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Application of energy methods in creep mechanics

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THE VALIDITY of several energy theorems well known in theory of plasticity is extended to creep mechanics for the case of general relationships between stress and strain rate tensors, restriction being made to incompressible, small motions and isotropic behaviour only. The influence of the third invariant of stress tensor is neglected.

Słuszność kilku twierdzeń energetycznych, dobrze znanych w teorii plastyczności, zostały rozszerzone na mechanikę pełzania dla przypadku ogólnej zależności pomiędzy tensorami naprężenia i prędkości odkształcenia. Ograniczono się do materiału nieściśliwego, małych ruchów i izotropowego zachowania się materiału. Wpływ trzeciego niezmiennika tensora naprężenia został pominięty.

Для случая общей зависимости между тензорами напряжения и скоростей деформирования, описывающей механику ползучести, доказана справедливость нескольких энергетических теорем, хорошо известных в теории пластичности. Исследованы случаи несжимаемости, малых перемещений и изотропного поведения материалов. При этом пренебрегается влиянием третьего инварианта тензора напряжений на механическое поведение в процессе ползучести.

IN THEIR work "Recent Trends in the Development of the Theory of Plasticity" (Pergamon Press, 1963), which covers what is generally known as time-independent plasticity, W. OLSZAK, Z. MRÓZ and P. PERZYNA, on p. 184 point out that problems of Rheology (which they simply define as "investigations of time-dependent phenomena of inelastic behaviour of bodies") would hardly admit of a coherent presentation from a unified standpoint. This may well be so. The following lines are meant to show that in certain fields of Rheology, the methods of plasticity theory could be applied with advantage and, in fact have been so applied for a long time. This statement refers to energy methods, applicable particularly in Creep Mechanics of structural metals, where time-dependent plasticity is associated with strongly nonlinear relationship between the tensors of stress and strain rate. Lack of analytic methods for the treatment of special problems here particularly calls for approximate methods.

If we exclude forerunners like A. HAAR and TH. V. KARMAN, the first attempt to apply energy methods in Plasticity is due to W. PRAGER and P. G. HODGE (1948). In Creep Mechanics Sovjet-Russian scientists started an early attack, see for instance L. M. KACHANOV (1949). This writer utilized what has subsequently become known as the elastic analogue, see N. J. HOFF (1953), deriving extremum theorems for the potential U and the complementary potential $\overline{U}$, cf. also F. K. G. ODQVIST (1966). These theorems will now be extended to general relationships between stress and strain rate tensors, restriction being made to incompressible, small motions and isotropic behaviour only. The influence of stress invariants of higher order than the second one will be neglected.

We introduce Cartesian tensors of stress σ_{ij} and strain ε_{ij}, using summation convention for the subscripts i, j, k only. Then,

$$\sigma_e^2 = \frac{3}{2} s_{ij} s_{ij}, \qquad \varepsilon_e^2 = \frac{2}{3} \varepsilon_{ij} \varepsilon_{ij} \tag{1}$$

are second-order invariants of the stress and strain tensors, where

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3} \sigma_{kk} \tag{2}$$

is the stress deviator and δ_{ij} is Kronecker's symbol. Obviously, we have

$$s_{kk} = 0. \tag{3}$$

We shall utilize the elastic analogue, implying that the strain rate $\dot{\varepsilon}_{ij}$ may, during the analysis, be replaced by the strain ε_{ij}, this being permitted if secondary creep under time-independent forces only is considered. If u_i is the displacement vector, the strain will be

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

The analysis being completed, we may again replace ε_{ij} by $\dot{\varepsilon}_{ij}$. In the following we will assume both ε_{ij} and $\dot{\varepsilon}_{ij}$ to be small. Incompressibility implies

$$\varepsilon_{kk} = 0. \tag{5}$$

As constitutive relation we may assume σ_e to be given as a function of ε_e, see Fig. 1. Together with the assumption that ε_{ij} be orthogonal to the surfaces σ_e = constant in stress

