{2.1}$$

where $\dot{\varepsilon}_{ij}$ and σ_{ij} denote the respective tensors of plastic strain rates and of stress, and γ is the viscosity coefficient (a material constant). The function F appearing in Eq. (2.1) is defined by $F = \sqrt{I_2}/K-1$, $I_2 = S_{ij}S_{ij}/2$ being the second invariant of the stress deviator with the components S_{ij} and $K = \sigma_0/\sqrt{3}$; σ_0 is the static yield limit of the material.

The physical equations of the plastic flow theory according to Saint Venant-Levy-Mises, $\dot{\varepsilon}_{ij} = \lambda S_{ij}$, are obtained from the Eqs. (2.1) if $\gamma = \infty$ and $\sqrt{I_2} = K$.

In the case of plane stress, the Eqs. (2.1) yield:

$$\dot{\varepsilon}_1 = \frac{\gamma}{3}\left(\frac{\sqrt{I_2}}{K}-1\right)\frac{2\sigma_1-\sigma_2}{\sqrt{I_2}}, \quad \dot{\varepsilon}_2 = \frac{\gamma}{3}\left(\frac{\sqrt{I_2}}{K}-1\right)\frac{2\sigma_2-\sigma_1}{\sqrt{I_2}}, \tag{2.2}$$

where the equation

$$I_2 = \frac{1}{3}(\sigma^2-\sigma_1\sigma_2+\sigma_2^2) \tag{2.3}$$

represents the Huber-Mises ellipse in the plane of principal stresses σ_1, σ_2, Fig. 3.

3. Formulation of the problem

Let us consider a cylindrical shell loaded by a pressure impulse directed radially and inwards, Fig. 1. In the state of compressive plastic and viscoplastic flow, the resulting homogeneous deformation (decreasing of the shell radius under unchanged circular symmetry conditions) is — in general — unstable, since every small perturbation of the deformation exhibits a tendency to increase during the subsequent deformation process. The perturbations are caused by imperfect (non-uniform) distribution of radial displacements and velocities. The problem is mathematically formulated similarly to [1–11] as a superposition of small perturbations on the fundamental unperturbed motion. The amplitude of the perturbed motion has to be chosen so as to ensure that the homogeneous compressive deformation is predominant over the local bending. Material incompressibility will be assumed and elastic deformations will be disregarded as compared with moderately large plastic deformations.

In the problem considered, the longitudinal deformation occurs independently of the action of axial forces at the ends of the shell and without external constraints counteracting the free deformation. Longitudinal inertia will also be disregarded. Thus, for sufficiently

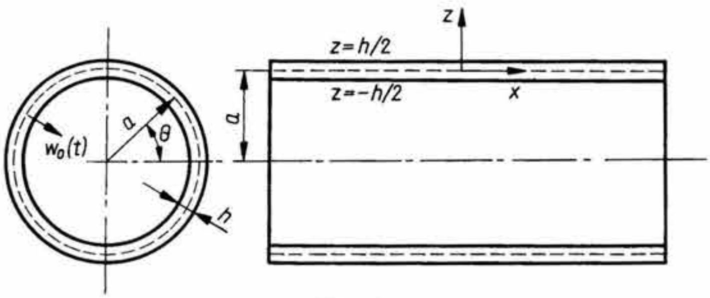

FIG. 1.

short shells, it may be assumed that $\sigma_x = 0$, on the middle surface $z = 0$. Making that assumption in the Eq. (2.2), we obtain $\dot{\varepsilon}_x/\dot{\varepsilon}_\theta = -1/2$.

In the case of very long shells, however, the assumption $\dot{\varepsilon}_x = 0$ should be made, yielding by means of the Eq. (2.2) the result $\sigma_x/\sigma_\theta = 1/2$. In order to take into account the effect of the length of the shell, let us introduce the relation

(3.1) $$\dot{\varepsilon}_x = -k\dot{\varepsilon}_\theta \quad \text{for} \quad z = 0,$$

where

$$0 \leqslant k \leqslant 1/2.$$

The relation was originally proposed in [8]. The change of the value of k from $k = 1/2$ to $k = 0$ corresponds to the variation of the length of the shell — from very short to infinitely long. On the basis of experimental results, the relation between k and L/D was determined in [8], Fig. 2,

(3.2) $$L/D = \ln \frac{1}{4}k^2,$$

Here L is the length of the shell, and D its diameter.

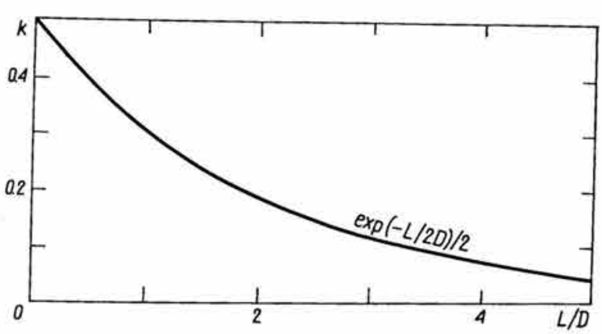

FIG. 2.

Again on the basis of experimental results, let us furthermore assume that the normal cross-sections of the shell remain plane during the deformation process, $\dot{\varepsilon}_x = \dot{\varepsilon}_x(t)$.