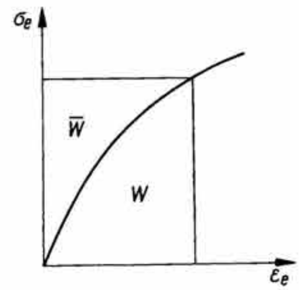

FIG. 1. Constitutive relationship between σ_e and ε_e.

space, assumed to be convex. Then, we obtain for the increment of elastic work per unit volume, remembering (2)

$$dW = \sigma_e d\varepsilon_e = \sigma_{ij} d\varepsilon_{ij} = s_{ij} d\varepsilon_{ij} \tag{6}$$

and, similarly, for the complementary work $\overline{W}$

$$d\overline{W} = \varepsilon_e d\sigma_e = \varepsilon_{ij} ds_{ij} = \varepsilon_{ij} d\sigma_{ij}. \tag{7}$$

Of the functional relationships $W(\varepsilon_e)$ and $\overline{W}(\sigma_e)$, we shall require that they are continuous with their derivatives to the second order and that

$$W(0) = 0, \qquad \overline{W}(0) = 0, \tag{8}$$

(9) $$\frac{d^2 W}{d\varepsilon_e^2} > 0, \quad \frac{d^2 \overline{W}}{d\sigma_e^2} > 0.$$

From (6) and (7) follows the relation

(10) $$W + \overline{W} = \sigma_e \varepsilon_e = s_{ij}\varepsilon_{ij}.$$

We shall now consider the following boundary value problem (B. P.) of mixed type. Given a domain V in space, bounded by the surface S with the outer normal n_j, we shall try to find the state of deformation u_i, ε_{ij} in the case when the body within V is subject to volume forces X_i and surface forces $\sigma_{ij} n_j = T_i$ over part S_T of the surface S, whereas u_i be given over the rest $S_u = S - S_T$ of S. Thus the boundary conditions are

(11) $$\sigma_{ij} n_j = T_i \text{ given on } S_T$$

(12) $$u_i \text{ given on } S_u,$$

where T_i is supposed vanish.

The equilibrium conditions read

(13) $$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = 0 \text{ within } V.$$

Equations (4), (5), (13) and the relationship $\sigma_e = \sigma_e(\varepsilon_e)$ enable us to obtain a system of four independent equations for the four unknown functions u_i and σ_{kk}, so the number of equations is sufficient. The constitutive relations being in general nonlinear, analytic methods usually will not be available for the treatment of special problems. In their stead, we shall utilize the extremum properties of U and $\overline{U}$.

The potential U and complementary potential $\overline{U}$ are defined

(14) $$U = \int_V [W(\varepsilon_e) - X_i u_i] dV - \int_{S_T} T_i u_i dS,$$

(15) $$\overline{U} = \int_V \overline{W}(\sigma_e) dV - \int_{S_u} T_i u_i dS.$$

As written down in (14) and (15), for the true solution u_i, ε_{ij}, σ_{ij} of the problem B.P., the quantities U and $\overline{U}$ are, of course, scalars. If u_i, etc. are replaced by other functions, U and $\overline{U}$ become functionals.

If we allow the solution u_i of B.P. to be varied in such a way (*admissible state of deformation*) that the varied displacement $u_i + \delta u_i = u_i^*$ obeys the same boundary conditions on the surface S_u as u_i, whereas the prescribed external forces, i.e., the volume force X_i in V and the surface tractions T_i on S_T remain unvaried, we may prove that U is a true minimum for $u_i^* = u_i$. Here and later, we may use Gauss-Green's theorem in the form

(16) $$\int_V \frac{\partial F}{\partial x_j} dV = \int_S F n_j dS,$$

where F is a function, continuous with $\partial F/\partial x_j$ in $V+S$.