In Fig. 3 is presented the actual dynamic yield condition, the stress profile and the strain rate vectors perpendicular to the ellipse for the points of the surface $z = 0$ and for three values of k.

The equations of motion and the geometric relations applied in the present paper are taken from [8]. In that paper were applied the constitutive relations for a rigid-plastic

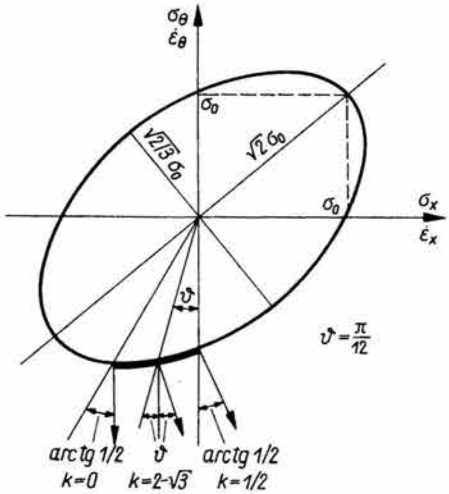

FIG. 3.

material with linear hardening and without the viscosity effects. In the present paper, however, we are using the constitutive equations for strain rate sensitive materials and hence the equation governing the problem, as also the final solutions, differ from those of the paper [8].

4. Unperturbed motion

The radial displacements of the middle surface of the shell directed inwards and produced by a uniform pressure impulse is assumed in the form $w_0(t)$. The components of the strain rate tensor are equal to [8],

$$\dot{\varepsilon}_x = \frac{k\dot{w}_0}{a}, \quad \dot{\varepsilon}_\theta = -\left(1-\frac{z}{a}\right)\frac{\dot{w}_0}{a}, \quad \dot{\varepsilon}_z = \left(1-k-\frac{z}{a}\right)\frac{\dot{w}_0}{a}, \tag{4.1}$$

where the dots denote differentiation with respect to time t. In the Eqs. (4.1) and in all other expressions, the terms z/a in powers higher than one are disregarded in comparison with unity. Introducing the notations

$$K_1 = 2(2-k+k^2), \quad K_2 = (3K_1/2)^{1/2}, \quad K_3 = (2-k)/K_1 \tag{4.2}$$

and inserting Eqs. (4.1) into Eqs. (2.2), we obtain the following components of the stress tensor:

$$\text{(4.3)}\qquad \begin{aligned}\sigma_x &= \frac{\sqrt{3}\sigma_0}{3\gamma}\left(2k-1+\frac{z}{a}\right)\frac{\dot{w}_0}{a}+\frac{\sigma_0}{K_2}\left(2k-1+\frac{3kK_3}{2-k}\,\frac{z}{a}\right),\\ \sigma_\theta &= \frac{\sqrt{3}\sigma_0}{3\gamma}\left(k-2+\frac{2z}{a}\right)\frac{\dot{w}_0}{a}+\frac{\sigma_0}{K_2}\left(k-2+\frac{3k^2K_3}{2-k}\,\frac{z}{a}\right).\end{aligned}$$

According to the assumptions of the technical shell theory the assumption $\sigma_z = 0$ has been made.

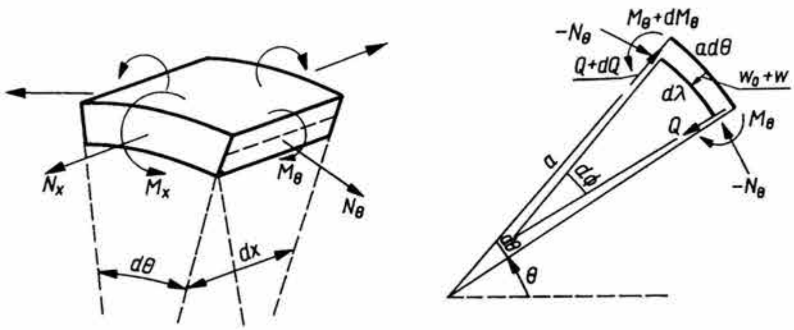

FIG. 4.

The components of internal forces (Fig. 4) are

$$\text{(4.4)}\qquad \begin{aligned}M_x &= \int_{-h/2}^{h/2}\sigma_x z\,dz = \left[\frac{\sqrt{3}\sigma_0}{3\gamma}\,\frac{\dot{w}_0}{a}+\frac{3k\sigma_0K_3}{(2-k)K_2}\right]\frac{h^3}{12a},\\ M_\theta &= \int_{-h/2}^{h/2}\sigma_\theta z\,dz = \left[\frac{2\sqrt{3}\sigma_0}{3\gamma}\,\frac{\dot{w}_0}{a}+\frac{3k^2\sigma_0K_3}{(2-k)K_2}\right]\frac{h^3}{12a},\\ N_x &= \int_{-h/2}^{h/2}\sigma_x\,dz = \left[\frac{\sqrt{3}\sigma_0}{3\gamma}(2k-1)\frac{\dot{w}_0}{a}+\frac{\sigma_0}{K_2}(2k-1)\right]h,\\ N_\theta &= \int_{-h/2}^{h/2}\sigma_\theta\,dz = \left[\frac{\sqrt{3}\sigma_0}{3\gamma}(k-2)\frac{\dot{w}_0}{a}+\frac{\sigma_0}{K_2}(k-2)\right]h.\end{aligned}$$

Substitution of $\gamma \to \infty$ in Eqs. (4.3), (4.4) yields the corresponding expressions for ideal plasticity.