In fact, let us form

$$\delta U = \int_S \left(\frac{\partial W}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij} - X_i \delta u_i \right) dV - \int_{S_T} T_i \delta u_i dS.$$

Here we have

$$\delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\delta u_i}{\partial x_j} + \frac{\partial\delta u_j}{\partial x_i}\right).$$

Utilizing (6), we obtain

$$\frac{\partial W}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} = \sigma_{ij}\delta\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial}{\partial x_j}(\sigma_{ij}\delta u_i) + \frac{\partial}{\partial x_i}(\sigma_{ij}\delta u_j) - \delta u_i\frac{\delta\sigma_{ij}}{\partial x_j} - \delta u_j\frac{\partial\sigma_{ij}}{\partial x_i}\right]. \tag{17}$$

Remembering that $\sigma_{ij} = \sigma_{ji}$, we obtain, utilizing (16),

$$\delta U = -\int_V\left(\frac{\partial\sigma_{ij}}{\partial x_j} + X_i\right)\delta u_i dV + \int_{S_T}(\sigma_{ij}n_j - T_i)\delta u_i dS + \int_{S_u}\sigma_{ij}n_j\delta u_i dS.$$

Here, the three integrands all vanish, due to (13), (11) and (12), respectively, and the result is

$$\delta U = 0. \tag{18}$$

Further, we have, due to (9),

$$\delta^2 U = \frac{1}{2}\int_V\frac{d^2W}{d\varepsilon_e^2}(\delta\varepsilon_e)^2 dV > 0. \tag{19}$$

Thus, as a consequence of (18) and (19), we have prooved

THEOREM 1. *The potential U has a true minimum for the solution of problem* B.P. *if the state of strain is varied within the field of admissible states of deformation.*

Further, we may vary the state of stress σ_{ij} in such a way that the varied stress system $\hat{\sigma}_{ij} = \sigma_{ij} + \delta\sigma_{ij}$ fulfils the differential Eqs. (13) and the boundary conditions (11) (*admissible state of stress*) and form, utilizing (7) and (2),

$$\delta\overline{U} = \int_V\varepsilon_{ij}\delta\sigma_{ij}dV - \int_{S_u}\delta T_i u_i dS.$$

After a similar partial integration as in (17), we obtain, using (16) and observing that $\delta\sigma_{ij} = \delta\sigma_{ji}$

$$\delta\overline{U} = -\int_V\frac{\partial\delta\sigma_{ij}}{\partial x_j}u_i dV + \int_{S_u}(n_j\delta\sigma_{ij} - \delta T_i)u_i dS + \int_{S_T}n_j\delta\sigma_{ij}u_i dS.$$

Here, the integrands will vanish due to (13), (12) and (11) respectively, and we obtain $\delta\overline{U} = 0$. Further, we have

$$\delta^2\overline{U} = \frac{1}{2}\int_V\frac{d^2\overline{W}}{d\sigma_e^2}(d\sigma_e)^2 dV > 0,$$

due to (9), and thus we have proved

THEOREM 2. *The complementary potential $\overline{U}$ has a true minimum for the solution of problem* B.P., *if the state of stress be varied within the field of admissible states of stress.*

The Theorems 1 and 2 were first proved by KACHANOV (1949) in the case of a special, time-hardening body and by ODQVIST (1966) for the special case

$$W(\varepsilon_e) = \frac{n\sigma_c}{n+1}\varepsilon_e^{1+1/n}, \quad \overline{W}(\sigma_e) = \frac{\sigma_c}{n+1}\left(\frac{\sigma_e}{\sigma_c}\right)^{n+1}, \tag{20}$$

where n and σ_c are material constants. Remembering (6), (7) and (10), we then also have

$$\varepsilon_e = \left(\frac{\sigma_e}{\sigma_c}\right)^n, \quad W = \frac{n}{n+1}\sigma_e\varepsilon_e, \quad \overline{W} = \frac{1}{n+1}\sigma_e\varepsilon_e. \tag{21}$$

The present proof allows the Theorems 1 and 2 to be extended to more general constitutive equations, holding true over much larger intervals of strain ε_e, for example to those published by ODQVIST (1970), see Fig. 2.

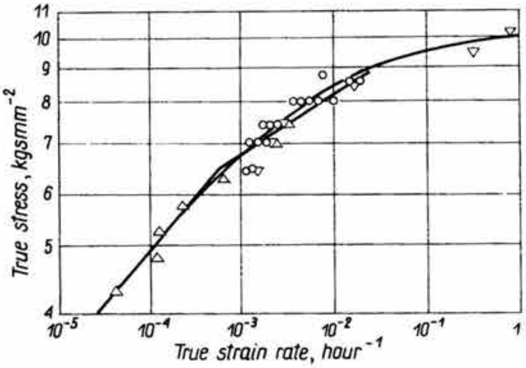

FIG. 2. Constitutive relationship for Mg alloy ZRE1 260°C (Söderquist, Storakers) containing 0.6 per cent Zr and 2.95 per cent rare earths.