The equation of motion of the shell element has the form

$$\text{(4.5)}\qquad N_\theta = a\varrho h\ddot{w}_0,$$

and after substitution in Eqs. (4.4) is transformed to

$$\text{(4.6)}\qquad \ddot{w}_0+\frac{\sqrt{3}\sigma_0(2-k)}{3\varrho\gamma a^2}\,\dot{w}_0 = -\frac{\sigma_0(2-k)}{a\varrho K_2};$$

ϱ denotes the density of the material.

The solution of Eq. (4.6) with the initial conditions

$$w_0(0) = 0, \quad \dot{w}_0(0) = V_0 \tag{4.7}$$

has the form

$$w_0 = \frac{\sqrt{3}\gamma\varrho a^2}{(2-k)\sigma_0}\left(V_0 + \frac{\sqrt{3}\gamma a}{K_2}\right)\left[1 - \exp\left(-\frac{(2-k)\sigma_0}{\sqrt{3}\gamma\varrho a^2}t\right)\right] - \frac{\sqrt{3}\gamma a}{K_2}t. \tag{4.8}$$

The unperturbed motion ceases at the instant $t = t_f$ when $\dot{w}_0(t_f) = 0$. The time of its duration is then found to be

$$t_f = -\frac{\sqrt{3}\gamma\varrho a^2}{(2-k)\sigma_0}\ln\frac{\sqrt{3}\gamma a}{V_0K_2 + \sqrt{3}\gamma a}. \tag{4.9}$$

In the limiting case of ideal plasticity $\gamma \to \infty$ and Eqs. (4.8), (4.9) are reduced to the respective two equations

$$w_0 = \frac{\sigma_0(k-2)}{2\varrho a K_2}t^2 + V_0 t, \tag{4.10}$$

$$t_f = \frac{V_0 \varrho a K_2}{(2-k)\sigma_0}. \tag{4.11}$$

Substituting $k = 1/2$ into the relations derived in this section, we obtain the results given in [11].

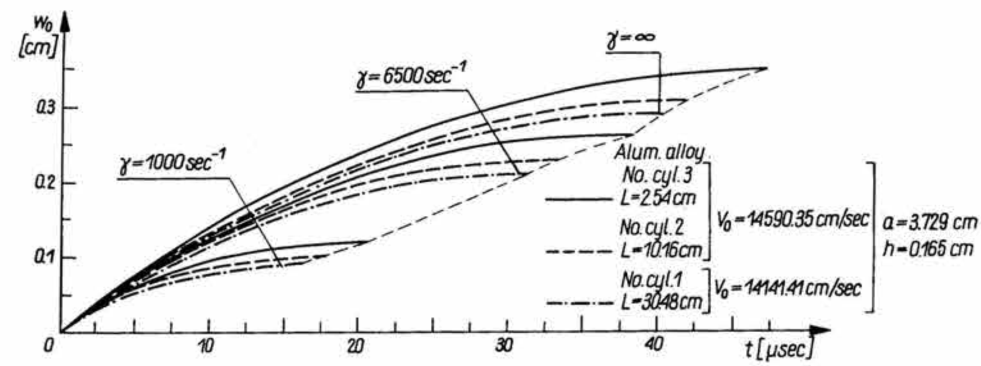

Fig. 5.

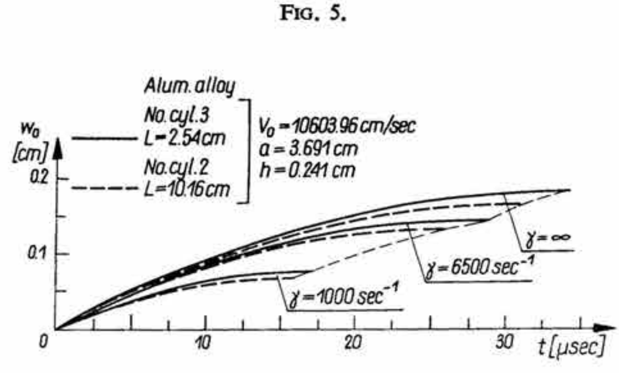

Fig. 6.

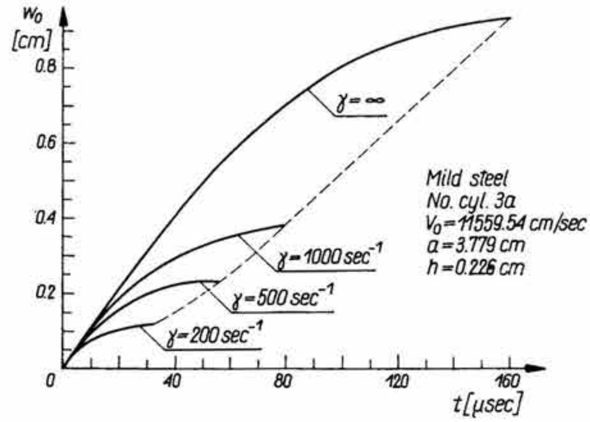

FIG. 7.

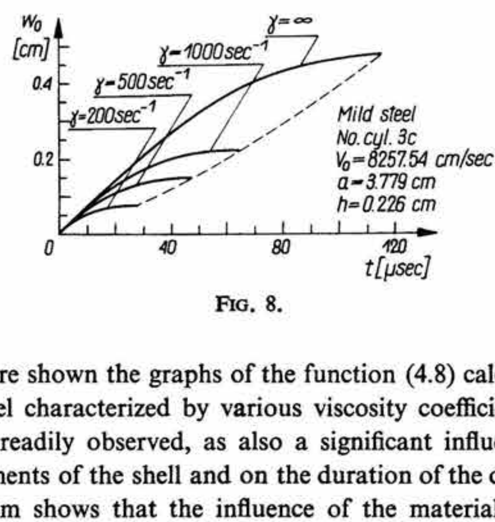

FIG. 8.

In Figs. 5, 6, 7, 8 are shown the graphs of the function (4.8) calculated for shells made of aluminium and steel characterized by various viscosity coefficients. The effect of the length of the shell is readily observed, as also a significant influence of viscosity upon the values of displacements of the shell and on the duration of the deformation. A detailed analysis of the problem shows that the influence of the material viscosity significantly increases in the course of deformation, especially when the velocity V_0 is sufficiently large.

Introducing the impulse magnitude $I = \varrho h V_0$, we obtain from the Eqs. (4.8), (4.9) in the general case

$$I - \frac{\sqrt{3}\gamma\varrho a h}{K_2} \ln\left(1 + \frac{I K_2}{\sqrt{3}\gamma\varrho a h}\right) = \frac{(2-k)\sigma_0 h}{\sqrt{3}\gamma a^2} w_0(t_f), \tag{4.12}$$

and in the limiting case $\gamma \to \infty$

$$I = h\left(2\sigma_0\varrho \frac{2-k}{K_2} \frac{w_0(t_f)}{a}\right)^{1/2}. \tag{4.13}$$

Equations (4.12), (4.13) may be used to calculate, for a given shell and the required final displacement value $w_0(t_f)$, the value of the impulse applied to the shell.

5. Perturbed motion

Let us now introduce the perturbed displacement function $w_0(\theta, t)$; the total displacement directed toward the center of the shell will be equal to $w_0(t)+w(\theta, t)$. Assume, moreover, that $w(\theta, t)$ is much smaller than $w_0(t)$.

The curvature of the deformed shell, Fig. 4, is equal to

$$\frac{\partial \Phi}{\partial \lambda} = \varkappa = \frac{1}{a} + \frac{w_0}{a^2} + \frac{w''}{a^2}, \tag{5.1}$$

primes denoting differentiation with respect to θ.

The strain rate tensor components are, [8],

$$\begin{aligned} \dot{\varepsilon}_x &= \frac{k\dot{w}_0}{a}, \\ \dot{\varepsilon}_\theta &= -\left(1-\frac{z}{a}\right)\frac{\dot{w}_0}{a} + \frac{z}{a}\frac{\dot{w}''}{a}, \\ \dot{\varepsilon}_z &= \left(1-k-\frac{z}{a}\right)\frac{\dot{w}_0}{a} - \frac{z}{a}\frac{\dot{w}''}{a}. \end{aligned} \tag{5.2}$$

Substituting Eqs. (5.2) into (2.2), we obtain the components of the stress tensor

$$\begin{aligned} \sigma_x &= \frac{\sqrt{3}\sigma_0}{3\gamma}\left[\frac{\dot{w}_0}{a}(2k-1) + \frac{z}{a}\frac{\dot{w}_0+\dot{w}''}{a}\right] + \sigma_0\left[\frac{2k-1}{K_2} + \frac{z}{a}\frac{3kK_3}{(2-k)K_2}\left(1+\frac{\dot{w}''}{\dot{w}_0}\right)\right], \\ \sigma_\theta &= \frac{\sqrt{3}\sigma_0}{3\gamma}\left[\frac{\dot{w}_0}{a}(k-2) + \frac{2z}{a}\frac{\dot{w}_0+\dot{w}''}{a}\right] + \sigma_0\left[\frac{k-2}{K_2} + \frac{z}{a}\frac{3k^2K_3}{(2-k)K_2}\left(1+\frac{\dot{w}''}{\dot{w}_0}\right)\right]. \end{aligned} \tag{5.3}$$

Integration of Eqs. (5.3) over the thickness of the shell yields the internal forces:

$$\begin{aligned} M_x &= \left[\frac{\sqrt{3}\sigma_0}{3\gamma}\frac{\dot{w}_0+\dot{w}''}{a} + \frac{3k\sigma_0K_3}{(2-k)K_2}\left(1+\frac{\dot{w}''}{\dot{w}_0}\right)\right]\frac{h^3}{12a}, \\ M_\theta &= \left[\frac{2\sqrt{3}\sigma_0}{3\gamma}\frac{\dot{w}_0+\dot{w}''}{a} + \frac{3k^2\sigma_0K_3}{(2-k)K_2}\left(1+\frac{\dot{w}''}{\dot{w}_0}\right)\right]\frac{h^3}{12a}, \\ N_x &= \frac{\sqrt{3}\sigma_0}{3\gamma}\left[\frac{\dot{w}_0}{a}(2k-1)\right]h + \sigma_0\left(\frac{2k-1}{K_2}\right)h, \\ N_\theta &= \frac{\sqrt{3}\sigma_0}{3\gamma}\left[\frac{\dot{w}_0}{a}(k-2)\right]h + \sigma_0\left(\frac{k-2}{K_2}\right)h. \end{aligned} \tag{5.4}$$

In the limiting case $\gamma \to \infty$ Eqs. (5.3), (5.4) yield the corresponding formulae of ideal plasticity.