If we add the two Eqs. (14) and (15), we obtain, using (10),

$$U+\overline{U} = \int_V (\sigma_{ij}\varepsilon_{ij} - X_i u_i)dV - \int_S T_i u_i dS.$$

Introducing (4) and repeating a transformation like the one in (17) in combination with (16), we get, using (11), (12) and (13),

$$U+\overline{U} = -\int_V \left(\frac{\partial\sigma_{ij}}{\partial x_j} + X_i\right) u_i dV + \int_S (\sigma_{ij}n_j - T_i)u_i dS = 0. \tag{22}$$

This relation in combination with the Theorems 1 and 2 may be used to give upper and lower bounds for U and $\overline{U}$. If we form U with any admissible state of strain u_i^*, ε_{ij}^*, it may be denoted with U^*. Similarly, $\overline{U}$ formed with any admissible state of stress $\hat{\sigma}_{ij}$, may be denoted with $\hat{\overline{U}}$. Then, we have

$$U^* > U, \quad \hat{\overline{U}} > \overline{U}, \tag{23}$$

hence also

$$-U^* < -U, \quad -\hat{\overline{U}} < -\overline{U}.$$

and then, from (22),

$$U^* > U > -\hat{\bar{U}}, \quad \hat{\bar{U}} > \bar{U} > -U^*. \tag{24}$$

Also, adding the inequalities (23), we have

$$U^* + \hat{\bar{U}} \geqslant 0, \tag{25}$$

equality being reserved for the case, when $u_i^* = u_i$ and $\hat{\sigma}_{ij} = \sigma_{ij}$.

If we now introduce Drucker's postulate (1951) in the form

$$\int_{\varepsilon''_{ij}}^{\varepsilon'_{ij}} (\sigma'_{ij} - \sigma''_{ij}) d\varepsilon_{ij} \geqslant 0, \tag{26}$$

where σ'_{ij}, ε'_{ij} and σ''_{ij}, ε''_{ij} are any two states of stress and the corresponding strains according to (6) and Fig. 1, we may carry out the integration in (26) while keeping σ''_{ij}, ε''_{ij} constant. Thus we obtain

$$W(\varepsilon'_{ij}) - W(\varepsilon''_{ij}) - \sigma''_{ij}(\varepsilon'_{ij} - \varepsilon''_{ij}) \geqslant 0.$$

Remembering (10), this finally yields

$$W(\varepsilon'_{ij}) + \bar{W}(\sigma''_{ij}) \geqslant \sigma''_{ij} \varepsilon'_{ij}, \tag{27}$$

an inequality, derived by J. B. MARTIN (1964), and used by him to estimate, in the case of constitutive relations of the type of (20), an upper bound for the displacement δ in an arbitrary point Q of the body, MARTIN (1966). MARTIN assumes ε'_{ij} to be an admissible state of strain. Neglecting volume forces, he assumes σ''_{ij} to be a state of stress in internal and external equilibrium with the given surface tractions T''_i. Integrating (27) over the volume V, he obtains

$$\frac{n}{n+1} \int_V \sigma'_{ij} \varepsilon'_{ij} dV + \frac{1}{n+1} \int_V \sigma''_{ij} \varepsilon''_{ij} dV \geqslant \int_V \sigma''_{ij} \varepsilon'_{ij} dV. \tag{28}$$

With a transformation like the one in (17), utilizing (16), he finds

$$\frac{1}{n+1} \int_V \sigma''_{ij} \varepsilon''_{ij} dV \geqslant \int_S \left(T''_i - \frac{n}{n+1} T_i \right) u'_i \, dS.$$

The surface traction T''_i are used to eliminate the unknown displacements u'_i corresponding to ε'_{ij} except the displacement δ. Adding a force P in the point Q and in the direction in which δ is required, from (28) he finally obtains

$$\frac{1}{n+1} \int_V \sigma''_{ij} \varepsilon''_{ij} dV \geqslant P\delta. \tag{29}$$

The upper bound for δ thus contains but the state of stress σ''_{ij}, which may be guessed in advance with the only requirement to be in internal and external equilibrium with the tractions $nT_i/n+1$, to which we have to add the force P, so that P occurs implicitely also on the left side of (29). The upper bound for δ may then be optimized with respect to P.