6. Solution and determination of the buckled form

The perturbed motion equation of the shell element, Fig. 4, considered in [8] and taking into account the transversal inertia effects, has the form:

$$\frac{M''_\theta}{a^2} - N_\theta\left(\frac{1}{a} + \frac{w_0}{a^2} + \frac{w''}{a^2}\right) + \varrho h(\ddot{w}_0 + \ddot{w}) = 0. \tag{6.1}$$

Using Eqs. (4.5), (5.4) in (6.1), and disregarding w_0/a as small compared with 1 in Eq. (5.1), we obtain the equation governing the problem of viscoplastic buckling:

$$(6.2)\qquad \left[\frac{2\sqrt{3}\sigma_0}{3\gamma}\frac{\dot{w}''}{a}+\frac{3k^2\sigma_0 K_3}{(2-k)K_2}\frac{\dot{w}''}{\dot{w}_0}\right]''\frac{h^3}{12a^3} - \left[\frac{\sqrt{3}\sigma_0(k-2)}{3\gamma}\frac{\dot{w}_0}{a}\frac{w''}{a}+\frac{\sigma_0(k-2)}{K_2}\frac{w''}{a}\right]\frac{h}{a}+\varrho h\ddot{w}=0,$$

$\dot{w}_0$ being determined by Eq. (4.8).

The following dimensionless quantities are now introduced

$$(6.3)\qquad u=\frac{w}{a},\quad u_0=\frac{w_0}{a},\quad \tau=\frac{V_0 t}{2a},\quad \alpha^2=\frac{h^2}{12a^2},\quad \beta=\frac{\sqrt{3}V_0}{3a\gamma},\quad \delta=1+\beta K_2,$$

enabling transformation of Eqs. (6.2) to the form

$$(6.4)\qquad \left[\beta\dot{u}''+\frac{3k^2K_3}{(2-k)K_2}\frac{\dot{u}''}{\dot{u}_0}\right]''\alpha^2+\left[\frac{\beta}{2}(2-k)\dot{u}_0\dot{u}''+\frac{2-k}{K_2}u''\right]-\frac{\beta\tau_f(2-k)}{2\ln\frac{1}{\delta}}\ddot{u}=0,$$

Dotted symbols denote now the derivatives with respect to dimensionless time τ.

The solution of the unperturbed motion (4.8) which will be used later, may be rewritten in terms of the notations (6.3),

$$(6.5)\qquad u_0=\frac{2}{1-\delta}\left\{\frac{\tau_f\delta}{\ln\frac{1}{\delta}}\left[1-\left(\frac{1}{\delta}\right)^{\tau/\tau_f}\right]+\tau\right\}.$$

The additional dimensionless variable is introduced for the sake of simplicity,

$$(6.6)\qquad \zeta=1-\frac{\tau}{\tau_f},\quad 0\leqslant\zeta\leqslant 1,$$

τ_f denoting the value of τ at the instant $t=t_f$ given by Eq. (4.9).

Equation (6.4) assumes, by means of Eqs. (6.5), (6.6), the form

$$(6.7)\qquad \ddot{u}+\frac{2\tau_f\ln(1/\delta)}{1-\delta}\delta^{\zeta}u''+\frac{2\alpha^2\ln(1/\delta)}{2-k}\left[1+\frac{3k^2K_3}{2(2-k)(\delta^{\zeta}-1)}\right]\dot{u}''''=0,$$

dots denoting the derivatives with respect to ζ.

The perturbed displacement motion can generally be expressed as a sine and cosine sum of a Fourier series. In view of the fact that the differential equations governing the time variation of the displacement amplitude $u_n(\zeta)$ are the same for both these series, assumption of the following form of the function proves to be sufficient:

$$(6.8)\qquad u(\theta,\zeta)=\sum_{n=1}^{\infty}u_n(\zeta)\sin n\theta.$$

The initial perturbation of the displacement

$$(6.9)\qquad u(\theta,1)=\sum_{n=1}^{\infty}a_n\sin n\theta,$$

may be produced, for instance, by mechanical working of the lateral surfaces of the cylinders, and then the magnitude a_n can be determined as the admissible tolerance of the working process. Inserting (6.8) into (6.7), we obtain the ordinary differential equation for the amplitude u_n,

$$\ddot{u}_n + (R_n \delta^\xi + Q_n)\dot{u}_n/(\delta^\xi - 1) - S_n^2 \delta^\xi u_n = 0. \tag{6.10}$$

Here

$$R_n = \frac{2\alpha^2 \ln(1/\delta)}{2-k} n^4,$$

$$Q_n = \frac{2\alpha^2 \ln(1/\delta)}{2-k}\left[-1 + \frac{3k^2 K_3}{2(2-k)}\right] n^4, \tag{6.11}$$

$$S_n^2 = \frac{2\tau_f \ln(1/\delta)}{1-\delta} n^2.$$

Assuming the representation of the initial perturbed velocity in the form

$$V = V_0\left(1 + \sum_{n=1}^{\infty} b_n \sin n\theta\right) \tag{6.12}$$

and the function (6.9), we obtain the initial conditions of Eq. (6.10)

$$u_n(1) = a_n, \quad \dot{u}_n(1) = -2\tau_f b_n. \tag{6.13}$$

Substitution of $k = 1/2$ into the Eq. (6.10) does not lead to the corresponding equation of the amplitude u_n derived in [11], since there the unperturbed motion and the circumferential force have been approximated by the solutions based on ideal plasticity.

7. Limiting passage to ideal plasticity

In ideal plasticity, $\gamma = 0$. The solution of the unperturbed motion (4.10), (4.11), written in terms of dimensionless quantities has the form:

$$u_0 = \tau(2 - \tau/\tau_f), \quad \tau_f = \frac{V_0^2 \varrho K_2}{2(2-k)\delta\sigma_0}. \tag{7.1}$$

It is readily seen that in this case τ_f is not only the dimensionless instant of time at which the motion ceases but also the final maximum hoop deformation of the middle surface of the shell, $u_0(\tau_f) = \tau_f$. Equation (6.4) leads, by means of Eqs. (7.1), (6.6) and (6.8), to the following differential equation of the amplitude:

$$\ddot{u}_n - Q_n \dot{u}_n/\zeta - S_n^2 u_n = 0. \tag{7.2}$$

Here

$$Q_n = \frac{3k^2\alpha^2 K_3}{(2-k)^2} n^4, \quad S_n^2 = 2\tau_f n^2. \tag{7.3}$$

Equation (7.2) is identical with the equation derived in [11] for the values of Q_n and S_n^2 corresponding to $k = 1/2$.

The solution of Eq. (7.2), together with the initial conditions (6.13) for $S_n^2 > 0$ [which is always true owing to Eq. (7.3)], has the following form [14]:

$$u_n(\zeta) = A_n(\zeta)a_n + B_n(\zeta)b_n, \tag{7.4}$$

with the notations

$$\begin{aligned} A_n(\zeta) &= \zeta^\nu[I_\nu(S_n\zeta)K_{\nu-1}(S_n) + K_\nu(S_n\zeta)I_{\nu-1}(S_n)]S_n, \\ B_n(\zeta) &= 2\tau_f\zeta^\nu[-I_\nu(S_n\zeta)K_\nu(S_n) + K_\nu(S_n\zeta)I_\nu(S_n)], \end{aligned} \tag{7.5}$$

I_ν and K_ν are the modified Bessel functions of the first and second kind and of order $\nu = (1+Q_n)/2$. It is seen from the solution (7.4) that the instantaneous amplitude u_n is equal to the sum of coefficients of the initial perturbations a_n and b_n multiplied by the hardening functions A_n and B_n.

In the general case $\zeta(0 \leqslant \zeta \leqslant 1)$, Eq. (7.2) can by means of numerical integration be more easily solved than Eq. (7.5). For a finite motion ($\zeta = 0$), however, the asymptotic form of Eq. (7.5) enables the derivation of certain useful formulae.

In the limiting case $\xi \to 0$ we have [14, 15],

$$\zeta^\nu I_\nu(S_n\zeta) \to 0, \qquad \zeta^\nu K_\nu(S_n\zeta) \to 2^{\nu-1}\Gamma(\nu)/S_n^\nu, \tag{7.6}$$

where $\Gamma(\nu)$ denotes the gamma-function. For S_n sufficiently larger than ν, we also obtain:

$$I_\nu(S_n) \approx e^{S_n}/(2\pi S_n)^{1\ 2}. \tag{7.7}$$

Substitution of (7.6), (7.7) into (7.5) yields

$$\begin{aligned} A_n(0) &\approx 2^{\nu-1}\Gamma(\nu)e^{S_n}/[(2\pi)^{1/2}S_n^{\nu-1/2}], \\ B_n(0) &\approx \tau_f \cdot 2^\nu\Gamma(\nu)e^{S_n}/[(2\pi)^{1/2}S_n^{\nu+1/2}]. \end{aligned} \tag{7.8}$$

If n were assumed to be a continuous variable [6], the determinantion of the maximum of (7.8) would lead to the following two equations

$$\begin{aligned} S_n - 2Q_n\ln(S_n/2) - (Q_n/2 - 2Q_n\psi) &= 0, \\ S_n - 2Q_n\ln(S_n/2) - (1 + Q_n/2 - 2Q_n\psi) &= 0, \end{aligned} \tag{7.9}$$

$\psi(\nu) = d\ln\Gamma(\nu)/d\nu$ being the Gauss representation of gamma function. Number n (the closest and smallest possible integers) satisfying Eq. (7.9) represent the numbers of half-waves. From the numerical solution (7.9), and from its comparison with the numerical solution (7.4) given in [11], it follows that for $k = 1/2$ the assumption $Q_n \approx 2$ is justified for the shell parameters considered. Consequently it should be assumed that $\nu = 3/2$ and $\Gamma(3/2) = (1/2)\pi^{1/2}$, the hardening functions (7.8) are

$$\overline{A}_n = e^{S_n}/2S_n) \qquad \overline{B}_n = \tau_f e^{S_n}/S_n^2, \tag{7.10}$$

and the solution (7.4) for $\zeta = 0$ has the form

$$u_n = \overline{A}_n a_n + \overline{B}_n b_n. \tag{7.11}$$

For $k = 0$ it follows from (7.3) that $Q_n = 0$, and hence the value $Q_n \approx 2$ may be assumed for larger values of k only. The lower bound of the interval for k at which the assumption $Q_n \approx 2$ is still feasible can be determined by solving the Eq. (7.9) numerically. In [8], this value was assumed to be $0.3 < k \leqslant 0.5$.