Inequalities of the type of (29) have been generalized to simultaneously elastic and creeping materials, i.e., to problems of relaxation and redistribution of stress. Further, also to strain-hardening creeping materials and to bodies creeping in the time-independent plastic range, see F. A. LECKIE and J. B. MARTIN (1967), A. R. S. PONTER (1969), F. A. LECKIE and A. R. S. PONTER (1970).

Thus, for example, these writers have treated the problem of simultaneous creep rate $\dot{v}_{ij}$ and time-independent plastic strain rate $\dot{p}_{ij}$, where the dots indicate differentiation with respect to the time, as before. The yield surface may be $f(\sigma_{ij}) = 0$, assumed to be convex, i.e.,

$$(\sigma_{ij}-\sigma^c_{ij})\dot{p}_{ij} > 0, \tag{30}$$

where σ_{ij} is on the yield surface in a point where $\dot{p}_{ij}$ is normal to this surface and σ^c_{ij} is any state of stress within or on the yield surface. Neglecting elastic strain, the total strain rate is $\dot{\varepsilon}_{ij} = \dot{v}_{ij}+\dot{p}_{ij}$, assumed to be derived from the velocity $\dot{u}_i$ according to (4), differentiated with respect to the time. Utilizing the inequality (28), LECKIE and PONTER (1970) have proved that estimates of the type of (29) may still be used if σ^c_{ij} in (30) be put

$$\sigma^c_{ij} = \frac{n+1}{n}\sigma''_{ij},$$

that is, if the stress σ''_{ij} be within the surface

$$f\left(\frac{n+1}{n}\sigma''_{ij}\right) = 0,$$

see Fig. 3. This result must be valued highly, as it gives the computer a possibility to estimate creep deformations in particularly interesting stress regions, which have been inaccessible so far.

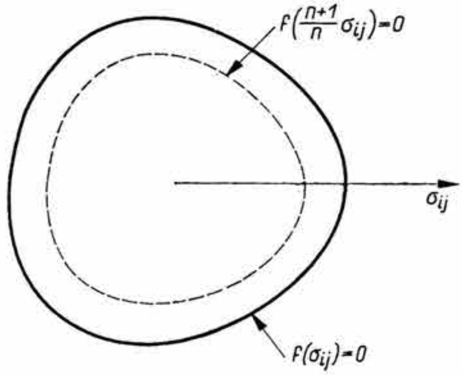

FIG. 3. Yield surface.

In further papers, independent of the assumption of the elastic analogue, A. R. S. PONTER has also treated more complicated loading cases, including problems of shakedown in the presence of creep, to establish deformation bounds for bodies approaching rupture, see A. R. S. PONTER (1971) and also F. A. LECKIE and A. R. S. PONTER (1971).

In order to demonstrate the use of the inequality (29), MARTIN treats the simple problem of a cantilever beam loaded uniformly with p per unit of length according to Fig. 5.

In this case, an exact solution is available for comparison. For the deflection w, positive upwards as in Fig. 4, we have

$$\frac{d^2w}{dx^2} = -\mu M^n, \tag{31}$$

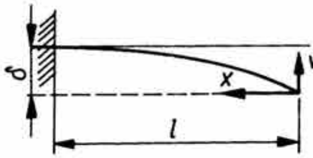

FIG. 4. Coordinate system.