8. Discussion of results and comparison with experiments

The theoretical results obtained are compared with the experimental data presented in [8] (aluminum shells) and [9] (steel shells).

The fundamental equations of amplitudes (6.10) and (7.2) have been solved, by means of the Runge-Kutta method, for various process parameters and $a_n = 0.01\, h/a$, $b_n = 0.1$. Theoretical values of the buckling forms n are given in Tables 1 and 2 for several viscosity parameters. The experimentally investigated shells [8] were made of 6061-T6 aluminium which is considered as a practically a strain rate insensitive material. Actually for $\gamma = \infty$ a good agreement of the theoretical and experimental results is achieved, Table 1. For small k the values of the coefficients at $\dot{u}_n$ in the amplitude Eqs. (6.10) approach zero and then the solution does not yield proper results. In the case of mild steel, Table 2, which is a very strain rate sensitive material, the discrepancies between the theoretical and experimental results are large. It should be emphasised that the solution was based on the relation (3.2) determined experimentally for a material practically strain rate insensitive. Taking into account the viscosity effects in (3.2), on the basis of possible experiments, we should obtain a better agreement. Generally speaking, in the case of shells made of materials exhibiting marked viscosity effects and at very smal values of k, the solution should employ nonlinear viscosity laws.

Table 1. Values of n for maximum values of u_n

Material	Cylinder No.	Time ζ	$k = -\varepsilon_x/\varepsilon_\theta$	Viscosity coefficients γ[sec^{-1}]			Experim. buckling form n
				1000	6500	∞	
Aluminium alloys	3 ($h = 0.165$ cm)	0.8	0.5	5	9	12	
		0.6		6	9	14	
		0.4		6	10	14	
		0.2		6	10	13	
		0		6	9	12	15
	2 ($h = 0.165$ cm)	0.8	0.2	5	9	20	
		0.6		6	11	22	
		0.4		6	12	21	
		0.2		7	13	20	
		0		7	12	19	16
	3 ($h = 0.241$ cm)	0.8	0.5	4	7	8	
		0.6		4	7	9	
		0.4		4	7	9	
		0.2		4	7	9	
		0		4	7	8	11
	2 ($h = 0.241$ cm)	0.8	0.3	4	7	13	
		0.6		4	8	15	
		0.4		4	8	15	
		0.2		5	9	14	
		0		5	8	13	12

Table 2. Values of n for the maximum values of u_n

Material	Cylinder No.	Time ζ	$k = -\varepsilon_x/\varepsilon_\theta$	Viscosity coefficients $\gamma[\sec^{-1}]$				Exper. buckling form n
				200	500	1000	00	
Mild steel	3a	0.8	0.5	4	5	6	13	
		0.6		4	6	7	14	
		0.4		4	6	8	14	
		0.2		4	6	8	13	
		0		4	6	7	12	13
	3c	0.8	0.5	3	4	5	10	
		0.6		4	5	6	12	
		0.4		4	5	6	12	
		0.2		4	5	6	12	
		0		4	5	6	10	14

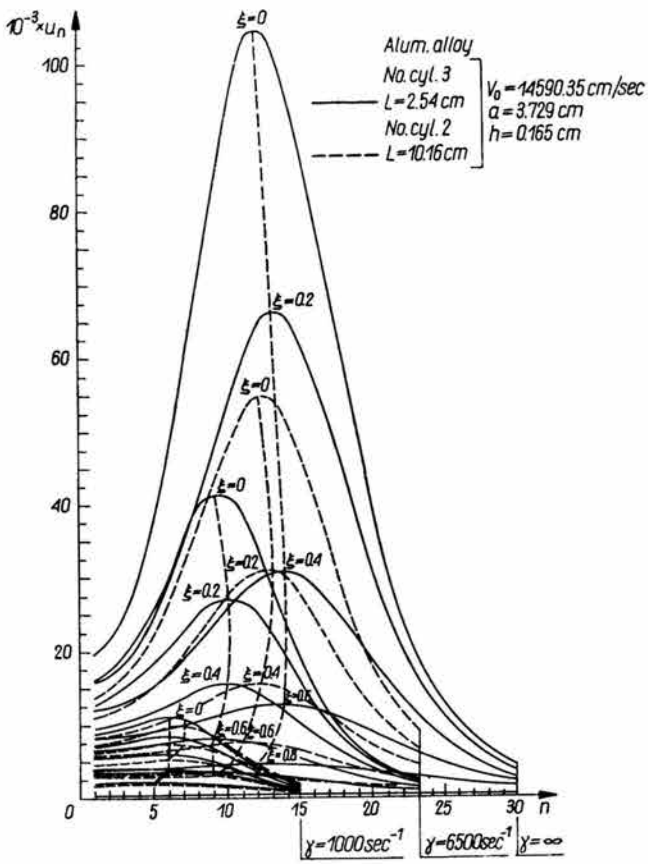

FIG. 9.

The linearized theory used in the present paper yields satisfactory results in the case of shells made of materials which are moderately strain rate sensitive and perfectly plastic, and for k lying in the interval $0.2 < k \leqslant 0.5$.

In Figs. 9 and 10 is shown the amplitude function increasing in time for consecutive values of n. The increase is larger for larger values of V_0 and γ. The change of the buckling form is shown by vertical dashed lines. Influence of viscosity is easily observed while the influence of the length of the shell on the amplitude is much smaller. The numerical results (Figs. 9, 10) indicate that the application of exact expressions for the circumferencial

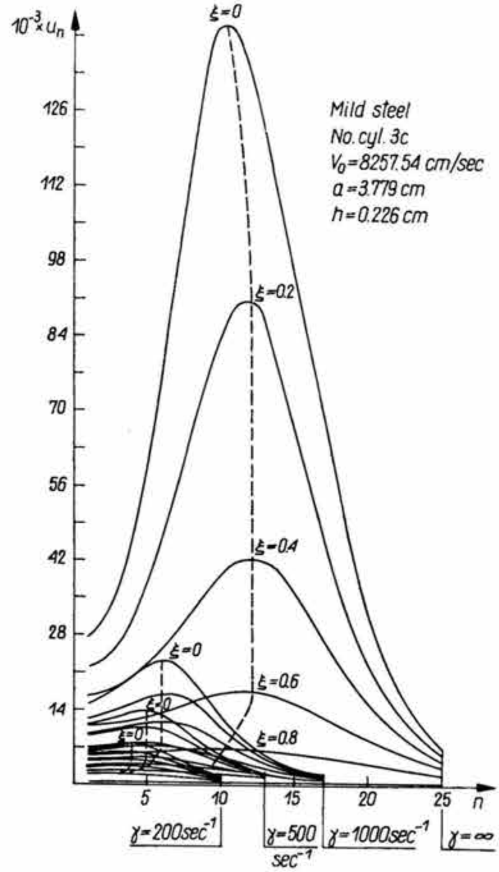

FIG. 10.

force and the unperturbed motion for $\gamma \neq \infty$ (instead of the approximation [11]) reduces the deflection amplitude u_n by the factor of 1.5–3, depending on the value of the viscosity coefficient.

In Figs. 11, 12 is shown the variation of the maximum amplitude as a function of the impulse applied. It is seen that these functions reach very large values in a certain narrow interval of the impulse variation; hence it is natural to determine the critical value of the impulse graphically as the abcissa of such a point of the curve at which a small increment

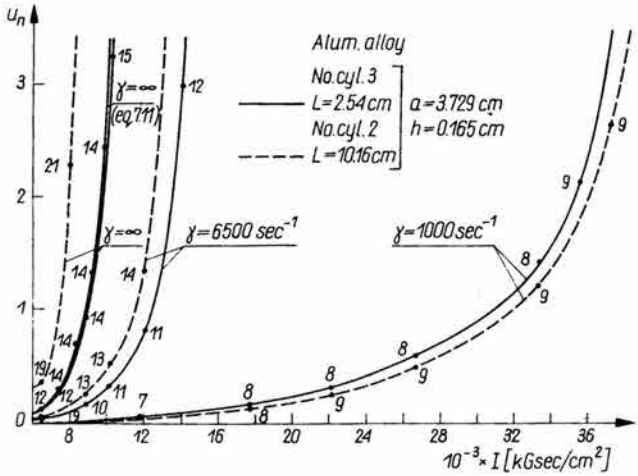

FIG. 11.

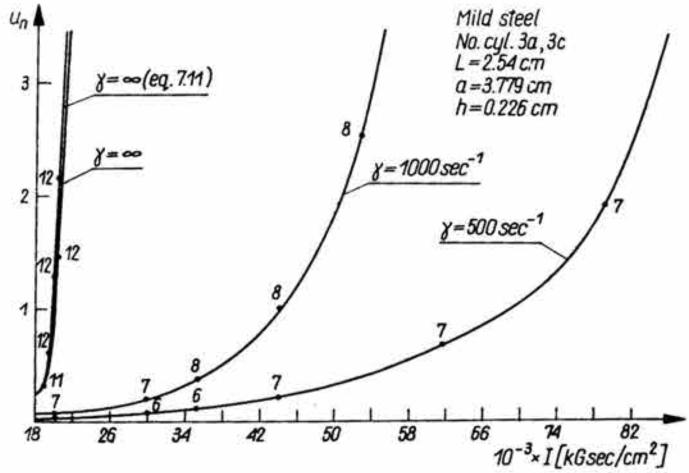

FIG. 12.

of the pulse begins to produce considerable increments of the deflection amplitude. It has also been established that the numbers at the dots distributed along the curves vary but little (usually by one) above the critical value of the impulse. In the case of ideal plasticity ($\gamma = \infty$), the asymptotic solution [Eq. (7.11)] lies very close to the exact solution (Figs. 11 and 12). For smaller γ the critical value of the impulse considerably increases, thus viscosity has a stabilizing effect on the buckling process. The length of the shell has a much smaller influence on the critical value.

It should be stressed once more that the influence of viscosity of the material on the form of buckling, the final displacements, the duration of the deformation process, as also on the critical impulse value in problems of dynamic buckling of shells is larger than the corresponding influence of the length of the shell.

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