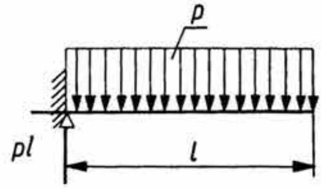

FIG. 5. Cantilever beam uniformly loaded.

where μ is a constant, and the minus sign corresponds to $M > 0$ for a beam bent concave from below. For simplicity, n is assumed to be an odd integer > 1.

$$\mu = \sigma_e^{-n} I_n^{-n}, \quad I_n = \int z^{1+1/n} dA.$$

Further, we have

$$M = \frac{px^2}{2}$$

and hence

$$\frac{d^2w}{dx^2} = -\mu\left(\frac{p}{2}\right)^n x^{2n}$$

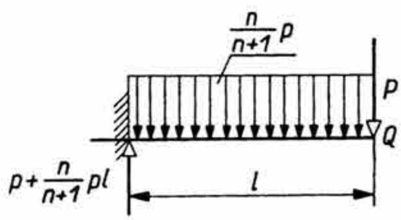

FIG. 6. J. B. Martin's auxiliary loading system.

to be integrated with the conditions $w(0) = 0$, $w(l) = \delta$, $(dw/dx)_{x=l} = 0$, yielding

$$\delta = \mu\left(\frac{p}{2}\right)^n \frac{l^{2n+2}}{2n+2}.$$

For the dimensionless deflection Δ, we obtain

$$\Delta = \frac{\delta}{\mu}\left(\frac{2}{p}\right)^n l^{-2n-2} = \frac{1}{2n+2}. \tag{32}$$

The equilibrium system considered by MARTIN is seen in Fig. 6. In this case, we have

$$M'' = \frac{pn}{2(n+1)} x^2 + Px.$$

Application of (29) yields

$$\frac{1}{n+1}\int_0^l \mu(M'')^{n+1}dx \geqslant P\delta. \tag{33}$$

Martin has maximized δ from (33) as a function of the parameter $2P/pl$, see the table below.

For comparison, we may utilize Theorem 1 and compute

$$U = \int_0^l \left(\frac{n}{n+1}\mu^{-1/n}\left|\frac{d^2w}{dx^2}\right|^{1+1/n} + pw\right)dx - pl\delta. \tag{34}$$

Here, we may use a form for w, capable of satisfying the boundary conditions, say

$$\frac{d^2w}{dx^2} = Cx^m, \qquad \frac{dw}{dx} = \frac{C}{m+1}(x^{m+1} - l^{m+1}),$$

$$w = \frac{C}{m+1}\left(\frac{x^{m+2}}{m+2} - xl^{m+1}\right), \qquad \delta = -\frac{Cl^{m+2}}{m+2}, \tag{35}$$

where the two constants C and m are free for minimizing U. Inserting in (34), we obtain

$$U = \frac{n}{n+1}\mu^{-1/n}\frac{l^{m(1+1/n)+1}}{m(1+1/n)+1}C^{1+1/n} + \frac{pl^{m+3}C}{2(m+3)}.$$

Putting $\partial U/\partial C = 0$, we obtain for the dimensionless deflection

$$\Delta = \frac{1}{m+2}\left[\frac{m(1+1/n)+1}{m+3}\right]^n. \tag{36}$$

Minimizing U with respect to m means maximizing Δ. This has been carried out, and the result is seen in the table below. Obviously, $m = 2n$ renders Δ maximum and corresponds to the exact solution (32).

	MARTIN (1966)		Theorem 1		Exact. Eq. (32, 36)
n	Δ_M	$2P/pl$	Δ_I	m	Δ
1	0.254	0.387	0.2500	2	0.2500
3	0.126	0.25	0.1250	6	0.1250
5	0.083	0.15	0.0833	10	0.0833

MARTIN produced this simple example just to show how the inequality (29) is to be used. It is possible that more interesting applications could be found. For the time being, I find the Theorem 1 as powerful as (29). Moreover, Theorem 1 yields the deflection in all points of the beam, whereas (29) gives it in one point only. In fact, Martin's method is more related to Theorem 2, which would need introduction of an auxiliary force P in the way done by MARTIN and is well-known in structural mechanics as Castigliano's method.

Still I think that the work initiated by J. B. MARTIN is well justified. Particularly, the further development, due mainly to F. A. LECKIE and A. R. S. PONTER, must be highly appreciated.

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